Optimal path planning for nonholonomic robotic systems via parametric optimisation

James Biggs

Advanced Space Concepts Laboratory, Department of Mechanical Engineering, University of Strathclyde.james.biggs@strath.ac.uk

Abstract. Motivated by the path planning problem for robotic systems this paper considers nonholonomic path planning on the Euclidean group of motions $SE(n)$ which describes a rigid bodies path in n-dimensional Euclidean space. The problem is formulated as a constrained optimal kinematic control problem where the cost function to be minimised is a quadratic function of translational and angular velocity inputs. An application of the Maximum Principle of optimal control leads to a set of Hamiltonian vector field that define the necessary conditions for optimality and consequently the optimal velocity history of the trajectory. It is illustrated that the systems are always integrable when $n = 2$ and in some cases when $n = 3$. However, if they are not integrable in the most general form of the cost function they can be rendered integrable by considering special cases. This implies that it is possible to reduce the kinematic system to a class of curves defined analytically. If the optimal motions can be expressed analytically in closed form then the path planning problem is reduced to one of parameter optimisation where the parameters are optimised to match prescribed boundary conditions. This reduction procedure is illustrated for a simple wheeled robot with a sliding constraint and a conventional slender underwater vehicle whose velocity in the lateral directions are constrained due to viscous damping.

1 Introduction

In recent years there has been a great deal of research that involves motion planning in the presence of nonholonomic constraints. Several books give a general overview of the nonholonomic motion planning problem (MPP) [1, 2] for nonholonomic mechanical systems, and [3–5] in the context of robotics. Nonholonomic motion planning is challenging because nonlinear control theory does not provide an explicit procedure for constructing controls. In addition linearisation techniques, effective for nonlinear systems, fail to be useful, as highlighted in [6] that linearisation renders such systems uncontrollable. The computation of feasible trajectories for nonholonomic systems is a complex task and generally treated using numerical methods. However, numerical methods have the drawback that they are inherently local and not guaranteed to find a feasible solution. Additionally, such methods as dynamic programming [7] may provide global optimal solutions, but are often computationally expensive. However, when the configuration space can be represented by a Lie group the motion planning algorithms can be designed to exploit the underlying structure of the system.

Sophus Lie introduced and developed his ideas on continuous transformation groups that leave certain mathematical structures invariant. This comprehensive theory of infinitesimal transformations is now known as the study of Lie groups and their accompanying Lie algebras. Lie focussed on using these transformation groups to solve differential equations, see [8]. However, since then Lie groups have made a significant contribution to many diverse areas of mathematics and theoretical physics and have more recently made an appearance in the control literature. In particular the seminal paper [9], puts the theory of Lie groups and Lie algebras into the context of nonlinear control theory to express notions of controllability and observability for invariant systems evolving on matrix Lie groups. One of the most important consequences of this work was the recognition that questions about these kind of systems on Lie groups can be reduced to questions about their associated Lie algebra. Lie algebras are simpler than matrix Lie groups, because the Lie algebra is a linear space. Thus, much can be understood about a Lie algebra by performing Linear algebra. For nonholonomic systems defined on Lie groups, the MPP methodologies are naturally based on Lie-algebraic techniques. The general idea is to use a family of control functions i.e. piecewise constant controls, periodic controls or polynomial controls to generate motions in the directions of iterated Lie brackets, that is, steering the system in directions that are not directly controlled. These types of methods were first initiated using piecewise constant inputs, see for example [10], where the authors propose a general motion planning algorithm for kinematic models of nonholonomic systems. The algorithm is based on expressing the flow, resulting from constant control inputs, as an exponential product expansion involving iterated Lie brackets. For nilpotent systems i.e. systems for which all iterated Lie brackets of high order are zero, the algorithm provides exact steering. The use of different families of control functions other than that of piecewise constant controls are used in [11], [12], [13] where open loop control laws are derived using a family of periodic controls at integrally related frequencies. The problem then amounts to finding appropriate frequencies and amplitudes for the periodic controls.

