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Uniform bounds on the 1-norm of the inverse of lower triangular Toeplitz matrices ☆

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A uniform bound on the 1-norm is given for the inverse of a lower triangular Toeplitz matrix with non-negative monotonically decreasing entries whose limit is zero. The new bound is sharp under certain specified constraints. This result is then employed to throw light upon a long standing open problem posed by Brunner concerning the convergence of the one-point collocation method for the Abel equation. In addition, the recent conjecture of Gauthier et al. is proved.

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1. Introduction

Consider the Abel equation

$$\int_0^t (t-s)^{-\alpha} y(s)ds = g(t), \quad t \in I := [0, T], \tag{1.1}$$

with $g(0) = 0$, and $\alpha \in [0, 1)$. Its unique solution $y \in C(I)$ is given by

$$y(t) = \frac{1}{\gamma_\alpha} \int_0^t (t-s)^{\alpha-1} g'(s)ds, \quad t \in I$$

with $\gamma_\alpha := \Gamma(\alpha)\Gamma(1-\alpha)$. Consider its associated difference equation

$$a_0(c;\alpha)y_{i+1} + \sum_{j=0}^{i-1} a_{i-j}(c;\alpha)y_{j+1} = b_i(c;\alpha), \quad i = 0, 1, \ldots, n, \tag{1.2}$$

where the parameter c lies in $(0, 1]$, and the coefficients are given by

$$\begin{aligned} &(a)\ a_0(c;\alpha) = \int_0^c (c-s)^{-\alpha} ds; \\ &(b)\ a_k(c;\alpha) = \int_0^1 (c+k-s)^{-\alpha} ds, \quad k = 1, 2, \ldots, n; \\ &(c)\ b_i(c;\alpha) = h^{\alpha-1} g(t_i + ch), \quad i = 0, 1, \ldots, n. \end{aligned} \tag{1.3}$$

Here we define $\sum_{j=k}^{\ell} \equiv 0$ if $\ell < k$.

The points t_i belong to a uniform mesh $I_i = \{t_i = ih \,:\, i = 0, 1, \ldots, n+1;\, t_{n+1} = T\}$.

This is a collocation method with a single collocation point t_{i+c} per sub-interval $[t_i, t_{i+1}]$.

Note that the case $c = 1$ gives rise to the implicit Euler product-integration method (see e.g. Weiss and Anderssen [14], Eggermont [7]).

In [3] Brunner posed the problem:

Problem 1.1. Given $\alpha \in [0, 1)$, for which values of $c = c(\alpha) \in (0, 1]$ do the solutions y_{i+1} of the difference equation (1.2) remain uniformly bounded as $n \to \infty,\ h \to 0$ with $(n+1)h = T$?

By evaluating the integrals in (a) and (b) of (1.3) the totality of the difference equations (1.2) may be written as

$$T_n\mathbf{y} = \begin{pmatrix} c^{1-\alpha} & & & & O \\ (1+c)^{1-\alpha} - c^{1-\alpha} & c^{1-\alpha} & & & \\ (2+c)^{1-\alpha} - (1+c)^{1-\alpha} & (1+c)^{1-\alpha} - c^{1-\alpha} & c^{1-\alpha} & & \\ \vdots & & & \ddots & \\ (n+c)^{1-\alpha} - (n-1+c)^{1-\alpha} & \ldots & & \ldots & c^{1-\alpha} \end{pmatrix} \mathbf{y} = \mathbf{d}, \tag{1.4}$$

where

$$\mathbf{y} = (y_1, \ldots, y_{n+1})^T, \quad \mathbf{d} = (d_1, \ldots, d_{n+1})^T.$$

We note that T_n is a Toeplitz matrix.

Furthermore, in the case of this class of lower triangular Toeplitz matrices T_n we have that

$$\|T_n\|_1 = \|T_n\|_\infty,$$

due to the fact that

$$\max_i \sum_{j=1}^{n+1} |t_{ij}| = \max_j \sum_{i=1}^{n+1} |t_{ij}|.$$

It is easy to see that $\|T_n^{-1}\|_1$ $(= \|T_n^{-1}\|_\infty)$ being uniformly bounded is a necessary condition for $\mathbf{y}$ to remain uniformly bounded. Notice that if $\alpha = 0$ in (1.1), then the complete answer to Problem 1.1 is known: the solutions of the difference equation (1.2) remain uniformly bounded if, and only if, $c \geq \frac{1}{2}$ (cf. Brunner [3–5]). For $0 < \alpha < 1$ is a sufficient condition for uniform boundedness is $c \geq c^*(\alpha) := \left(\frac{1}{2}\right) [\alpha(1-\alpha)\gamma_\alpha]^{1/(1-\alpha)}$ (see [3]). For example, $c^*(\alpha) \simeq 0.3084$ when $\alpha = \frac{1}{2}$.

The following conjecture suggests the existence of an upper bound on the inverse of a Toeplitz matrix with a similar, but different structure to that of (1.4).

