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Stress relaxation and elastic follow-up using a stress range-dependent constitutive model

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Abstract: Despite the availability of detailed non-linear finite element analysis (FEA), some aspects of high-temperature design can still be best addressed through more simplified methods. One such simplified method relates to the problem of elastic follow-up where, typically in strain-controlled situations, elastic behaviour in one part of a structure can lead to large strain accumulation in another. Over the past 30 years, it has been shown that in regions with significant elastic follow-up, a plot of maximum stress against strain (a 'stress-strain trajectory') is virtually independent of the constitutive relation – a characteristic which can be used to estimate elastic follow-up for design purposes without detailed non-linear FEA. The majority of studies which have reported this independence on material behaviour have used simple constitutive models for creep strain, primarily based on power-law creep or variations. Recently, studies of the behaviour of high-temperature structures with a stress range-dependent constitutive law have begun to emerge. This article examines the problem of elastic follow-up using such a constitutive law for a classic two-bar structure and for a more complex structure using FEA. It is found that the independence of the stress–strain trajectory on constitutive equation is lost with a stress range-dependent relation.

Keywords: creep, power-law breakdown, structural analysis, stress relaxation, elastic follow-up, high-temperature design

1 INTRODUCTION

Comprehensive assessment rules for high-temperature design are usually, of necessity, a compromise between detailed inelastic analysis and simplified, generic rules. One such area of compromise relates to the issue of elastic follow-up in complex structures, where dominant elastic behaviour in one part of a structure can lead to excessive creep strain accumulation in local regions in another part. This can be a particular problem in structures which are predominantly under stress relaxation conditions. Elastic follow-up is not a special problem if a detailed inelastic analysis can be performed to evaluate, and thereby limit, the level of inelastic strain. However, the aim in current high-temperature assessment design codes is to avoid such detailed inelastic analysis for localized

regions and instead rely upon more simplified rules. For example, the *JSME NC2-2005* [1] treats elastic follow-up through a so-called 'elastic follow-up parameter', q , unique to each structure, which can be estimated from simplified elastic analysis and uniaxial creep data. In fact, the JSME standard takes a 'conservative' estimate of this factor as $q=3$ for fast breeder reactor components based on more detailed inelastic studies. The *R5 Assessment Procedure for the High-Temperature Response of Structures* [2] similarly defines an 'elastic follow-up factor' for general stress relaxation, which can be estimated by simplified analysis. A common feature of these elastic follow-up factors is the assertion that these are approximately independent of the creep law and represent a predominantly geometrical effect. The aim of this article is to demonstrate that, for certain types of

creep law, there can be a strong dependency on the form of the law and that, in these cases, design and assessment procedures may need to be re-examined.

Elastic follow-up was introduced in 1955 by Robinson [3] in a study of the creep of high-temperature steam piping systems. He demonstrated that despite being deformation controlled, large creep strains could, in fact, occur and explained this type of behaviour, which he called 'follow-up elasticity' by the tendency of certain configurations to maintain stress in a highly strained component through an elastic action even though the overall loads relaxed – thereby slowing down the expected rate of creep relaxation. The terminology 'elastic follow-up' initially passed into piping design – the ASME Boiler and Pressure Vessel Code has repeatedly cautioned the piping designer about the potential design problems associated with this behaviour. Nevertheless, design guidance in the Code remained unhelpful for some time since. Despite reminding the designer that such a condition should be avoided, no means was given for determining if elastic follow-up existed either qualitatively or quantitatively. Over the years, several researchers, including the author, attempted to resolve this problem, not only for piping systems but also for any component which exhibits this behaviour, in particular structural discontinuities. A full review will not be given here, but reference can be made to other reviews of elastic follow-up over the past two decades by Boyle and Nakamura [4], Kasahara [5, 6] and more recently by Hadidi-Moud and Smith [7]. Of relevance to the current work described in this study, Kasahara [5, 6] analysed elastic follow-up in structural discontinuities, extending previous work to include plasticity in addition to creep. To begin with, a simple two-bar structure (first introduced in that form by Boyle and Nakamura [4]) was analysed to demonstrate the independence of elastic follow-up from the plastic or creep law. This was followed by detailed inelastic finite element analyses (FEA) of an axisymmetric Y-piece under thermal loading. Various creep laws were used and it was demonstrated that in terms of quantifying elastic follow-up '... the structures have the unique characteristic of being insensitive to the creep strain equations ...' [6, Section 2.2]. Further, Hadidi-Moud and Smith, following their review paper [7], have written a series of papers [8–10] which extend the simple two-bar structure of Boyle and Nakamura [4] to a number of similar simple 'benchmark' bar structures, representative of the behaviour of real structures, as part of a study into the relaxation of residual stress in a range of structural components. The concept of an 'elastic follow-up factor' from R5 [2] is further developed and it is argued that the factor '... is

independent of the creep law and is reflecting a purely geometrical effect ...' [10, p. 363]. In reference [8], a series of experimental studies of these benchmark bar structures are reported – these experiments reflected the essential features of the theoretical studies, although the elastic follow-up factors were greater than predicted. In addition, it was found that initial residual stresses did not significantly contribute to elastic follow-up.

Most studies of elastic follow-up have been based on simple creep constitutive models. These simple constitutive models, for example the time- and strain-hardening constitutive equations, are based on adaptations for time-varying stress of equally simple models for the secondary creep stage from constant load/stress uniaxial tests where the minimum creep rate is constant. In fact, the majority of studies of the characteristics of elastic follow-up (the exception being the work of Kasahara [5, 6]) simply use a secondary creep law combined with elastic behaviour. The most common secondary creep constitutive model has been the Norton–Bailey law, which gives a power-law relationship between minimum creep rate and (constant) stress. The unique mathematical properties of the power-law allowed the development of robust simplified methods, many of which can be found in high-temperature design codes. Now that detailed FEA for creep is readily accomplished on the desktop, it is perhaps surprising that the simple time- or strain-hardening constitutive models based on power-law creep remain the most widely available in common commercial finite element software, such as ANSYS or ABAQUS, even though more comprehensive time-dependent non-linear constitutive models are available (and can be included as user-defined materials). The most common reason for persisting with the more simple constitutive models is the ease with which material constants can be derived from experiments, the ability to check detailed solutions with simplified (robust) methods, and an underlying understanding of the expected behaviour of simple structures subject to power-law creep [11–13]. Nevertheless, it has long been known that creep over a range of stress does not follow one simple power-law relationship, typically following one power-law at low stress and another at high stress – a phenomenon known as 'power-law breakdown'. A common observation is a shift from a power-law (usually dislocation) mechanism at 'moderate' stress to a diffusion mechanism at 'low' stress, characterized by a linear viscous relationship between creep rate and stress [14, 15] with a more significant power-law breakdown at 'high' stress. Such a stress range-dependent constitutive model, with a

