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Fractional Calculus of Periodic Distributions

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Abstract

Two approaches for defining fractional derivatives of periodic distributions are presented. The first is a distributional version of the Weyl fractional derivative in which a derivative of arbitrary order of a periodic distribution is defined via Fourier series. The second is based on the Grünwald-Letnikov formula for defining a fractional derivative as a limit of a fractional difference quotient. The equivalence of the two approaches is established and an application to a fractional diffusion equation posed in a space of periodic distributions is also discussed.

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1 Introduction

It is generally regarded that the modern theory of fractional calculus began with the work of Riemann and Liouville. Motivated by the fact that $D^n(e^{ax}) = a^n e^{ax},\ n = 0, 1, 2, \ldots,$ Liouville, in 1832, defined the fractional derivative $D^\alpha \phi$ of the function

$$\phi(x) = \sum_{k=0}^{\infty} c_k e^{a_k x}$$

by

$$(D^\alpha \phi)(x) = \sum_{k=0}^{\infty} c_k a_k^\alpha e^{a_k x}, \tag{1.1}$$

provided the series converges. Other pioneering ideas due to Liouville which influenced later work on fractional calculus include the fractional integration formula

$$(D^{-\alpha}\phi)(x) = \frac{1}{(-1)^\alpha \Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1}\phi(t)\,dt, \qquad x \in \mathbb{R},\, \alpha > 0, \tag{1.2}$$

and also the notion of a fractional derivative as a limit of a fractional difference quotient. In 1847 Riemann produced an alternative formula to (1. 2) for a fractional integral of order α, namely

$$(L^\alpha \phi)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\phi(t)\,dt, \qquad x > 0,\, \alpha > 0. \tag{1.3}$$

In modern terminology, (1. 3) is referred to as the Riemann-Liouville fractional integral of order α, while (1. 2) (with the factor $(-1)^\alpha$ omitted) is the Weyl fractional integral of order α.

Later, Grünwald (1867) and Letnikov (1868) pursued Liouville's earlier idea of obtaining a theory of fractional differentiation via the limit of some fractional difference quotient. Both used the formula

$$(D^\alpha \phi)(x) = \lim_{t\to 0} \frac{(\triangle_t^\alpha \phi)(x)}{t^\alpha}\,, \tag{1.4}$$

where $\triangle_t^\alpha$ is a fractional difference operator defined by

$$(\triangle_t^\alpha \phi)(x) = \sum_{j=0}^\infty (-1)^j \binom{\alpha}{j} \phi(x-tj), \quad \alpha > 0. \tag{1.5}$$

Liouville's first formula, (1. 1), can also be regarded as the forerunner of a theory of fractional calculus introduced by Weyl in 1917. In this case, the α^{th} fractional integral and α^{th} fractional derivative of a 2π-periodic function ϕ are defined, respectively, by

$$(D^{-\alpha}\phi)(x) = \sum_{l=-\infty}^{\infty} (il)^{-\alpha}\hat{\phi}(l)e^{ilx}, \quad (D^{\alpha}\phi)(x) = \sum_{l=-\infty}^{\infty} (il)^{\alpha}\hat{\phi}(l)e^{ilx}, \tag{1.6}$$

where $\alpha > 0$ and

$$\hat{\phi}(l) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ilt}\phi(t)dt, \qquad l = \pm 1, \pm 2, \ldots, \quad \hat{\phi}(0) = 0.$$

This use of Fourier series, or, equivalently, the finite (or discrete) Fourier transform can be regarded as the most natural way of defining fractional integrals and derivatives of a periodic function ϕ. Moroever, if we consider formula (1. 3) for the case when ϕ is a 2π-periodic function,and proceed formally, then we obtain

$$L^\alpha \phi \sim \sum_{l=-\infty}^{\infty} \hat{\phi}(l)\, L^\alpha \psi_l\,, \tag{1.7}$$

where

$$\psi_l(x) = e^{ilx}. \tag{1.8}$$

From results in [2, pp.420-427], when $0 < \alpha < 1$ and the lower limit of integration in the definition of $L^\alpha\phi$ is replaced by $-\infty$, it can be shown that

$$L^\alpha \psi_l = (il)^{-\alpha}\psi_l\,, \quad \forall\, l \in \mathbb{Z} : l \neq 0\,. \tag{1.9}$$

Hence, if we define

$$\widehat{L^\alpha \phi}(0) = 0,$$

then it follows that

$$L^{\alpha}\phi \sim \sum_{l\neq 0}(il)^{-\alpha}\,\hat{\phi}(l)\,\psi_{l}, \tag{1.10}$$

showing that the definition of $D^{-\alpha}\phi$ given in (1.6) coincides with $L^{\alpha}\phi$ for $0<\alpha<1$. Thus the formulae in (1.6) emerge as the natural candidates for defining fractional derivatives and integrals of periodic functions for any $\alpha>0$.