1.1. GKM conjecture

If $T_n \in \mathbb{R}^{(n+1)\times(n+1)}$ is the lower triangular Toeplitz matrix whose first column is

$$\frac{1}{\sqrt{2}}(1, \sqrt{3}-1, \sqrt{5}-\sqrt{3}, \ldots, \sqrt{2n+1}-\sqrt{2n-1})^T, \tag{1.5}$$

then for all n, $\|T_n^{-1}\|_1 < 3$.

Recently, Gauthier et al. [8] proved that an algorithm due to Chen and Mangasarian [6] for solving a mixed linear complementarity problem arising from the discretization of a special system of singular Volterra integral equations would converge if the above conjecture were true.

These and related problems have a long history dating back to the work of Holyhead [10], Weiss and Anderssen [14] and others; yet they appear to have evaded resolution.

Essentially the problem may be regarded as requiring that the 1-norm of the inverse of the lower triangular Toeplitz matrix with specific constraints on its elements be uniformly bounded with respect to its order. Consider the lower triangular $(n+1) \times (n+1)$ Toeplitz matrix

$$T_n = \begin{pmatrix} b_0 & & & & \\ b_1 & b_0 & & & \\ b_2 & b_1 & b_0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ b_n & \ldots & \ldots & b_1 & b_0 \end{pmatrix}, \tag{1.6}$$

which may be characterized by its first column $(b_0, b_1, \ldots, b_n)^T$ where $b_0 \geq b_1 \geq \cdots \geq b_n \geq b \geq 0$.

The upper bounds for $\|T_n^{-1}\|_\infty$ given in [1,12] (see Section 2) are not uniform with respect to n in the case where $\lim_{n\to\infty} b_n = b = 0$. Therefore, in this paper we shall provide a sharp uniform upper bound for $\|T_n^{-1}\|_\infty$ for this case, subject to specified constraints on the elements of T_n. We shall demonstrate that the GKM conjecture is a special case of this result. Furthermore, sharp necessary conditions for Brunner's collocation problem will be provided.

This paper will be organized as follows. In Section 2, a uniform bound is given for the 1-norm of the inverse of the lower triangular Toeplitz matrix (1.6) subject to certain constraints on its elements. Under these restrictions, it is shown that this new bound is sharp. The GKM conjecture is proved in Section 3. In the last section, Brunner's one-point collocation problem is also partially answered. Furthermore, we prove that the 1-norm of Brunner's associated T_n matrix is not uniformly bounded when $\alpha = 0$.

2. Uniform upper bound

Interesting results have already been obtained for matrices of the type defined by (1.6): the main result is given below.

Theorem ([1,12] See, also the more recent papers [2,13]). *An upper bound on $\|T_n^{-1}\|_\infty$ is given by*

$$\|T_n^{-1}\|_\infty \leq \begin{cases} \frac{2}{b}\left[1-\left(1-\frac{b}{b_0}\right)^{\left[\frac{n}{2}\right]+1}\right], & \text{if } b > 0, \\ \frac{2}{b_0}\left(\left[\frac{n}{2}\right]+1\right), & \text{if } b = 0. \end{cases} \tag{2.1}$$

In particular if $b > 0$,

$$\|T_n^{-1}\|_\infty \leq \frac{2}{b}$$

independently of n and b_0.

Note from (2.1) that the upper bound is dependent on n when $b = 0$. The GKM conjecture and Problem 1.1 both involve the case of $b = 0$. However, numerical tests clearly show that $\|T_n^{-1}\|_\infty$ is bounded independently of n. Thus, this paper will deal with obtaining a uniform bound for T_n^{-1} in the 1-norm (or ∞-norm) when $b = 0$ subject to specified constraints on the elements of T_n.

The following lemma, due to Jurkat [11], is an extension of a result by Hardy [9] on inclusion theorems for Norlund means. For clarity and completeness we include Jurkat's concise proof.

Lemma 2.1. *If a_i and b_i satisfy the conditions*

$$b_i > 0, \quad i = 0, 1, 2, \ldots, \quad \frac{b_{i+1}}{b_i} \geq \frac{b_i}{b_{i-1}}, \quad i = 1, 2, \ldots \tag{2.2}$$

and

$$\frac{a_i}{b_i} \geq \frac{a_{i-1}}{b_{i-1}}, \quad i = 1, 2, \ldots, \text{ with } a_0 > 0, \tag{2.3}$$

then all the coefficients, k_n, of the Taylor expansion of

$$\sum_{i=0}^{\infty} a_i x^i \Big/ \sum_{i=0}^{\infty} b_i x^i \tag{2.4}$$

are non-negative. Furthermore, these coefficients of the Taylor expansion are all positive if (2.3) holds as a strict inequality for all $i = 1, 2, \ldots$