These methodologies provide a constructive procedure for motion planning based on the iterated Lie bracket in that they provide a feasible path between two configurations (at least approximately). However, in general there will be more than one feasible path between two configurations with some more practically desirable than others. As in dynamic programming a specific path can be selected by optimising the manoeuvre with respect to some pre-specified cost function. Despite the numerous optimisation tools available, the use of optimal control theory to tackle the MPP has had little impact on practical applications, presumably because the delicate numerical treatment of optimal control problems is often less suited to practical implementation than other methods. However, since the development of geometric control theory, new approaches have arisen, distinguished from numerical methods, in that they exploit the systems underlying analytic structure. The use of optimal control theory to tackle the MPP for nonholonomic systems on Lie groups is studied in [14–18, 2, 9]. In Brockett's seminal papers [15, 16] it is shown that a number of these optimal control problems are completely solvable in closed form where the optimal angular velocities can be expressed as analytic functions such as trigonometric or Jacobi elliptic functions. In this paper the focus is placed on the group $SE(n)$ where $n = 2, 3$ which have direct applications to robotic systems. The cost function used is an integral function of angular velocities. The function yields smooth curves where the velocity along the curve is minimised and therefore the forces required to track are theoretically small. The emphasis in this paper is placed on integrable cases that not only lead to closed form expressions for the optimal velocity inputs but also closed form expression for the corresponding motions. Contrary to integrable systems in mechanics, which are few and far between, this setting allows us to render the closed-loop system integrable by appropriate manipulation of the cost function. This can be achieved by setting weights of the cost function to be equal which essentially introduces enough symmetry into the resulting Hamiltonian system for it to be integrable. However, setting weights to be equal reduces the generality of the cost function and therefore the class of solution. Reducing the MPP to an analytic closed form solution essentially reduces the MPP to a problem of optimising the available parameters of these analytic functions to match prescribed boundary conditions. In addition an iterative approach can be used to select the parameters to avoid stationary and known obstacles.

1.1 Nonholonomic Systems on the Euclidean Groups $SE(n)$

From a practical point of view the kinematics of robotic systems can often be framed as nonholonomic control systems on the Euclidean Lie groups $SE(2)$ (position and orientation in the plane) and $SE(3)$ (position and orientation in space). Examples of these are the "vertical" rolling disk or Unicycle [18], hopping robots [19] on $SE(2)$ and Autonomous Underwater Vehicles [6], Unmanned Air Vehicles [20, 21] and robotic manipulators [22] on $SE(3)$. The nonholonomic kinematics of these systems can be expressed as:

$$\frac{dg(t)}{dt} = \sum_{i=1}^{s} u_i(t) X_i \tag{1}$$

where the curve $g(t) \in SE(n)$ describes the motions of the system in the configuration manifold $SE(n)$. $X_1,...,X_n$ are arbitrary vector fields in the tangent space $TSE(n)$ at $g(t)$, denoted $T_{g(t)}SE(n)$ and $u_1,...,u_s$ are the control functions. It follows that $X_1,...,X_s \in T_{g(t)}SE(n)$ are the controlled vector fields on the manifold $SE(n)$ and $X_0 = \sum_{i=1}^{m} X_i \in T_{g(t)}SE(n)$ is the drift vector on $SE(n)$. For nonholonomic systems $n > s$. The cases considered are when X_i are left (respectively right) invariant vector fields on $SE(n)$, see [17], the vector fields can be expressed as $X_1 = g(t)A_1,...,X_n = g(t)A_n \in T_{g(t)}SE(n)$, where $A_1,...,A_n \in T_I SE(n)$ are basis elements of the tangent space at the identity $I \in SE(n)$. The tangent space at the identity is called the Lie algebra denoted $\mathfrak{se}(n)$. It follows that the differential equation (1) can be expressed as:

$$\frac{dg(t)}{dt} = g(t)(\sum_{i=1}^{s} u_i(t) A_i) \tag{2}$$

where $A_1,...,A_n \in \mathfrak{se}(n)$ where the Lie algebra $\mathfrak{se}(n)$, is a vector space together with the matrix commutator, the Lie bracket:

$$[X,Y] = XY - YX \tag{3}$$

where $X,Y \in \mathfrak{se}(n)$ where for $n=2$ the basis of the Lie algebra is

$$A_1 = \begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, A_2 = \begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix}, A_3 = \begin{pmatrix} 0&-1&0 \\ 1&0&0 \\ 0&0&0 \end{pmatrix} \tag{4}$$

satisfying the following relations

$$[A_1,A_3] = -A_2,\ [A_2,A_3] = A_1,\ [A_1,A_2] = 0. \tag{5}$$

and when $n=3$ the basis for the six-dimensional Lie algebra is:

$$A_1 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&1&0 \end{pmatrix}, A_2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&-1&0&0 \end{pmatrix}, A_3 = \begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix}$$
$$A_4 = \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, A_5 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \end{pmatrix}, A_6 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}$$

and satisfies the commutative table:

[,]	A_1	A_2	A_3	A_4	A_5	A_6
A_1	0	A_3	-A_2	0	A_6	-A_5
A_2	-A_3	0	A_1	-A_6	0	A_4
A_3	A_2	-A_1	0	A_5	-A_4	0
A_4	0	A_6	-A_5	0	0	0
A_5	-A_6	0	A_4	0	0	0
A_6	A_5	-A_4	0	0	0	0

Note that the driftless system in (2) can be augmented to include systems with drift by setting one of the controls u_i to a constant a priori. The method described in this paper is also applicable to these cases. Controllability for systems of the form (2) can be assessed through computations performed at the level of the Lie algebra involving the Lie bracket (3), see [17, 2]. For example if $[X,Y] = Z$ for some $Z \in \mathfrak{se}(n)$, then it is possible to move in the direction of the vector field Z by controlling only the vector fields X and Y. From a control theory viewpoint this is particularly useful if $Z \notin span\{X,Y\}$. Indeed if it is possible to directly control the vector fields $X_1,...,X_s$ then motions can be generated in the direction of its iterated Lie brackets:

$$X_i, [X_i,X_j], [X_i,[X_j,X_k]], ..., \tag{6}$$

$1 \leqslant i,j,k,... \leqslant s$. In other words, although not all directions are controlled, it may be possible to obtain motions in all of them, by taking sufficiently many Lie brackets. If the system is controllable then it is possible to construct a well defined optimal control problem. This is formalised in the following subsection.

1.2 Problem Statement

Subject to the kinematic nonholonomic constraint given by (2) and given that the system is controllable the problem is then to find a trajectory $g(t) \in SE(n)$ from an initial position and orientation $g(0) \in SE(n)$ to a final position and orientation $g(T) \in SE(n)$ where T is some fixed final time that minimises the functional

$$J = \frac{1}{2}\int_0^T c_i u_i^2 dt \tag{7}$$

where $i = 1, ..., s$ and c_i are constant weights. This cost function is not conventional but it is meaningful as it ensures smooth motions which, given a reasonably long enough fixed final time, will not require large torques and forces to track them. In addition it enables the MPP to be formulated in the context of geometric optimal control and this enables us to ask questions of integrability and in some cases solve the system in closed form. Furthermore, obtaining a closed form solution essentially reduces the MPP to a problem of optimising the available parameters to match the prescribed boundary conditions.

2 Methodology

The methodology for MPP comprises of the following phases:

1. Lifting the optimal control problem on $SE(n)$ to a Hamiltonian setting via the maximum principle of optimal control and Poisson calculus.
2. Solving integrable cases of the Hamiltonian vector fields analytically in the most general form of the cost function (7).
3. Given the optimal velocities derive the corresponding motions in $SE(n)$ analytically reducing the MPP to a parameter optimisation problem.
4. As the boundary conditions are not contained in the cost function it is necessary to numerically optimise the available parameters of the analytic solutions to match the prescribed boundary conditions (This stage is not covered in this paper.)

2.1 General Hamiltonian lift on SE(n)

The application of the coordinate free Maximum Principle to left-invariant optimal control problems are well known, see [23], [17]. As the Hamiltonian is left-invariant the cotangent bundle $T^*SE(n)$ can be realised as the direct product $SE(3) \times \mathfrak{se}(n)^*$ where $\mathfrak{se}(n)^*$ is the dual of the Lie algebra $\mathfrak{se}(n)$ of $SE(n)$. Therefore, the original Hamiltonian defined on $T^*SE(n)$ can be expressed as a reduced Hamiltonian on the dual of the Lie algebra $\mathfrak{se}(n)^*$ as $T^*SE(n)/SE(n) \cong \mathfrak{se}(n)^*$. The appropriate Hamiltonian for the constraint (2) with respect to minimizing the cost function (7) is given by (see [17] for details):

$$H(p,u,g) = \sum_{i=1}^{s} u_i p(g(t)A_i) - \rho_0 \frac{1}{2}\sum_{i=1}^{s} c_i u_i^2 \tag{8}$$