transition from linear to power-law behaviour, has recently been studied by Naumenko *et al.* [16]: stress analyses using this modified power-law were compared to linear and pure power-law over a range of stress and load for several simple structures – simple stress relaxation, the beam in bending, and a pressurized thick cylinder. In this article, the first analysis – stress relaxation – is extended to examine the implications for elastic follow-up.

2 SECONDARY CREEP CONSTITUTIVE MODEL

The minimum creep rate ($\dot{\epsilon}_{\min}$) during the secondary (or steady state) deformation stage is frequently related to the (constant) applied stress (σ) by a power-law relationship in the form

$$\dot{\epsilon}_{\min} = B\sigma^n \quad (1)$$

where B and n are constants determined from uniaxial creep testing. Use of a power-law relation reflects an almost linear relationship between $\log(\text{minimum creep rate})$ and $\log(\text{stress})$ which is often found in creep tests: typical results for an austenitic stainless steel AISI 316L(N) taken from Rieth *et al.* [17] are shown in Fig. 1.

However, many metals and alloys typically exhibit different regimes with $n \approx 1$ at low stresses and $n \approx 4$ or 5 at higher stress levels with n increasing again in the power-law breakdown regime [14]. This is illustrated in Fig. 2, taken from reference [14] based on data on 0.5Cr0.5Mo0.25 V steel from Evans *et al.* [18]. Indeed, at lower temperatures (although still above that for creep), even the data from reference [17] show similar behaviour, Fig. 3. Numerous attempts have been made to find a continuous curve to

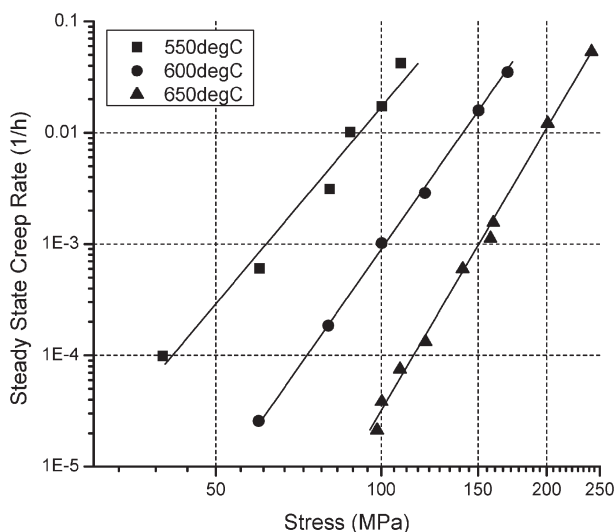


Fig. 1 Steady creep of austenitic AISI 316L(N) 550–650°C after reference [17]

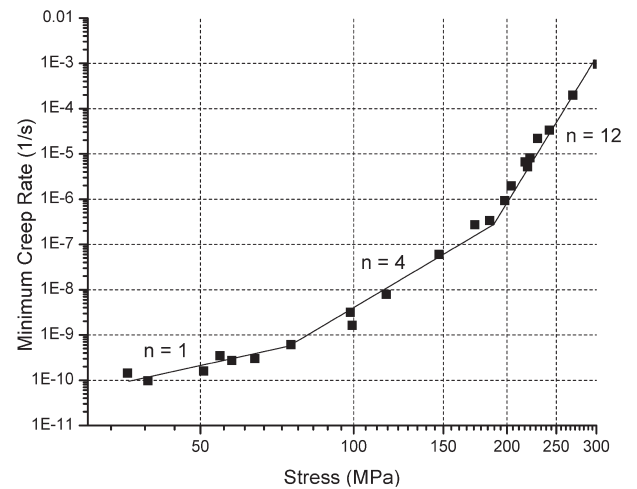


Fig. 2 Steady creep of 0.5Cr0.5Mo0.25 V after reference [14]

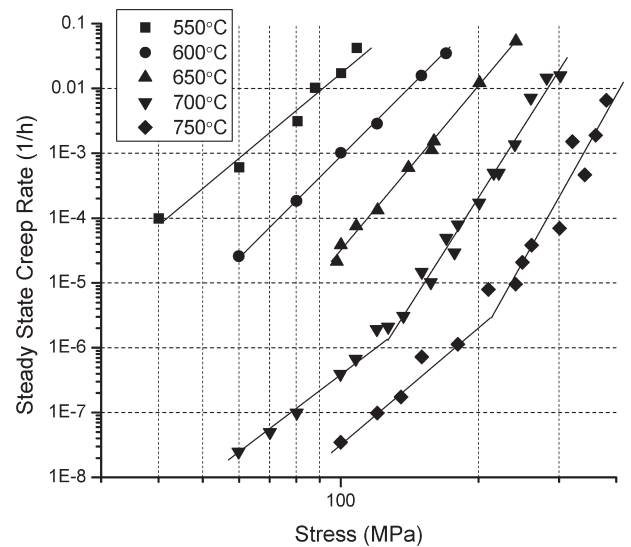


Fig. 3 Steady creep of austenitic AISI 316L(N) 550–750°C after reference [17]

describe this behaviour over the complete stress range, principal among these being the hyperbolic sine relationship

$$\dot{\epsilon}_{\min} = B \sinh(C\sigma) \quad (2)$$

and the equation proposed by Garofalo [19]

$$\dot{\epsilon}_{\min} = B \sinh(C\sigma)^n \quad (3)$$

where B , C , and n are constants. A more complete summary can be found in reference [20].