Proof. We first note that

$$\sum_{\nu=0}^{n} k_\nu b_{n-\nu} = a_n \quad \text{for } n \geq 0, \tag{2.5}$$

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and

$$k_0 = a_0/b_0 \geq 0. \tag{2.6}$$

Assume that $k_1, \ldots, k_{n-1} \geq 0$. Then we have to show that $k_n \geq 0$. From (2.5) we have the identity

$$\sum_{\nu=0}^{n-1} k_\nu \left(\frac{b_{n-\nu}}{b_n} - \frac{b_{n-1-\nu}}{b_{n-1}} \right) + k_n \frac{b_0}{b_n} = \frac{a_n}{b_n} - \frac{a_{n-1}}{b_{n-1}}, \tag{2.7}$$

and from (2.2)

$$\frac{b_{n-\nu}}{b_n} - \frac{b_{n-1-\nu}}{b_{n-1}} \begin{cases} = 0, & \text{for } \nu = 0, \\ \leq 0, & \text{for } 1 \leq \nu \leq n-1. \end{cases} \tag{2.8}$$

Substituting (2.8) into (2.7) and using (2.3), it follows that

$$k_n \frac{b_0}{b_n} \geq \frac{a_n}{b_n} - \frac{a_{n-1}}{b_{n-1}} \geq 0.$$

In a similar way, we can show that these coefficients are strictly positive if (2.3) holds as a strict inequality for all $i = 1, 2, \ldots$ $\square$

Corollary 2.1. *Let b_i, $i = 0, 1, 2, \ldots$, be a positive sequence such that $\frac{b_{i+1}}{b_i}$, $i = 0, 1, 2, \ldots$, is non-decreasing, then all the coefficients of the Taylor expansion of*

$$\frac{1}{\sum_{i=0}^{\infty} b_i x^i} \tag{2.9}$$

are non-positive except the constant term. Furthermore, all these coefficients except the constant term are negative if $\frac{b_{i+1}}{b_i}$, $i = 0, 1, 2, \ldots$, is strictly increasing for all $i = 0, 1, 2, \ldots$

Proof. It is easy to see that

$$\frac{1}{\sum_{i=0}^{\infty} b_i x^i} = \frac{1}{b_0} - \frac{x}{b_0} \frac{\sum_{i=0}^{\infty} b_{i+1} x^i}{\sum_{i=0}^{\infty} b_i x^i}. \tag{2.10}$$

This relation and the previous lemma with $a_i = b_{i+1}$ imply that the corollary is true. $\square$

Now we come to a more general case.

Corollary 2.2. *Let b_i, $i = 0, 1, 2, \ldots$, be a positive sequence such that $\frac{b_{i+1}}{b_i}$, $i = 2, 3, \ldots$, is non-decreasing and that the relation*

$$\frac{b_2}{b_1} < \frac{b_1}{b_0} \leq \frac{b_3}{b_2} \tag{2.11}$$

holds. Then all the coefficients of the Taylor expansion of (2.9) are non-positive except the first and third terms.

Proof. Again the following identity holds:

$$\frac{1}{\sum_{i=0}^{\infty} b_i x^i} = \frac{1}{b_0}\left[1 - \gamma_1 x - \gamma_2 x^2 - x^3 \frac{\sum_{i=0}^{\infty} a_i x^i}{\sum_{i=0}^{\infty} b_i x^i}\right], \tag{2.12}$$

where

$$\gamma_1 = \frac{b_1}{b_0}, \quad \gamma_2 = \gamma_1\left(\frac{b_2}{b_1} - \frac{b_1}{b_0}\right), \tag{2.13}$$

and

$$a_i = b_{i+3} - \gamma_1 b_{i+2} - \gamma_2 b_{i+1}, \tag{2.14}$$

for all $i = 0, 1, 2, \ldots$. Assumption (2.11) implies that $\gamma_2 < 0$. Consequently, it follows from relation (2.14) that

$$a_i \geq b_{i+3} - \gamma_1 b_{i+2} = b_{i+2}\left[\frac{b_{i+3}}{b_{i+2}} - \frac{b_1}{b_0}\right] \geq b_{i+2}\left[\frac{b_3}{b_2} - \frac{b_1}{b_0}\right] \geq 0, \tag{2.15}$$

for all $i = 0, 1, 2, \ldots$. Moreover, we have

$$\begin{aligned}
a_{i+1} &= b_{i+4} - \gamma_1 b_{i+3} - \gamma_2 b_{i+2} \\
&= b_{i+3}\frac{b_{i+4}}{b_{i+3}} - \gamma_1 b_{i+3} - \gamma_2 b_{i+2} \\
&\geq \frac{b_{i+3}}{b_{i+2}}[b_{i+3} - \gamma_1 b_{i+2}] - \gamma_2 b_{i+2} \\
&\geq \frac{b_{i+2}}{b_{i+1}}[b_{i+3} - \gamma_1 b_{i+2}] - \gamma_2 b_{i+2} \\
&= \frac{b_{i+2}}{b_{i+1}}[b_{i+3} - \gamma_1 b_{i+2} - \gamma_2 b_{i+1}] \\
&= \frac{b_{i+2}}{b_{i+1}}\, a_i \\
&\geq \frac{b_{i+1}}{b_i}\, a_i,
\end{aligned} \tag{2.16}$$