where $p \in T^*SE(n)$ (where $n = 1$ or $n = 2$) and $\rho_0 = 1$ for regular extremals and $\rho_0 = 0$ for abnormal extremals. In this paper we consider only the regular extremals, therefore we set $\rho_0 = 1$. The Hamiltonian (8) defined on $T^*SE(n)$ can be expressed as a reduced Hamiltonian on the dual of the Lie algebra $\mathfrak{se}(n)^*$. It follows that $p(g(t)A_i) = \hat{p}(A_i)$ for any $p = (g(t), \hat{p})$ and any $A_i \in \mathfrak{se}(n)$. Defining the extremal (linear) functions explicitly as $\lambda_i = \hat{p}(A_i)$, where $\hat{p} \in \mathfrak{se}^*(n)$ the Hamiltonian (8) can be expressed on $\mathfrak{se}(n)^*$ as

$$H = \sum_{i=1}^{s} u_i \lambda_i - \frac{1}{2} \sum_{i=1}^{s} c_i u_i^2 \tag{9}$$

Through the Maximum Principle and the fact that the control Hamiltonian (9) is a concave function of the control functions u_i, it follows by calculating $\frac{\partial H}{\partial u_i} = 0$ and $\frac{\partial H}{\partial v_i} = 0$ that the optimal kinematic control inputs are:

$$u_i^* = \frac{1}{c_i} \lambda_i, \tag{10}$$

where $i = 1, ..., s$. Substituting (10) back into (9) gives the appropriate left-invariant quadratic Hamiltonian:

$$H = \frac{1}{2} (\sum_{i=1}^{s} \frac{\lambda_i^2}{c_i}) \tag{11}$$

where λ_i are the extremal curves. For each quadratic Hamiltonian (11), the corresponding vector fields are calculated using the Poisson bracket $\{\hat{p}(\cdot), \hat{p}(\cdot)\} = -\hat{p}([\cdot, \cdot])$ where $(\cdot) \in \mathfrak{se}(n)$. Then the Hamiltonian vector fields are given by:

$$\frac{d(\cdot)}{dt} = \{\cdot, H\} \tag{12}$$

where $(\cdot) \in \mathfrak{se}(3)^*$. Finally, substituting (10) into (2) yields:

$$\frac{dg(t)}{dt} = g(t) \nabla H \tag{13}$$

where ∇H is the gradient of the Hamiltonian and $g(t) \in SE(n)$ are the corresponding paths. The MPP is thus reduced to solving for $g(t) \in SE(n)$ such that the boundary conditions $g(0) \in SE(n)$ and $g(T) \in SE(n)$ in some final time T are matched.

2.2 A note on integrability

The motion planning problem has been reduced to the problem of finding solutions to the equations (13) where $g(t) \in SE(n)$ with $n = 1$ or $n = 2$. The natural question to now ask is if it is possible to reduce the equations to quadratures (as integrals of analytic functions) or better still solve them in closed form. In order for the equations to be at least reducible to quadratures the system has to be integrable. Moreover, a Hamiltonian function on a symplectic manifold N of dimension $2n$ is said to be integrable if there exist constant functions $\varphi_2, ..., \varphi_n$ on N that together with the Hamiltonian $H = \varphi_1$ satisfy the following two properties:

- $\varphi_1, ..., \varphi_n$ are functionally independent i.e the differentials $d\varphi_1, ..., d\varphi_n$ are linearly independent for an open subset of N.

– The functions $\varphi_1, ..., \varphi_n$ Poisson commute with each other.

Thus, in identifying the $(n-1)$ functions φ_i the Hamiltonian function is completely integrable and in general the system can be reduced to quadratures and in some cases can be solved in closed form. The implications of this are that an approximately optimal analytic solution can be derived (when the system can be reduced to quaratures) and an exactly optimal analytic solution when completely solvable. In terms of the motion planning problem if the system is integrable it means that the problem can be reduced to one of parameter optimisation where the initial values of the extremal curves can be optimised to match prescribed boundary conditions. For all left-invariant Hamiltonian systems defined on $SE(2)$ and $SE(3)$ we can be more specific about integrability:

Lemma 1 *For any left (respectively right) invariant Hamiltonian system defined on $SE(2)$, there exist three integrals of motion, the Hamiltonian H the Casimir function $M = \lambda_1^2 + \lambda_2^2$, and the integral of motion φ_3 corresponding to a right-invariant vector field.*