It is often argued that the change in behaviour from low to moderate stress can be explained by diffusional creep theories, while the transition from moderate to high stress (power-law breakdown) can be accounted for by diffusion-controlled mechanisms and movement of lattice dislocations, for example

in pure metals. These explanations are not generally agreed to [14]. Nevertheless, there remains an obvious need to perform stress analysis with this type of constitutive behaviour and reliable (if not perfect) constitutive models are required. Williams and Wilshire [21] proposed the ‘transition stress’ model for power-law breakdown

$$\dot{\epsilon}_{\min} = B(\sigma - \sigma_0)^p \quad (4)$$

where B and p are constants and σ_0 the transition stress. Unfortunately, the transition stress cannot be reliably measured. To model the transition from low to moderate stress, Naumenko *et al.* [16] proposed a constitutive relationship which assumed that the physical mechanisms were independent and that the corresponding creep rates could simply be added

$$\frac{\dot{\epsilon}_{\min}}{\dot{\epsilon}_0} = \frac{\sigma}{\sigma_0} + \left(\frac{\sigma}{\sigma_0}\right)^n \quad (5)$$

where σ_0 , ϵ_0 , and n are material constants. The stress σ_0 is a kind of transition stress different from that studied by Williams and Wilshire [21], since it specifies the stress level at which the behaviour changes from linear (viscous) to power-law (Fig. 4). Equation (5), which shall be referred to as a ‘modified power-law’ for simplicity, was used by Naumenko *et al.* [16] to examine how the stress system in simple components – uniaxial stress relaxation, a beam in bending and a pressurized thick cylinder – would change compared to pure linear and pure power-law behaviour. In this article, the stress relaxation problem will be re-examined in more detail in the context of a more detailed analysis of *elastic follow-up* – the beam and thick cylinder problems are considered in more detail elsewhere [22], in particular the applicability of simplified methods and deformation behaviour.

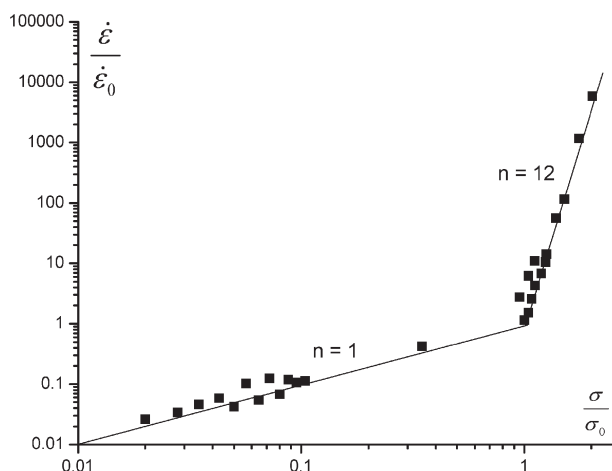


Fig. 4 Steady creep of 9 per cent Cr steel at 600°C after reference [16]

3 EFFECT OF THE MODIFIED POWER-LAW ON STRESS RELAXATION

The classic problem of simple uniaxial stress relaxation is well known [11–13]. The problem was briefly re-examined using the modified power-law, equation (5), by Naumenko *et al.* [16]. While equation (5) includes the necessary features for stress relaxation of elastic follow-up, namely hardening and non-linear steady state creep, primary creep is neglected. Thus, reference [16] and the further studies to be described in this article are limited to a qualitative analysis of essential characteristics.

Stress relaxation of a single uniaxial bar occurs when the strain is held *constant* – that is the problem is deformation, or strain, controlled. Then, the creep strain rate, equation (5), is added to the elastic strain rate such that

$$\frac{\dot{\sigma}}{E} + \dot{\epsilon}_0 \left[\frac{\sigma}{\sigma_0} + \left(\frac{\sigma}{\sigma_0}\right)^n \right] = 0 \quad (6)$$

where E is the Young’s modulus. Equation (6) can be solved subject to a suitable initial condition for $\sigma(0)$.

Introducing the normalized quantities

$$\gamma = \frac{E\dot{\epsilon}_0}{\sigma_0} \quad \tau = \gamma t$$

Equation (6) can be re-written in the form

$$\frac{dS}{d\tau} + S + \alpha^{n-1} S^n = 0 \quad (7)$$

where $S(\tau) = \sigma(\tau)/\sigma(0)$ and a load parameter, $\alpha = \sigma(0)/\sigma_0$, the ratio of the initial elastic stress in the bar to the transition stress in the modified power-law. The initial condition for equation (7) is simply $S(0) = 1$; so its solution is

$$S(\tau) = \frac{e^{-\tau}}{\left[1 + \alpha^{n-1}(1 - e^{-(n-1)\tau})\right]^{\frac{1}{n-1}}} \quad (8)$$

The result is plotted in Fig. 5 for various values of n and α . Comparison may be made with the special cases of pure linear viscous creep, where the non-linear component is ignored in equation (5), and pure power-law where the linear part is ignored – the latter of course being the classic solution for creep stress relaxation.

For linear viscous creep, the solution is

$$S_{\text{Linear}}(\tau) = e^{-\tau} \quad (9)$$

and for pure power-law creep, the solution is [11–13]

$$S_{\text{Power}}(\tau) = \frac{1}{\left[1 + (n-1)\alpha^{n-1}\tau\right]^{\frac{1}{n-1}}} \quad (10)$$

It should be noted that the latter is not the conventional form for power-law stress relaxation since the