for all $i = 0, 1, 2, \ldots$. □

We have demonstrated that, in (2.15),

$$a_i \geq 0, \quad i = 0, 1, 2, \ldots \tag{2.17}$$

and, in (2.16),

$$a_{i+1} \geq \frac{b_{i+1}}{b_i}\, a_i, \quad i = 0, 1, 2, \ldots \tag{2.18}$$

Thus, we may appeal to (2.7) in Lemma 2.1 to show that the coefficients of the Taylor expansion of

$$\frac{\sum_{i=0}^{\infty} a_i x^i}{\sum_{i=0}^{\infty} b_i x^i}$$

in (2.12) are all non-negative, and consequently that the coefficients of the Taylor expansion of (2.9) are all non-positive.

We may note that a consequence of this corollary is that

$$b_{i+3} - \gamma_1 b_{i+2} - \gamma_2 b_{i+1} \geq 0, \quad i = 0, 1, 2, \ldots \tag{2.19}$$

We now impose an additional condition on the sequence $\{b_i\}$ and provide an estimate of the sum of the coefficients of the Taylor expansion of (2.9).

Lemma 2.2. *Suppose that the sequence b_i, $i = 0, 1, 2, \ldots$, satisfies the conditions of Corollary 2.2 and suppose that*

$$\frac{b_{i+1}}{b_i} \leq 1, \quad i = 0, 1, 2, \ldots \tag{2.20}$$

Then,

$$\phi(x) = \frac{1}{\sum_{i=0}^{\infty} b_i x^i} = \frac{1}{b_0}\left[1 - \gamma_1 x - \gamma_2 x^2\right] - x^3 \sum_{i=0}^{\infty} \theta_i x^i, \tag{2.21}$$

where γ_1 and γ_2 are given by (2.13), and $\theta_i \geq 0$ for all $i = 0, 1, 2, \ldots$ Furthermore, we have that

$$\frac{1}{b_0}\left[1 - \gamma_1 - \gamma_2\right] - \beta = \sum_{i=0}^{\infty} \theta_i, \tag{2.22}$$

where

$$\beta = \begin{cases} 0, & \text{if} \quad \sum_{i=0}^{\infty} b_i = \infty; \\ \frac{1}{\sum_{i=0}^{\infty} b_i}, & \text{otherwise.} \end{cases} \tag{2.23}$$

Proof. From Corollary 2.2, (2.21) gives the Taylor expansion of $\phi(x)$. Due to the assumption (2.20), it follows that $b_i \leq b_0$ for all $i \geq 0$. Thus, for any given $\delta \in (0, 1)$, the sequence $\sum_{i=0}^{\infty} b_i x^i$ is uniformly convergent on the interval $|x| \leq \delta$. Hence, we see that the relation (2.21) holds for all $|x| < 1$. Now (2.22) follows from (2.21) by letting $x \to 1_-$. This completes the proof.

Now, we apply the above results to provide an estimate of the 1-norm of lower triangular Toeplitz matrices. First, we can write the matrix T_n given in (1.6) in the following form:

$$T_n = b_0 I + \sum_{i=1}^{n} b_i J^i = b_0 I + \sum_{i=1}^{\infty} b_i J^i, \tag{2.24}$$

where J is the Jordan matrix

$$J = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}, \tag{2.25}$$

since $J^k = O$ for all $k \geq n + 1$.

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Thus, we have that

$$T_n^{-1} = \phi(J), \tag{2.26}$$

where $\phi(x)$ is defined in (2.21). $\square$

The following result follows directly from Corollary 2.1 and Lemma 2.2.

Theorem 2.1. *Suppose that T_n is defined by (1.6) with $b_i > 0$ for $i \geq 0$. If b_{i+1}/b_i is non-decreasing and bounded above by 1 for $i = 0, 1, 2, \ldots$, we have the bound:*

$$\|T_n^{-1}\|_1 \leq \frac{2}{b_0} - \beta, \tag{2.27}$$

where β is defined in (2.23). If, on the other hand, b_i, $i = 0, 1, 2, \ldots$, is a positive sequence, $b_{i+1}/b_i (i \geq 2)$ is non-decreasing and bounded above by 1 and the condition (2.11) holds then

$$\|T_n^{-1}\|_1 \leq \frac{2}{b_0}\left[1 - \frac{b_2}{b_0} + \left(\frac{b_1}{b_0}\right)^2\right] - \beta. \tag{2.28}$$

Proof. If b_{i+1}/b_i is non-decreasing and bounded above by 1 for $i = 0, 1, 2, \ldots$, then it follows from Corollary 2.1 that there exists $\tilde{\theta}_i \geq 0$ such that

$$\frac{1}{\sum_{i=0}^{\infty} b_i x^i} = \frac{1}{b_0} - x\sum_{i=0}^{\infty}\tilde{\theta}_i x^i.$$