Lemma 2 *For any left (respectively right) invariant Hamiltonian system defined on $SE(3)$, there exist five constants of motion the Hamiltonian H the Casimir functions $I_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ and $I_3 = \lambda_1\lambda_4 + \lambda_2\lambda_5 + \lambda_3\lambda_6$ and the right-invariant vector fields φ_4, φ_5.*

details of this can be found in [17]. This implies that left-invariant systems on $SE(2)$ are always integrable and on $SE(3)$ integrable if an additional constant of motion can be found. However, it is easy to induce an additional conserved quantity by introducing a symmetry into the problem by setting two of the weights of the cost function (7) to be equal. Although this restricts the generality of the cost function it ensures that the system is integrable. This procedure of manipulating the cost function is illustrated in example 2 of this paper. For integrable cases on $SE(2)$ and $SE(3)$ it is often possible to solve these systems in closed form as shown in the following example. In addition integrability is an intrinsic property of the system as it implies that all motions will be regular as opposed to chaotic. This ensures that all of the reference motions derived will be regular.

3 Examples

3.1 Example 1-The wheeled robot

A wheeled robot's configuration space can be described by a curve $g(t) \in SE(2)$

$$g(t) = \begin{pmatrix} R(t) & x \\ 0 & 1 \end{pmatrix} \tag{14}$$

where $R(t) \in SO(2)$ and $x \in \Re^2$, where

$$R(t) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{15}$$

we assume that the wheeled robot has a sliding constraint and can move backward or forwards at a velocity s which can be controlled. This velocity constraint can be expressed as:

$$\frac{dx}{dt} = R(t)\begin{bmatrix} s \\ 0 \end{bmatrix} \tag{16}$$

Furthermore, the robot can rotate at an angular velocity $u = \dot{\theta}$. Differentiating equation (14) and taking into the account the constraint (16) it is easily shown that the nonholonomic kinematic constraint can be expressed as a left-invariant differential equation:

$$g(t)^{-1}\frac{dg(t)}{dt} = \begin{pmatrix} 0 & -u & s \\ u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{17}$$

this can be expressed in the form:

$$g(t)^{-1}\frac{dg(t)}{dt} = sA_1 + uA_3 \tag{18}$$

where the Lie algebra is given by (4) and the cost function (7) is given by:

$$J = \int_0^1 s^2 + cu^2 dt \tag{19}$$

where c is a constant weight and the time t is scaled such that in real time τ with final fixed time t_f is $t = \tau/t_f$. In relation to the general form (7) $u_1 = s, c_1 = 1, u_2 = u, c_2 = c$. The Hamiltonian function corresponding to the constraint (17) that minimises the cost function (19) is:

$$H = s\lambda_1 + u\lambda_3 - \frac{1}{2}(s^2 + cu^2) \tag{20}$$

then Pontragin's maximum principle says that if

$$\frac{\partial H}{\partial s} = 0, \frac{\partial H}{\partial v} = 0, \frac{\partial^2 H}{\partial s^2} < 0, \frac{\partial^2 H}{\partial v^2} < 0, \tag{21}$$

then the functions s and u are optimal. These conditions are satisfied if:

$$s = \lambda_1,\ u = \tfrac{\lambda_3}{c} \tag{22}$$

substituting these values into (20) yields the optimal Hamiltonian H^*:

$$H^* = \frac{1}{2}\left(\lambda_1^2 + \frac{\lambda_3^2}{c}\right) \tag{23}$$

The corresponding Hamiltonian vector fields which implicitly define the extremal solutions are given by the Poisson bracket $\frac{d\lambda_i}{dt} = \{\lambda_i, H\}$ this yields the differential equations:

$$\begin{aligned} \dot{\lambda}_1 &= \tfrac{\lambda_2\lambda_3}{c}, \\ \dot{\lambda}_2 &= -\tfrac{\lambda_1\lambda_3}{c}, \\ \dot{\lambda}_3 &= -\lambda_1\lambda_2. \end{aligned} \tag{24}$$

in addition observe that the Casimir function

$$M = \lambda_1^2 + \lambda_2^2 \tag{25}$$

is constant along the Hamiltonian flow i.e. $\{M, H^*\}$. These extremal curves can be solved analytically and allow us to state the following Lemma:

Lemma 3 *The optimal velocity s in the surge direction and angular velocity u that minimises the cost function (19) subject to the kinematic constraint (17) are Jacobi elliptic functions $sn(\cdot,\cdot), dn(\cdot,\cdot)$ of the form:*

$$\begin{aligned} s &= \sqrt{M} sn\left(\frac{\sqrt{2H^*}}{\sqrt{cH^*}}t, \frac{M}{2H^*}\right) \\ u &= \sqrt{\frac{2H^*}{c}} dn\left(\frac{\sqrt{2H^*}}{\sqrt{cH^*}}t, \frac{M}{2H^*}\right) \end{aligned} \tag{26}$$

where H^* and M are constants defined by (23) and (25) respectively and c is the constant weight in the cost function (19) with the corresponding path:

$$\begin{aligned} x_1 &= -\frac{\sqrt{2}\sqrt{cH}}{\sqrt{M}} dn\phi \\ x_2 &= \frac{2Ht}{\sqrt{M}} - \sqrt{M}\left(\frac{\sqrt{cH}E\left(am\phi, \frac{M}{2H}\right)\sqrt{2 - \frac{Msn^2\phi}{H}}}{Mdn\phi}\right) \end{aligned} \tag{27}$$

where $E(\cdot,\cdot)$ is the elliptic integral of the second kind and $am(\cdot)$ is the Jacobi amplitude and where the rotation of the body along the path is:

$$R(t) = \begin{pmatrix} cn\phi & -sn\phi \\ sn\phi & cn\phi \end{pmatrix} \tag{28}$$

with $\phi = \left(\frac{\sqrt{2H}}{\sqrt{cH}}t, \frac{M}{2H}\right)$.

Proof.
The conserved quantities (25) can be parameterised by the Jacobi elliptic functions:

$$\lambda_1 = rsn(\alpha t, m), \; \lambda_2 = rcn(\alpha t, m) \tag{29}$$

then substituting (29) into (25) yields $r = \sqrt{M}$. Then (23) can be parameterised by defining:

$$\lambda_3 = \sqrt{Mp} dn(\alpha t, m) \tag{30}$$

substituting (30) and (29) into (23) gives $m = \frac{M}{2H}$ and $p = mc$, then:

$$\lambda_1 = rsn\left(\alpha t, \frac{M}{2H}\right), \; \lambda_2 = rcn\left(\alpha t, \frac{M}{2H}\right) \; \lambda_3 = \sqrt{2Hc} dn\left(\alpha t, \frac{M}{2H}\right) \tag{31}$$

finally to obtain α substitute (32) into (24) which yields:

$$\lambda_1 = rsn\left(\frac{\sqrt{2H}}{\sqrt{cH}}t, \frac{M}{2H}\right), \; \lambda_2 = rcn\left(\frac{\sqrt{2H}}{\sqrt{cH}}t, \frac{M}{2H}\right) \; \lambda_3 = \sqrt{2Hc} dn\left(\frac{\sqrt{2H}}{\sqrt{cH}}t, \frac{M}{2H}\right) \tag{32}$$

then from (22) yields (26), as $u = \dot{\theta}$ it follows from (26) that:

$$\theta = am(\phi) + C_1 \tag{33}$$

where C_1 is a constant of integration. For simplicity we set $C_1 = 0$ such that the rotation matrix $R(t)$ emanates from the origin. This yields (28) then substituting (28) and (26) into equation (16) yields:

$$\begin{aligned} \tfrac{dx_1}{dt} &= \sqrt{M} sn\phi cn\phi \\ \tfrac{dx_2}{dt} &= \sqrt{M} sn^2\phi \end{aligned} \tag{34}$$

these can be integrated analytically to yield (27). $\square$ The final step in the procedure would be to optimise the parameters to match prescribed boundary conditions $g(0) = g_0$ and $g(T) = g_T$.

3.2 Example 2-A conventional autonomous underwater vehicle (AUV)

This example is taken from [24] where a conventional slender AUV travels at arbitrary speed $v = \frac{d\gamma}{dt}$ constrained to travel in the surge direction and where the lateral motions are damped out quickly due to viscous friction. The kinematics of the AUV are then approximately described by

$$\frac{dg(t)}{dt} = g(t)(u_4A_4 + u_1A_1 + u_2A_2 + u_3A_3) \tag{35}$$

where u_4 is the translational velocity in the surge direction and u_1, u_2, u_3 are the angular velocities in the yaw, pitch and roll directions and $g(t) \in SE(3)$ describes the configuration space of the vehicle (position and orientation). The basis $A_1, A_2, A_3, A_4, A_5, A_6$ of the lie algebra $\mathfrak{se}(3)$ correspond physically to infinitesimal motion of the AUV in the yaw, pitch, roll, surge, sway and heave directions respectively. An application of the maximum principle described in Section 4 leads to the optimal velocity inputs $u_1 = \lambda_1/c_1, u_2 = \lambda_2/c_2, u_3 = \lambda_3/c_3, u_4 = \lambda_1/c_4$ where λ_i are defined implicitly by the Hamiltonian vector fields:

$$\begin{cases} \dfrac{d\lambda_1}{dt} = \dfrac{-\lambda_2\lambda_3}{c_2} + \dfrac{\lambda_2\lambda_3}{c_3}, & \dfrac{d\lambda_2}{dt} = \dfrac{\lambda_1\lambda_3}{c_1} - \dfrac{\lambda_1\lambda_3}{c_3} + \dfrac{\lambda_4\lambda_6}{c_4} \\ \dfrac{d\lambda_3}{dt} = \dfrac{-\lambda_1\lambda_2}{c_1} + \dfrac{\lambda_1\lambda_2}{c_2} - \dfrac{\lambda_4\lambda_5}{c_4}, & \dfrac{d\lambda_4}{dt} = \dfrac{-\lambda_2\lambda_6}{c_2} + \dfrac{\lambda_5\lambda_3}{c_3} \\ \dfrac{d\lambda_5}{dt} = \dfrac{\lambda_1\lambda_6}{c_1} - \dfrac{\lambda_4\lambda_3}{c_3}, & \dfrac{d\lambda_6}{dt} = -\dfrac{\lambda_1\lambda_5}{c_1} + \dfrac{\lambda_4\lambda_2}{c_2} \end{cases} \tag{36}$$

these equations are not integrable as an additional conserved quantity is required for integrability. Therefore, to solve these equations numerical integration methods are required. However, in this case the system cannot be reduced to a problem of parameter optimisation to match the boundary conditions. However, we can manipulate the weights of the cost function to render the system integrable. To demonstrate how this system can be solved by manipulation of the cost function we enforce a constraint on the weights of the cost function $c_1 = c_2 = c_3 = 1$ to yield :

$$\begin{cases} \dfrac{d\lambda_1}{dt} = 0, & \dfrac{d\lambda_2}{dt} = \dfrac{\lambda_4\lambda_6}{c_4} \\ \dfrac{d\lambda_3}{dt} = -\dfrac{\lambda_4\lambda_5}{c_4}, & \dfrac{d\lambda_4}{dt} = -\lambda_2\lambda_6 + \lambda_5\lambda_3 \\ \dfrac{d\lambda_5}{dt} = \lambda_1\lambda_6 - \lambda_4\lambda_3, & \dfrac{d\lambda_6}{dt} = -\lambda_1\lambda_5 + \lambda_4\lambda_2 \end{cases} \tag{37}$$

by exploiting the fact that λ_1 is constant call $\lambda_1(0)$ and using the conserved quantity $I_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ which for this case is $I_2 - \lambda_1^2(0) = \lambda_2^2 + \lambda_3^2$ suggests using polar coordinates $\lambda_2 = r\sin\theta, \lambda_3 = r\cos\theta$. It is then easily shown that the solution to (37) is:

$$\begin{array}{lll} \lambda_1 = \lambda_1(0), & \lambda_2 = r\sin\vartheta t, & \lambda_3 = r\cos\vartheta t \\ \lambda_4 = \lambda_4(0), & \lambda_5 = s\sin\vartheta t, & \lambda_6 = s\cos\vartheta t \end{array} \tag{38}$$

where the constants r, s, ϑ are defined as:

$$r = I_2 - \lambda_1^2(0), \quad s = \frac{rc_4}{\lambda_4(0)}(\lambda_1(0) - \lambda_4(0)r), \quad \vartheta = \lambda_1(0) - \frac{\lambda_4(0)r}{s} \tag{39}$$

to compute the corresponding motions $g(t) \in SE(3)$ analytically can then be obtained by following the procedure in [25]. This essentially reduces the MPP to a problem of parameter optimisation to match the boundary conditions.

4 Conclusion

In this paper a procedure is presented for reducing the complexity of the motion planning problem for some robotic systems (whose kinematics can be represented by left-invariant vector fields). The procedure assigns a meaningful cost function to the systems (nonholonomic) kinematics which ensures feasible and smooth motions. Furthermore, it is shown how it is possible to manipulate the weights of the quadratic cost function to obtain an analytic solution to this optimal control problem. This illustrates that for many robotic motion planning problems it is possible to construct analytically defined optimal motions in closed-form. This essentially reduces the motion planning problem of robotic systems to a problem of optimising the parameters of analytic functions to match prescribed boundary conditions. Future work will develop this method to take into account obstacle avoidance by using an iterative approach to select the parameters of the analytically defined curves.