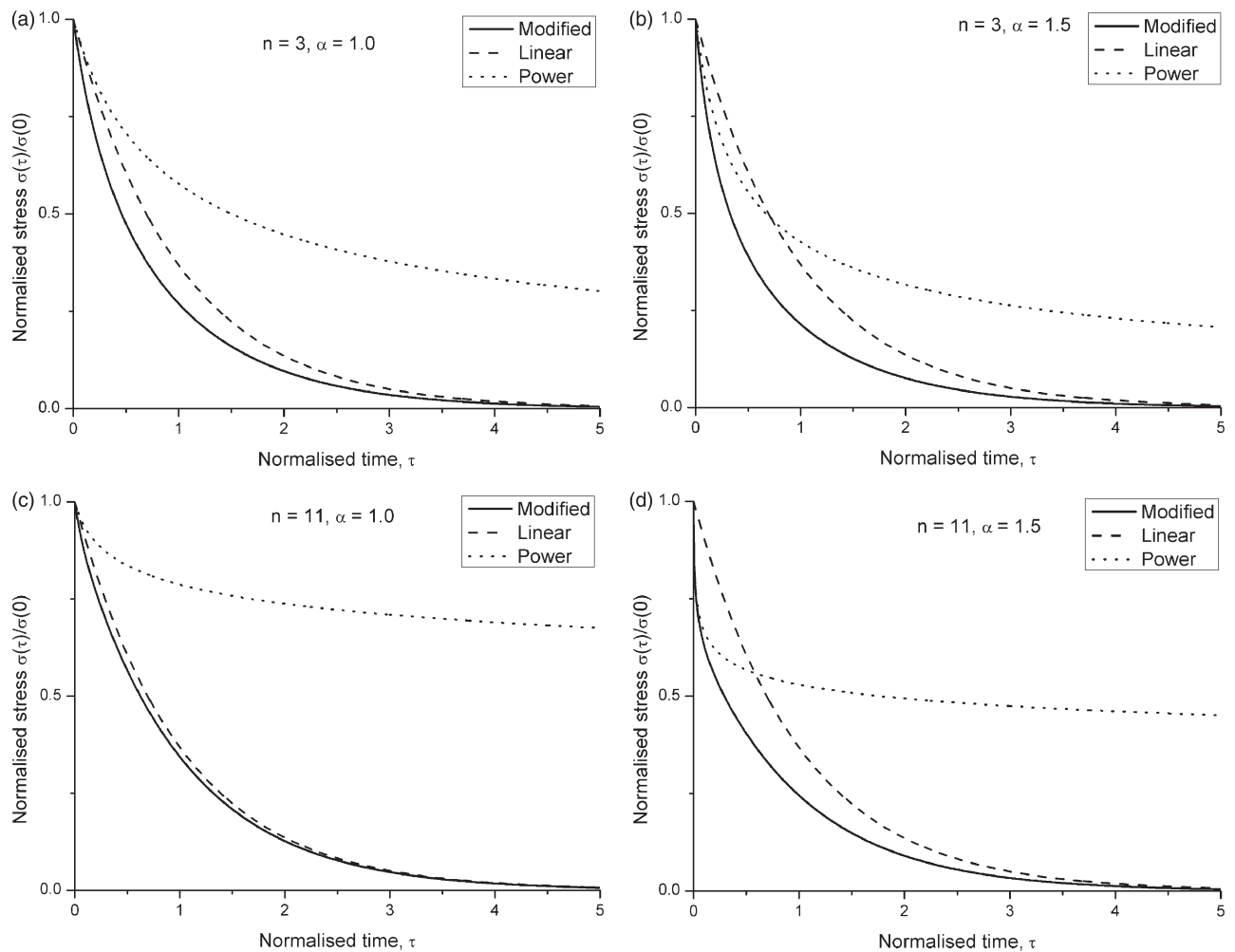


Fig. 5 Stress relaxation of a uniaxial bar: comparison of modified power law with linear viscous and pure power law for various values of n and α

timescale has been normalized with respect to the parameter τ rather than using $\tau = E\sigma(0)^{n-1}t$, which would render the solution independent of α . The pure viscous, equation (9), and pure power-law, equation (10), solutions are also shown in Fig. 5.

Results are shown in Fig. 5 for $n=3$ and 11 and $\alpha=1.0$ and 1.5. The values of α reflect two different cases, where the initial stress equals the modified power-law creep transition stress, σ_0 , and where it is 50 per cent above the transition stress. As noted by Naumenko *et al.* [16], it could be expected that, at least in the latter case, the power-law solution, equation (10), would dominate, but the results of Fig. 5 show otherwise. The relaxation curve for the modified power-law only follows the pure power-law solution for values of normalized time, $\tau \ll 1$, but it tends to approach the linear viscous solution, equation (9), as normalized time increases. For $\alpha=1.0$, it can be seen that the linear viscous solution dominates and indeed for values of $\alpha < 1$, the relaxation curves are practically identical. This result may seem surprising at first. In

reference [22], it is shown in a study of a beam in bending and a pressurized thick cylinder that, for load-controlled steady creep and values of maximum elastic stress 50 per cent above the transition stress in the modified power-law, the pure power-law solution does dominate as expected. However, this is apparently not the case for stress relaxation (at least in the present simple example). In fact, in reference [16], it is suggested that the influence of linear creep on a presumed power-law material could be detected by performing a relaxation test; this has been further investigated by Altenbach *et al.* [23]. The aim of this study is to examine this effect in the context of elastic follow-up.

4 EFFECT OF THE MODIFIED POWER-LAW ON ELASTIC FOLLOW-UP

4.1 Elastic follow-up of a two-bar structure: Power-law creep

The problem of the relaxation of a simple two-bar structure was introduced by Boyle and Nakamura [4]

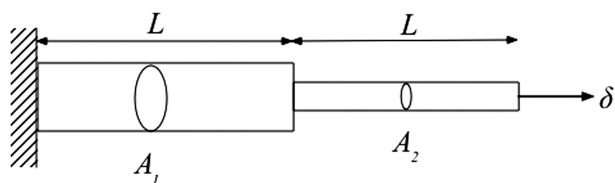


Fig. 6 Simple two-bar structure subject to a fixed axial displacement, δ

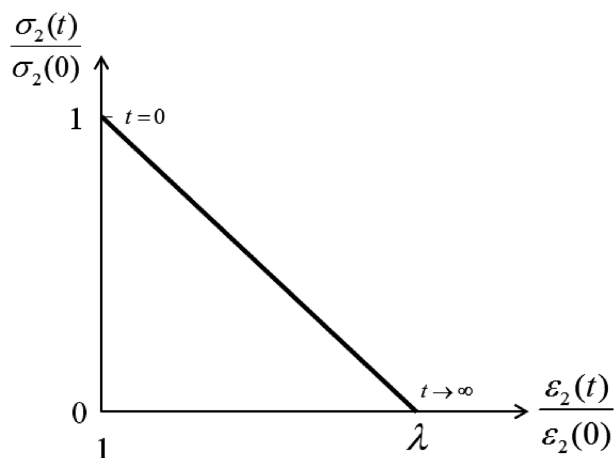


Fig. 7 Two-bar structure: isochronous stress–strain curve

to illustrate the characteristic behaviour of elastic follow-up. Consider a structure consisting of two bars of the same material and length L but of different cross-sectional areas A_1 and A_2 , connected in series (Fig. 6). One end of the structure is fully fixed, while the other suffers a fixed displacement in the axial direction, δ . To begin with, assume that the total strain in each bar is composed of an elastic component and a pure power-law creep component