Thus, by a similar argument to Lemma 2.2, we have

$$\sum_{i=0}^{\infty}\tilde{\theta}_i = \frac{1}{b_0} - \beta. \tag{2.29}$$

Therefore,

$$\begin{aligned} \|T_n^{-1}\|_1 &= \frac{1}{b_0} + \sum_{i=0}^{\infty}|\tilde{\theta}_i| \\ &= \frac{1}{b_0} + \left(\frac{1}{b_0} - \beta\right) \\ &= \frac{2}{b_0} - \beta. \end{aligned} \tag{2.30}$$

This proves (2.27). $\square$

Now we assume that $b_{i+1}/b_i (i \geq 2)$ is non-decreasing (and bounded above by 1) and, additionally, that the condition (2.11) is satisfied. In this case, we also have the relation (2.29). The assumptions on the sequence $\{b_i, i = 0, 1, 2, \ldots\}$ and Corollary 2.2 imply that $\gamma_1 \geq 0$, $\gamma_2 \leq 0$ and $\theta_i \geq 0$ for all $i \geq 0$. Therefore, it follows that

$$\begin{aligned} \|T_n^{-1}\|_1 &= \frac{1}{b_0}\left[1 + |\gamma_1| + |\gamma_2|\right] + \sum_{i=0}^{\infty}|\theta_i| \\ &= \frac{1}{b_0}\left[1 + \gamma_1 + |\gamma_2|\right] + \sum_{i=0}^{\infty}\theta_i = \frac{2}{b_0}(1 + |\gamma_2|) - \beta. \end{aligned} \tag{2.31}$$

Since it is clear that $\gamma_2 < 0$, the above equality and (2.13) yield (2.28).

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Note that the bound given in Theorem 2.1 depends on b_0, b_1, and b_2 but not on n. Consequently it is a uniform bound with respect to n.

3. GKM conjecture

Since the GKM conjecture is relatively straightforward to demonstrate, we shall deal with it first. Using the results given in the previous section, we can easily prove that this conjecture is true.

Theorem 3.1. *Let $T_n \in \mathbb{R}^{(n+1)\times(n+1)}$ be the lower triangular Toeplitz matrix whose first column is given by (1.5). Then the inequality*

$$\|T_n^{-1}\|_1 \leq 2\sqrt{2}(5-\sqrt{3}-\sqrt{5}) = 2.91860082 < 3 \tag{3.1}$$

holds for all $n = 1, 2, \ldots$

Proof. If $n = 1$, we have that

$$\begin{aligned}\|T_1^{-1}\|_1 &= \left\| \sqrt{2}\begin{pmatrix} 1 & 0 \\ -(\sqrt{3}-1) & 1 \end{pmatrix} \right\|_1 \\ &= \sqrt{2}[(\sqrt{3}-1)+1] < 2\sqrt{2}(5-\sqrt{3}-\sqrt{5}),\end{aligned} \tag{3.2}$$

which shows that (3.1) holds for $n = 1$. Similarly, direct calculations then provide

$$\begin{aligned}\|T_2^{-1}\|_1 &= \left\| \sqrt{2}\begin{pmatrix} 1 & 0 & 0 \\ -(\sqrt{3}-1) & 1 & 0 \\ 4-\sqrt{3}-\sqrt{5} & -(\sqrt{3}-1) & 1 \end{pmatrix} \right\|_1 \\ &= \sqrt{2}[1+(\sqrt{3}-1)+(4-\sqrt{3}-\sqrt{5})] \\ &= \sqrt{2}(4-\sqrt{5}) < 2\sqrt{2}(5-\sqrt{3}-\sqrt{5}),\end{aligned} \tag{3.3}$$

which demonstrates that (3.1) holds for $n = 2$ as well.

Now we consider the case when $n \geq 3$. Define the sequence $b_i,\ i = 0, 1, 2, \ldots,$ as follows:

$$b_0 = \frac{1}{\sqrt{2}}, \quad b_i = \frac{\sqrt{2i+1}-\sqrt{2i-1}}{\sqrt{2}}, \quad i = 1, 2, \ldots \tag{3.4}$$

We can easily check that all the conditions for b_i given in Lemma 2.2 (and, consequently, those given in Corollary 2.2) are satisfied. It follows from (2.13) and (3.4) that

$$\gamma_2 = \frac{b_2}{b_0} - \left(\frac{b_1}{b_0}\right)^2 = \sqrt{5}-\sqrt{3}-(\sqrt{3}-1)^2 = \sqrt{5}+\sqrt{3}-4. \tag{3.5}$$

Moreover, (3.4) also gives $\beta = 0$. Thus, from Theorem 21, we have that

$$\|T_n^{-1}\|_1 \leq 2\sqrt{2}(1+|\sqrt{5}+\sqrt{3}-4|) = 2\sqrt{2}(5-\sqrt{3}-\sqrt{5}) < 3. \tag{3.6}$$

This shows that the conjecture is true.