References

1. Zexiang, L., Canny, J. F., 'Nonholonomic Motion Planning', Kluwer Academic Publishers, 1993.
2. Bloch, A. M., 'Nonholonomic Mechanics and Control'. Springer-Verlag, New York,2003.
3. Murray, R., Li, Z., Sastry, S., 'A Mathematical Introduction to Robotic Manipulation'. CRC Press, 1994.
4. Canny, J. F., 'The Complexity of Robot Motion Planning'. Cambridge, MA: MIT press, 1988.
5. Latombe, J. C., 'Robot Motion Planning'. Boston: Kluwer, 1991.
6. Leonard, N., Krishnaprasad, P. S., 'Motion control of drift free, left-invariant systems on Lie groups'. IEEE Transactions on Automatic control, vol. 40, pp.1539-1554, 1995.
7. Luenberger, D., 'Linear and Nonlinear Programming'. Addison-Wesley Publishing Company, Reading, Massachusetts, 1984.
8. Olver, P. J., 'Applications of Lie groups to Differential equations'. Graduate Texts in Mathematics, Springer, 1986.
9. Brockett, R. W., 'Systems Theory on Group Manifolds and Coset spaces'. SIAM Journal of Optimization and Control, Vol. 10, pp. 265-284, 1972.

10. Lafferriere, G., Sussmann, H., 'Motion Planning for Controllable systems without drift', Proceedings IEEE International Conference on Robotics and Automation, pp. 1148-1153, Sacramento CA, 1991.
11. Gurtvis, L., Z, Li.,'Smooth time-periodic feedback solutions for nonholonomic motion planning'. in Nonholonomic Motion Planning, Kluwer Academic Publishers, 1993.
12. Sussmann, H. J., Liu, W., 'Limits of highly oscillatory controls and approximations of general paths by admissible trajectories'. Proceedings of 30th IEEE Conference on Decision and Control, pp. 437-442, 1991.
13. Murray, R., Sastry, S., 'Steering nonholonomic systems using sinusoids'. Proceedings of 29th IEEE Conference on Decision and Control, pp. 2097-2101. Honolulu, Hawaii, 1990.
14. Baillieul, J., 'Geometric Methods for Nonlinear Optimal control problems'. Journal of Optimization theory and Applications: Vol. 25, No. 4, 1978.
15. Brockett, R. W., 'Lie Theory and Control Systems Defined on Spheres'. SIAM Journal on Applied Mathematics, Vol.25, No. 2, 1973.
16. Brockett, R. W., Dai, L., 'Non-holonomic Kinematics and the Role of Elliptic Functions in Constructive Controllability'. in Nonholonomic Motion Planning, Kluwer Academic Publishers, 1993.
17. Jurdjevic, V., 'Geometric Control Theory'. Advanced Studies in Mathematics, Cambridge University Press, 52,1997.
18. Bullo, F., Lewis, A.,'Geometric Control of Mechanical Systems'. Springer-Verlag, New York,2005.
19. Carinena, J., Ramos, A., 'Lie systems in Control Theory' in Nonlinear Geometric Control Theory. World Scientific, 2002.
20. Justh, E. W., Krishnaprasad, P. S., 'Natural frames and interacting particles in three dimension'. Proceedings of 44th IEEE Conference on Decision and Control and the European Control Conference, 2005.
21. Biggs, J. D., Holderbaum, W., Jurdjevic, V., 'Singularities of Optimal Control Problems on some 6-D Lie groups'. IEEE Transactions on Automatic Control, pp. 1027-1038, June 2007.
22. Selig, J. M., 'Geometric Fundamentals of Robotics'. Monographs in Computer Science, Springer, 2006.
23. Sussmann, H. J., 'An introduction to the coordinate-free maximum principle'. In Geometry of Feedback and Optimal Control,B. Jakubczyk and W. Respondek Eds., Marcel Dekker, New York, pp. 463-557, 1997.
24. Biggs, J.D., Holderbaum, W. 'Optimal kinematic control of an autonomous underwater vehicle'. IEEE Transactions on Automatic Control, 54, (7), pp. 1623-1626, July 2009.
25. Biggs. J., Holderbaum, W., 'Integrable quadratic Hamiltonians on the Euclidean group of motions'. Journal of Dynamical and Control Systems, Vol. 16, No. 3, pp. 301317, 2010,