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \dot{\varepsilon}_c \quad (11)$$

$$\frac{\dot{\varepsilon}_c}{\dot{\varepsilon}_0} = \left(\frac{\sigma}{\sigma_0} \right)^n$$

As in Section 3, primary creep is neglected. It can easily be shown that the maximum stress in bar no. 2 relaxes in time according to

$$\frac{\sigma_2(t)}{\sigma_2(0)} = \frac{1}{\left[1 + (n-1)\zeta \frac{1}{\lambda} \right]^{\frac{1}{n-1}}} \quad (12)$$

assuming $A_2 \leq A_1$, where

$$\zeta = E\dot{\varepsilon}_0 \left(\frac{\sigma_2(0)}{\sigma_0} \right)^{n-1}$$

The geometrical parameter λ is given by

$$\lambda = \frac{1 + A_2/A_1}{1 + (A_2/A_1)^n}$$

In reference [4], it was shown that there is a simple *linear* relationship between stress and strain at any instant given by

$$\frac{\varepsilon_2(t)}{\varepsilon_2(0)} = 1 + (\lambda - 1) \left(1 - \frac{\sigma_2(t)}{\sigma_2(0)} \right) \quad (13)$$

which is plotted in Fig. 7 as a ‘stress–strain trajectory’, or isochronous stress–strain curve, of the type used for elastic follow-up by several earlier writers and is now common in the literature.

In the case of this simple structure, the total strain is limited, since from compatibility

$$\varepsilon_1 + \varepsilon_2 = \frac{\delta}{L}$$

where δ is the applied displacement. Initially, the maximum elastic strain accounts for the fraction $1/(1 + A_2/A_1)$ of the total strain δ/L ; however, as time progresses, this fraction increases

$$\varepsilon_2(0) = \frac{1}{1 + A_2/A_1} \frac{\delta}{L} \quad \varepsilon_2(\infty) = \frac{1}{1 + (A_2/A_1)^n} \frac{\delta}{L}$$

Therefore, as $t \rightarrow \infty$, $\varepsilon_2(t)/\varepsilon_2(0) \rightarrow \lambda$. As time progresses, the strain in bar no. 1 decreases – this is the phenomenon of elastic follow-up as described by Robinson [3].

It should be particularly noted that the stress–strain trajectory in Fig. 7 is almost independent of n for this simple structure. This feature indicates that the elastic follow-up phenomenon is largely geometrical in nature, since it is independent of material. Indeed, if primary creep is added to the creep law used, for example using time hardening in equation (11)

$$\frac{\dot{\varepsilon}_c}{\dot{\varepsilon}_0} = g(t) \left(\frac{\sigma}{\sigma_0} \right)^n$$

then the stress–strain trajectory, equation (13), remains the same [4].

The elastic follow-up parameter, q , of JSME NC2-2005 [1, 5] is related to the slope of the graph shown in Fig. 7

$$q = \frac{1}{\lambda - 1}$$

This definition was derived from an extensive study of this two-bar structure for both plasticity and creep, together with several variations on this model by Kasahara *et al.* [5]. A similar definition is given for the elastic follow-up factor in R5 [2].

4.2 Independence of constitutive relation

The near independence of the stress–strain trajectory of the constitutive equation is significant here. This characteristic behaviour has been found in several

more detailed studies of elastic follow-up in other, often more complex, structures. In a recent series of publications by Hadidi-Moud and Smith, summarized in reference [9], this feature is also noted. Hadidi-Moud and Smith analysed three sets of uniaxially loaded multi-bar structures – extensions of the two-bar structure studied in the above. The models considered included bars in series, parallel, and combined series/parallel assemblages; elastic behaviour coupled with simple power-law creep, equation (11), was assumed as the basic material behaviour. In all the cases, it was found that the elastic follow-up factor was independent of creep law, once again highlighting the geometric nature of elastic follow-up. As mentioned previously, Kasahara [6] carried out detailed inelastic FEA of an axisymmetric Y-piece under thermal loading using various creep laws – a power-law with different stress indices but no primary creep, combined elastic–plastic creep and a Blackburn-type primary creep law coupled with power-law secondary creep. On plotting stress–strain trajectories for each at the location of maximum strain over a time period of 210 000 h, it was found that, regardless of the constitutive equation used, the trajectories were essentially identical, although there was slight variation with the Blackburn-type creep law at larger times. It is important to recognize that it is the stress–strain trajectories which are independent of constitutive relation – strain accumulation at a specific time does vary, as would be expected. This problem will be re-examined later in this article to investigate the effect of a modified secondary creep law.

The usefulness of the stress–strain trajectory has been common in other areas of inelastic design as well as elastic follow-up. One of the first applications was reported by Neuber [24] in an attempt to estimate elastic–plastic strain concentration using elastic analysis. The concept was also found useful in the identification of primary and secondary stresses in pressure vessel design by analysis [25]. Further, Seshadri [26] developed the generalized local stress–strain (GLOSS) analysis method, which gave an estimation of the stress–strain trajectory at a stress concentration from two elastic analyses alone. Since only elastic analysis is required, the trajectory is independent of the constitutive relation in the GLOSS method. More recently, Ando *et al.* [27] compared several simplified methods to predict strain range, used in high-temperature design to estimate fatigue and creep-fatigue damage, at localized (peak) stress concentrations. The methods used were Neuber's Rule [24], the JSME elastic follow-up approximation ($q=3$), and the stress reduction locus (SRL), which is based on elastic analysis coupled with a uniaxial stress–strain curve. The simplified methods were compared to experimental

studies and detailed elastic–plastic analyses of notch bar specimens with various notch sizes. Ando *et al.* [27] concluded that, in terms of estimating the stress–strain trajectory, the SRL method gave good comparisons with the experimental results while Neuber's Rule and the JSME elastic follow-up method were quite conservative at higher strains, the latter perhaps being overly conservative. Like Seshadri's GLOSS method, by definition, the SRL method '... is insensitive to the constitutive equations ...' [27, p. 2] in estimating the stress–strain trajectory.