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4. Brunner's one-point collocation problem

Consider first the lower triangular Toeplitz matrix T_n whose first column is

$$(1,\ 2^{1-\alpha}-1,\ 3^{1-\alpha}-2^{1-\alpha},\ \ldots,(n+1)^{1-\alpha}-n^{1-\alpha})^T, \tag{4.1}$$

where $\alpha \in (0,\ 1)$. This corresponds to the case $c=1$, which is the case of the implicit Euler product-integration method applied to the Abel equation. Define the corresponding sequence

$$b_0=1, \quad b_i=(i+1)^{1-\alpha}-i^{1-\alpha}, \quad i=1,2,\ldots \tag{4.2}$$

It follows from (2.27) of Theorem 2.1 that $\|T_n^{-1}\|_1 \le 2$. Matlab 7.0 with Laptop Sony Vaio VGN TZ370 was used to test the Abel matrices in this case. The numerical results are displayed in Table 1.

We now turn our attention to Brunner's problem where the Toeplitz matrix T_n is given in (1.4). We define $T_n = c^{1-\alpha}\,\tilde{T}_n$ so that the elements of $\tilde{T}_n$ become

$$b_0=1, \quad b_i=\left(\frac{c+i}{c}\right)^{1-\alpha}-\left(\frac{c+i-1}{c}\right)^{1-\alpha}, \quad i=1,2,\ldots \tag{4.3}$$

First, we consider $\alpha \in (0,1)$. It is clear that $b_i>0$ for all $i \ge 0$. Moreover,

$$\frac{b_{i+1}}{b_i}=\frac{\left(1+\frac{1}{c+i}\right)^{1-\alpha}-1}{1-\left(1-\frac{1}{c+i}\right)^{1-\alpha}}, \quad i \ge 1. \tag{4.4}$$

Define the function

$$f(x)=\frac{(1+x)^{1-\alpha}-1}{1-(1-x)^{1-\alpha}}. \tag{4.5}$$

A simple calculation yields

$$f'(x)=(1-\alpha)\,\frac{(1-x)^{\alpha}+(1+x)^{\alpha}-2}{[1-(1-x)^{1-\alpha}]^2(1-x^2)^{\alpha}}. \tag{4.6}$$

It is easy to verify that $f'(x)<0$ for all $\alpha \in (0,1)$ and $x \in (0,1)$. Therefore, if $\alpha \in (0,1)$ and $c>0$, we have that b_{i+1}/b_i for $i \ge 2$ is monotonically increasing and converges to 1. In order to apply Theorem 2.1, we have to consider two separate cases.

(*Case A*) $b_2/b_1 \ge b_1/b_0$. This inequality reduces to

$$\psi_1(\alpha,c)=\left(\frac{2+c}{c}\right)^{1-\alpha}-\left(\frac{1+c}{c}\right)^{1-\alpha}-\left[\left(\frac{1+c}{c}\right)^{1-\alpha}-1\right]^2 \ge 0. \tag{4.7}$$

It follows from (2.27) of Theorem 2.1 that $\|\tilde{T}_n^{-1}\|_1 \le 2$ so that

$$\|T_n^{-1}\|_1 \le \frac{2}{c^{1-\alpha}}. \tag{4.8}$$

Table 1
1-Norm of the inverse of the Abel matrices when $c=1$.

n	$\alpha=0$	$\alpha=0.4$	$\alpha=0.5$	$\alpha=0.7$
50	2	1.9512	1.9091	1.7331
100	2	1.9682	1.9360	1.7838
1000	2	1.9920	1.9799	1.8912
2000	2	1.9947	1.9858	1.9122
2500	2	1.9954	1.9873	1.9179
3500	2	1.9962	1.9892	1.9258
5000	2	1.9970	1.9910	1.9333

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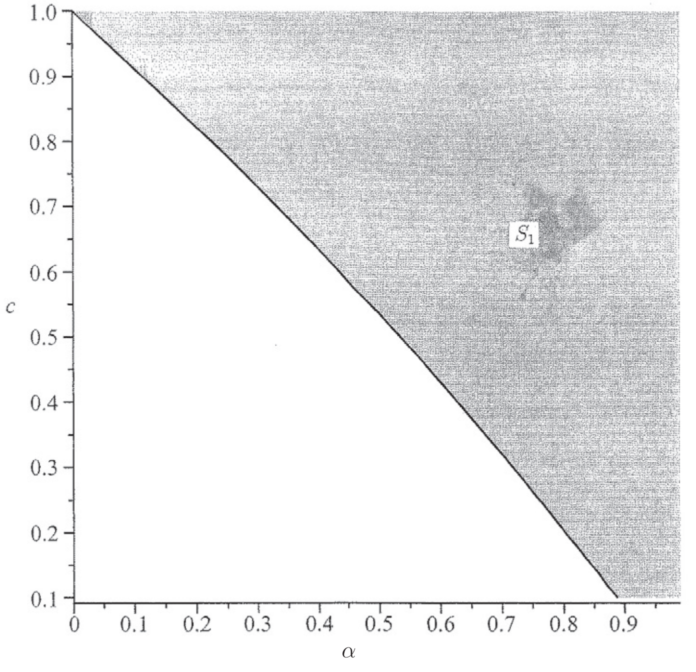

Graph 1. Case A displaying S_1 in the (c, α)-plane for which $\|T_n^{-1}\|_1$ is uniformly bounded. ($\psi_1(c, \alpha) = 0$ – solid line.)

(*Case B*) $\dfrac{b_2}{b_1} < \dfrac{b_1}{b_0} \le \dfrac{b_3}{b_2}$. Thus, from Theorem 2.1, we have the following bound

$$\|T_n^{-1}\|_1 \le 2\left(1 + b_1^2 - b_2\right)c^{\alpha-1} = \frac{2}{c^{1-\alpha}}\left[2 + \left(\frac{1+c}{c}\right)^{2(1-\alpha)} - \left(\frac{1+c}{c}\right)^{1-\alpha} - \left(\frac{2+c}{c}\right)^{1-\alpha}\right]. \tag{4.9}$$

The inequality (2.11) implies that (4.7) does not hold and $b_3/b_2 \ge b_1/b_0$. The latter is the following inequality

$$\frac{(3+c)^{1-\alpha} - (2+c)^{1-\alpha}}{(2+c)^{1-\alpha} - (1+c)^{1-\alpha}} \ge \frac{(1+c)^{1-\alpha} - c^{1-\alpha}}{c^{1-\alpha}}. \tag{4.10}$$

The above inequality may be written as

$$\psi_2(\alpha, c) = \frac{(3+c)^{1-\alpha} - (2+c)^{1-\alpha}}{(2+c)^{1-\alpha} - (1+c)^{1-\alpha}} - \frac{(1+c)^{1-\alpha} - c^{1-\alpha}}{c^{1-\alpha}} \ge 0. \tag{4.11}$$

Therefore, from Theorem 2.1, we have the following result.

Theorem 4.2 *Let T_n be defined in* (1.4). *Then we have that*

$$\|T_n^{-1}\|_1 \le \frac{2}{c^{1-\alpha}} \tag{4.12}$$

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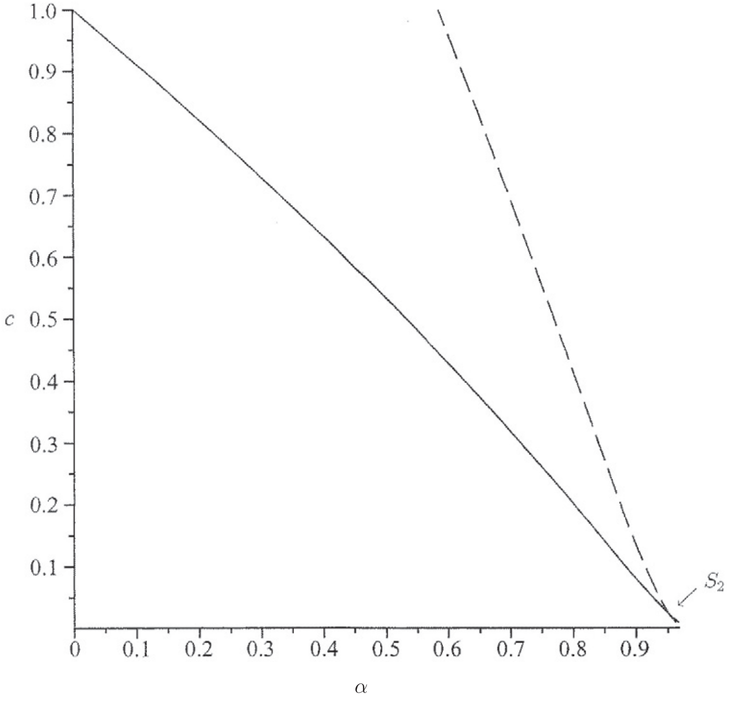


Graph 2. Case B displaying the (small) region S_2 in the (c, α)-plane for which $\|T_n^{-1}\|_1$ is uniformly bounded. ($\psi_1(c, \alpha) = 0$ – solid line; $\psi_2(c, \alpha) = 0$ – dashed line.)

if (4.7) *holds. If, however,* (4.7) *is not satisfied (i.e.* $\psi_1(\alpha, c) < 0$), *then we have that*

$$\|T_n^{-1}\|_1 \leq \frac{2}{c^{1-\alpha}} \left[2 + \left(\frac{1+c}{c}\right)^{2(1-\alpha)} - \left(\frac{1+c}{c}\right)^{1-\alpha} - \left(\frac{2+c}{c}\right)^{1-\alpha} \right], \tag{4.13}$$

provided that α *and* c *satisfy* $\psi_2(\alpha, c) \geq 0$.