All these studies seem fairly conclusive – for a number of varying simple, detailed finite element, and experimental structural problems, the stress–strain trajectory is essentially independent of the constitutive relation: this is an observation which has been put to good use in high-temperature design rules. In the following section, the effect of using a modified power-law on elastic follow-up will be examined.

4.3 Elastic follow-up of a two-bar structure: Modified power-law creep

The two-bar structure of Section 4.1 will now be re-examined by introducing the modified power-law creep model, equation (5), so that equation (11) takes the form

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + \dot{\epsilon}_c \quad (14)$$

$$\frac{\dot{\epsilon}_c}{\epsilon_0} = \frac{\sigma}{\sigma_0} + \left(\frac{\sigma}{\sigma_0}\right)^n$$

for each bar in Fig. 6.

Using the same normalized timescale as for the bar relaxation problem in Section 3 $\gamma = \frac{E\dot{\epsilon}_0}{\sigma_0}$ $\tau = \gamma t$ and defining $S(\tau) = \frac{\sigma_2(\tau)}{\sigma_2(0)}$, it can be shown that the stress in bar no. 2 relaxes according to the first-order differential equation

$$\frac{dS}{d\tau} + S + \frac{\alpha^{n-1}}{\lambda} S^n = 0 \quad (15)$$

subject to the initial condition, $S(0) = 1$. This has solution (verified by MAPLE)

$$S(\tau) = \frac{e^{-\tau}}{\left[1 + \frac{\alpha^{n-1}}{\lambda} (1 - e^{-(n-1)\tau})\right]^{\frac{1}{n-1}}} \quad (16)$$

where, as before, a geometrical parameter λ is defined

$$\lambda = \frac{1 + A_2/A_1}{1 + (A_2/A_1)^n}$$

together with a load parameter

$$\alpha = \frac{\sigma_2(0)}{\sigma_0}$$

similar to that used in Section 3 for bar relaxation. The load parameter represents the ratio of maximum initial elastic stress in the structure to the transition stress. Equation (16) can be contrasted with equation (8) for bar relaxation noting similarity but the addition of the geometrical parameter, λ .

In order to plot a stress–strain trajectory, it is necessary to calculate the strain variation. It can be shown that the corresponding maximum strain in bar no. 2 satisfies the equation

$$\frac{dE}{d\tau} = \left(1 - \frac{1}{\lambda}\right) \alpha^{n-1} S^n \quad (17)$$

where $E(\tau) = \frac{\varepsilon_2(\tau)}{\varepsilon_2(0)}$, subject to initial condition $E(0) = 1$. Equation (17) can be numerically solved as a first-order differential equation on combining with equation (16) (both Mathcad and Matlab were used for comparison). Stress–strain trajectories for various values of power index, n , area ratio, A_2/A_1 , and load factor, α , are shown in Fig. 8.

Solutions are given for three values of the load factor $\alpha = 1.0, 1.5$, and 2.0 in Fig. 8(a), (b), and (c), respectively. The first represents an initial stress in bar no. 2 equal to the modified power-law transition stress, the last an initial stress double that of the transition stress. For each load factor, two values of area ratio, $A_2/A_1 = 0.2$ and 0.5 , and three values of power index, $n = 3, 7$, and 11 are used. It can be immediately seen that, *for the modified power-law, independence of the stress–strain trajectories on the constitutive equation is lost*. As the load factor increases, a linear relation between normalized stress and strain becomes more evident, but in each case, the linear viscous part of the modified power-law eventually dominates so that the normalized strain tends to a constant value. For higher values of the load factor, Fig. 8(b) and (c), the initial stress–strain trajectory is reasonably linear, becoming more evident as the power-law index, n , increases. However, although the trajectory is linear, it varies with the power index and is different from that found for pure power-law (Fig. 7). The slope of the trajectory in Fig. 7 is determined by the value of the geometry factor such that $E(\tau) \rightarrow \lambda$ as time increases. For the values of area ratio and power index shown in Fig. 8, the geometry factor, λ , takes the values

$$\lambda = \frac{1 + A_2/A_1}{1 + (A_2/A_1)^n}$$

as given in Table 1.

In fact as $n \rightarrow \infty, \lambda \rightarrow 1 + A_2/A_1$; in Fig. 8, the maximum range of the normalized strains is taken as this limiting case for comparison purposes. Noticeably, the trajectories for the modified power-law indicate reduced strain accumulation due to the presence of

the linear viscous component of the constitutive relation as the result of a more rapid relaxation of stress in bar no. 2. In this sense, *the elastic follow-up effect is reduced* in a structure with material corresponding to a modified power-law.

4.4 Axisymmetric Y-piece: Modified power-law creep

As a more complex example, the axisymmetric Y-piece structure under thermal loading discussed by Kasahara [6] will now be similarly re-examined using the modified power-law. Sufficiently detailed information is given in reference [6] to allow the analysis to be reconstructed. The Y-piece is composed of a vertical thin cylinder of length 2.25 m, radius 3 m, and thickness 50 mm together with a skirt at a distance 0.75 m from the bottom and at an angle of 30° , with wall thickness 30 mm. The initial temperature of the whole structure is 50°C ; subsequently, the temperature of the inner surface of the cylinder is increased to 550°C while the skirt edge is maintained at 50°C . Quasi-static conditions are assumed and the structure is allowed to creep for 210 000 h under strain-controlled conditions, thus leading to stress relaxation. This complete structure was re-modelled using ABAQUS: the finite element mesh at the junction between cylinder and skirt, where maximum strain occurs, is shown in Fig. 9.

Several creep laws are used in reference [6]: Norton's law with varying power index, a Blackburn-type equation, and a Ramberg–Osgood elastic-creep law. Here, the results will only be reproduced using the power-law

$$\dot{\varepsilon}_c = B\sigma^n \quad (18)$$

where $B = 5.86 \times 10^{-15}$, with the power index taking the values $n = 3, 5$, and 7 (units for time are hour and stress megapascal).

A trajectory of equivalent stress against equivalent strain at an evaluation point, Fig. 10, being the location of maximum elastic strain, is given in reference [6, Fig. 4] showing the initial elastic response and stress relaxation during creep. Results obtained from a new creep analysis using ABAQUS are shown in Fig. 10: these are very similar to those given in reference [6] again, showing that the stress–strain trajectory is effectively insensitive to the power index as found for the two-bar structure (Fig. 7). However, the trajectory is no longer a straight line – this feature can be found in some other simple structures but in general, it has been found that a non-linear relationship, with an initial steep fall representing rapid rise in strain followed by a slower accumulation of strain as time progresses, would be more common.

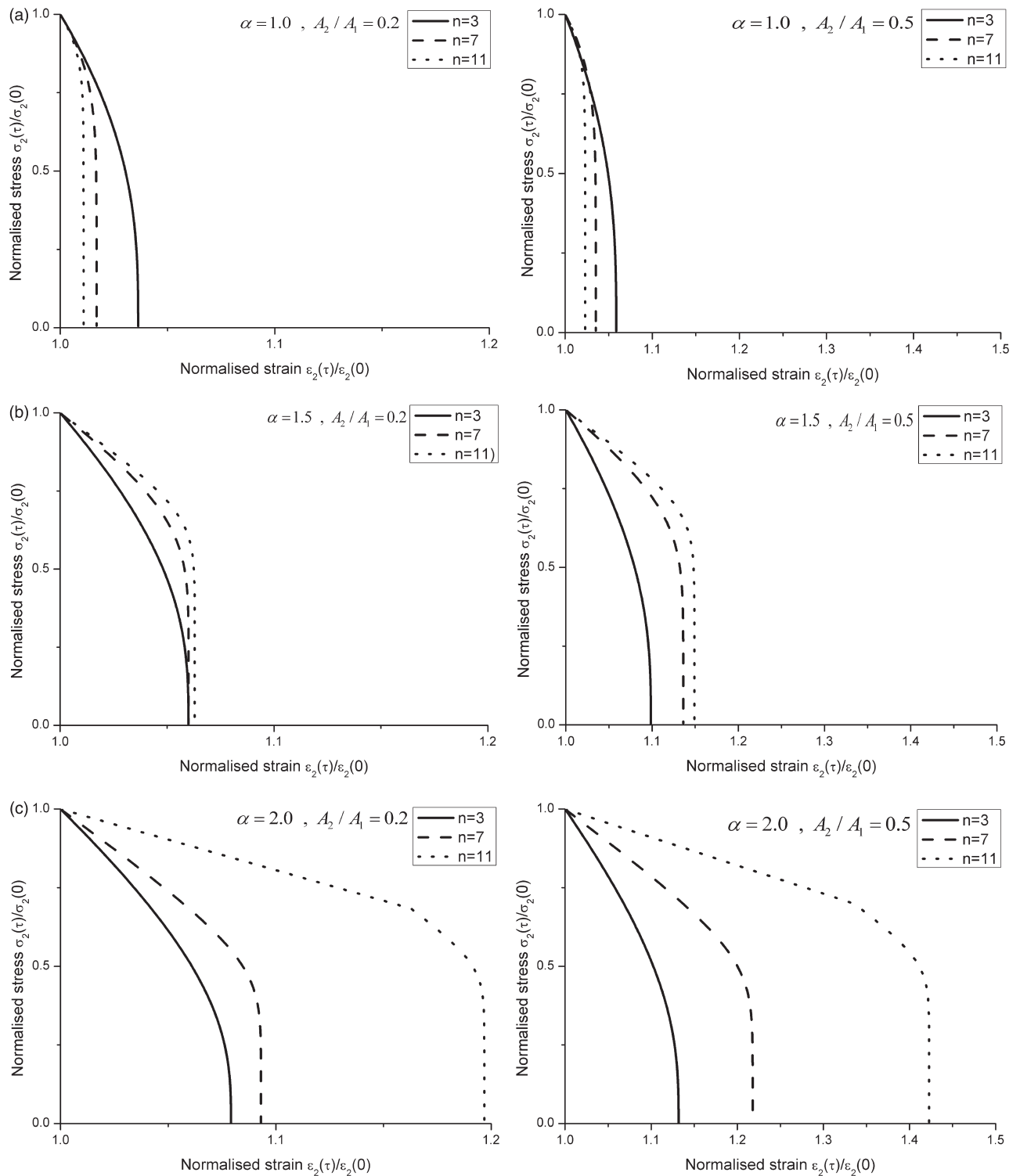


Fig. 8 Two-bar structure: stress–strain trajectory using modified power law: (a) $\alpha = 1.0$, (b) $\alpha = 1.5$, and (c) $\alpha = 2.0$

This analysis is now repeated for the modified power-law, equation (14). This was included as a user material in ABAQUS. For the purposes of comparison, the following nominal values of the material parameters σ_0 and $\dot{\epsilon}_0$ are adopted

$$\dot{\epsilon}_0 = 2 \times 10^{-7} 1/h, \quad \sigma_0 = 100\text{MPa}$$

but with the power index again taken as $n = 3, 5$, and 7 . The resulting stress–strain trajectories are shown in Fig. 11. It can be seen that the stress–strain trajectories now deviate somewhat from those corresponding to the pure power-law (Fig. 10). During the early stages of stress relaxation, the trajectories do follow