Proof. If (4.7) holds, it follows from (2.27) in Theorem 2.1 that (4.12) is satisfied. If, however, (4.7) does not hold but the inequality (4.11) does, it follows from Theorem 2.1 that (2.28) is satisfied. The bound (2.28) and the form of the elements (4.3) then imply (4.13).

Thus, by plotting the curves $\psi_i(\alpha, c) = 0$, $i = 1, 2$, we can easily see the region of $\{(\alpha, c)\}$ for which $\|T_n^{-1}\|_1$ is uniformly bounded. Indeed, $\|T_n^{-1}\|_1$ is bounded uniformly for all n, if (α, c) is contained within the following two regions:

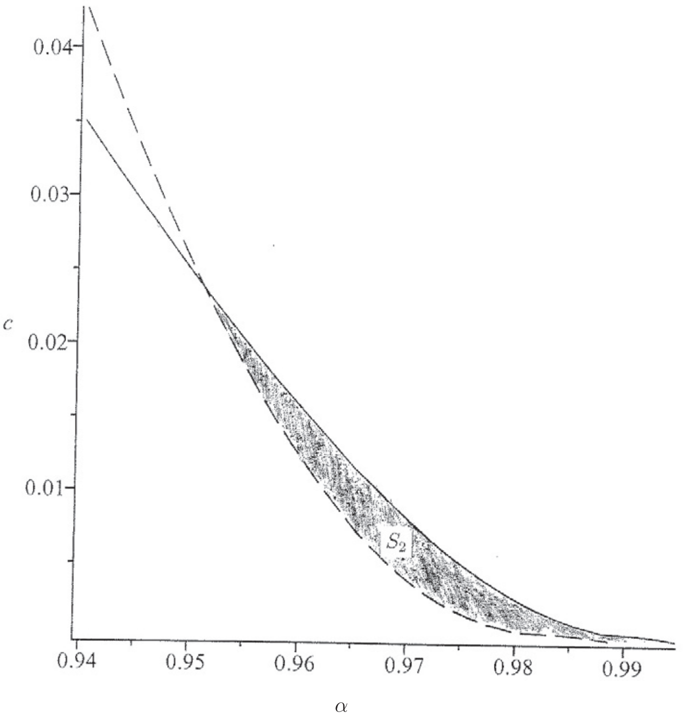
$$S_1 = \{(\alpha, c) | \psi_1(\alpha, c) \geq 0, \ \alpha \in (0, 1), \ c \in (0, 1].\} \tag{4.14}$$

$$S_2 = \{(\alpha, c) | \psi_1(\alpha, c) < 0, \ \psi_2(\alpha, c) \geq 0, \ \alpha \in (0, 1), \ c \in (0, 1].\} \tag{4.15}$$

Thus, for any point in S_1 or S_2, $\|T_n^{-1}\|_1$ is bounded uniformly with respect to n. □

Three graphs are displayed below. Graph 1 corresponds to case A while Graph 2 corresponds to case B. The region S_1 for which $\|T_n^{-1}\|_1$ is uniformly bounded is the shaded region in Graph 1. The region S_2 in Graph 2 is extremely small and so a "close-up" (Graph 3) is provided: S_2 is the shaded region in Graph 3.

Finally, we note that in the case $\alpha = 0$, the matrix does not satisfy the conditions of this paper.



Graph 3. Case B displaying a "close-up" of the region S_2 in the (c, α)-plane for which $\|T_n^{-1}\|_1$ is uniformly bounded. ($\psi_1(c, \alpha) = 0$ – solid line, $\psi_2(c, \alpha) = 0$ – dashed line.) S_2 is the region defined by $\psi_1(c, \alpha) < 0$ and $\psi_2(c, \alpha) > 0$.

However, if $\alpha = 0$, we have that

$$T_n = cI + J + J^2 + \cdots + J^n. \tag{4.16}$$

Direct calculations show that

$$T_n^{-1} = \frac{1}{c}\left[I - \frac{1}{c}\sum_{i=1}^{n}\left(1 - \frac{1}{c}\right)^{i-1} J^i\right]. \tag{4.17}$$

Thus,

$$\begin{aligned} \|T_n^{-1}\|_1 &= \frac{1}{c}\left[1 + \frac{1}{c}\sum_{i=1}^{n}\left|\left(1 - \frac{1}{c}\right)^{i-1}\right|\right] \\ &= \left\{\begin{array}{ll} 2(2n+1), & \text{if } c = 1/2; \\ \dfrac{\left(\frac{1}{c} - 1\right)^n - 2c}{c(1-2c)}, & \frac{1}{2} < c \le 1 \end{array}\right\}. \end{aligned} \tag{4.18}$$

Thus, we can conclude that with respect to n, $\|T_n^{-1}\|_1$ is uniformly bounded, if $c > \frac{1}{2}$; and not uniformly bounded, when $c = \frac{1}{2}$.

Acknowledgement

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