Table 1. Values of λ for various values of n and cross-sectional area ratio

λ		
n	$A_2/A_1 = 0.2$	$A_2/A_1 = 0.5$
3	1.19	1.333
7	1.2	1.488
11	1.2	1.499

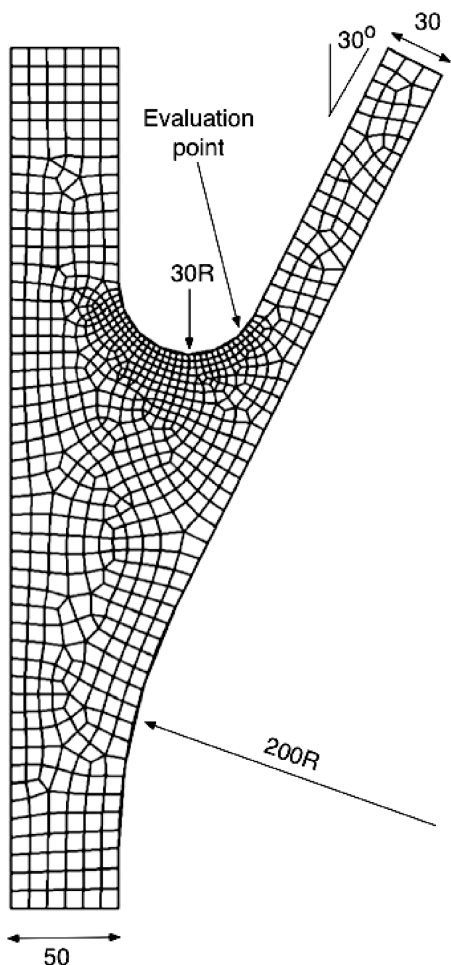


Fig. 9 Detail of ABAQUS finite element mesh at junction between cylinder and skirt of Y-piece

a slope similar to pure power-law, but start to deviate as time progresses with an indication that the strain accumulation is not as significant as that for the pure power-law.

5 CONCLUSIONS

The treatment of elastic follow-up in high-temperature design has, for several years, been based on the use of stress–strain trajectories which can be estimated from simplified analysis, avoiding detailed non-linear FEA. An underlying assumption, which has been observed and validated on a range of

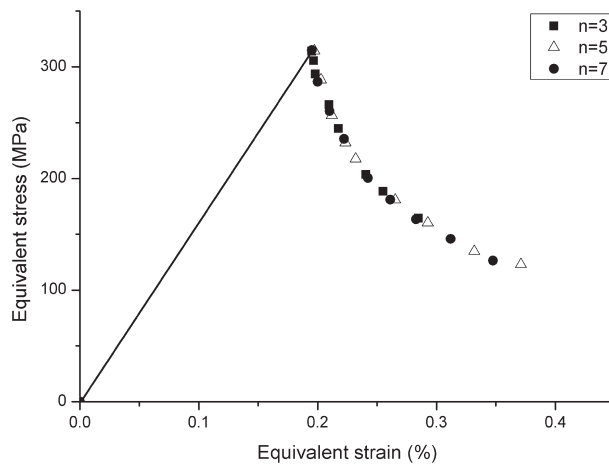


Fig. 10 Equivalent stress–strain trajectories for the evaluation point of Y-piece from ABAQUS FEA with pure power-law creep

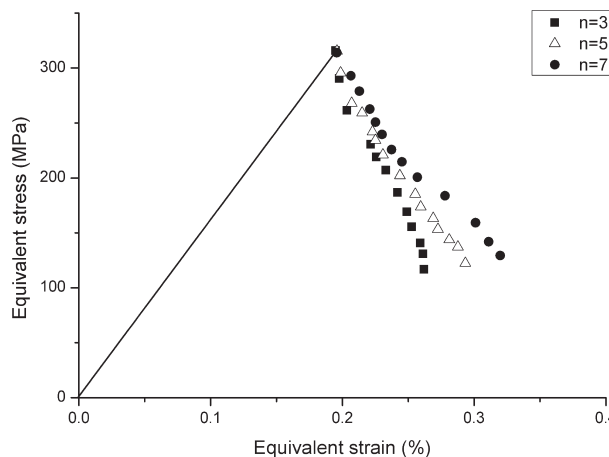


Fig. 11 Equivalent stress–strain trajectories for evaluation point of Y-piece from ABAQUS FEA with modified power-law creep

structures both simple and complex, has been the insensitivity of these stress–strain trajectories to the constitutive model. However, the majority of studies which exhibit this insensitivity have been based on familiar creep constitutive relations usually derived from power-law creep, or variations. In this study, the work of Naumenko *et al.* [16] on the nature of stress systems in structures composed of stress range-dependent constitutive models, such as those shown in Figs 2 to 4, has been extended to examine the consequences for elastic follow-up. In reference [16], it was shown that stress relaxation of a simple bar with a constitutive equation which included both linear viscous and power-law creep, equations (5) and (14), was significantly different from the classic solution based on power-law creep and further did not approach expected limits for high or low initial stress. This result has been re-examined in this study

in Section 3 and it is seen that the linear viscous part of the modified power-law has a significant effect in stress relaxation, more so than what is observed for constant load steady creep [16, 22]. The simple two-bar structure [4] has long been used as a reference benchmark in studies of elastic follow-up [4–10]: assuming a creep law based on power-law secondary creep alone, it can be shown that this structure has a stress–strain trajectory which is indeed independent of the power index. However, it is shown in Section 4.3 that the relaxation characteristics of the stress–strain trajectory are radically different when using the modified power-law. Specifically, insensitivity to the constitutive relation is lost and as a result, there is a strong dependency on some load factor (the ratio of maximum elastic stress to the transition stress in the modified power-law). To investigate this further, a finite element stress relaxation analysis of an axisymmetric Y-piece under thermal loading studied by Kasahara [6] has been re-examined. In reference [6], it was determined that insensitivity to creep law was also found, as in the simple two-bar structure. If a modified power-law is used, this insensitivity is again lost, although not as extensively as in the two-bar structure. In conclusion, for materials which exhibit stress-range dependency with appropriate load conditions, some care should be taken if simplified approaches used to estimate elastic follow-up are used for design. Some reassurance can be taken from the observation that strain accumulation at large times is less than that for the pure power-law, which could perhaps be used as a (hopefully not over) conservative estimate.

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