A rigorous treatment of the Weyl approach to fractional calculus, based on (1.6), can be carried out in the Banach spaces

$$L^{p}_{2\pi} := \{\phi \text{ is } 2\pi\text{-periodic on } \mathbb{R} \text{ and } \|\phi\|_{L^{p}_{2\pi}}<\infty\} \tag{1.11}$$

where

$$\|\phi\|_{L^{p}_{2\pi}} := \left\{\frac{1}{2\pi}\int_{0}^{2\pi}|\phi(x)|^{p}\,dx\right\}^{1/p}, \quad 1\leq p<\infty\,. \tag{1.12}$$

The factor $1/2\pi$ is convenient for Fourier analysis. As described in [3], formula (1.10) can be used to motivate an alternative definition of a fractional integral, $I^{\alpha}\phi$, expressed as a convolution integral, that will make sense for any $\alpha>0$ and $\phi\in L^{p}_{2\pi}$. The key to obtaining a formula for I^{α} is to rewrite the right-hand side of (1.10) as

$$\sum_{l=-\infty}^{\infty}\widehat{\eta_{\alpha}}(l)\hat{\phi}(l)\,\psi_{l} \tag{1.13}$$

where

$$\widehat{\eta_{\alpha}}(l)=\begin{cases}(il)^{-\alpha} & \text{when } l\neq 0,\\ 0 & \text{when } l=0.\end{cases}$$

The corresponding function η_{α} which has the above sequence $\{\widehat{\eta_{\alpha}}\}$ as its Fourier coefficients is given by

$$\eta_{\alpha}(x) := \sum_{l\neq 0}(il)^{-\alpha}e^{ilx}. \tag{1.14}$$

Since $\eta_{\alpha}\in L^{1}_{2\pi}$ for all $\alpha>0$, it can be used as the kernel for an associated convolution integral operator, I^{α}, defined by

$$(I^{\alpha}\phi)(x)=(\eta_{\alpha}*\phi)(x)=\frac{1}{2\pi}\int_{0}^{2\pi}\eta_{\alpha}(x-u)\,\phi(u)\,du\,. \tag{1.15}$$

It follows from [2, pp.8-10] that I^{α} is a bounded linear operator on $L^{p}_{2\pi}$ for all $p\in[1,\infty)$ with

$$\|I^{\alpha}\phi\|_{L^{p}_{2\pi}}\leq\|\eta_{\alpha}\|_{L^{1}_{2\pi}}\,\|\phi\|_{L^{p}_{2\pi}} \quad \forall\phi\in L^{p}_{2\pi}\,.$$

Moreover, from the convolution theorem for periodic functions [2, Theorem 4.1.3], we obtain

$$\begin{aligned}\widehat{(\eta_{\alpha}*\phi)}(l) &= \widehat{\eta_{\alpha}}(l)\hat{\phi}(l) \quad \forall l\in\mathbb{Z}\\ &= \begin{cases}(il)^{-\alpha}\hat{\phi}(l) & \text{when } l\neq 0,\\ 0 & \text{when } l=0.\end{cases}\end{aligned} \tag{1.16}$$

On comparing this with (1.6) and (1.10), it follows that I^{α} can justifiably be regarded as a fractional integral operator on periodic functions and can be defined by

$$(I^{\alpha}\phi)(x)=\sum_{l\neq 0}(il)^{-\alpha}\,\hat{\phi}(l)\,\psi_{l}. \tag{1.17}$$

Although $I^\alpha \phi$ is well defined as a function in $L^p_{2\pi}$ for all $\phi \in L^p_{2\pi}$, the same cannot be said for the Weyl fractional derivative of order α, given by

$$D^\alpha \phi \sim \sum_{l=-\infty}^{\infty} (il)^\alpha \hat{\phi}(l)\, \psi_l\,, \quad \alpha > 0, \tag{1. 18}$$

where $(il)^\alpha$ is taken to be 0 when $l = 0$. For example, in the particular case $p = 2$, when we have the Hilbert space $L^2_{2\pi}$, Parseval's equation can be used to show that the maximal domain for the fractional differential operator in (1. 18) is

$$Dom(D^\alpha) := \left\{\phi \in L^2_{2\pi} : D^\alpha \phi \in L^2_{2\pi}\right\} = \left\{\phi \in L^2_{2\pi} : \sum_{l=-\infty}^{\infty} l^{2\alpha} |\hat{\phi}(l)|^2 < \infty\right\}. \tag{1. 19}$$

As we shall establish later, one way of extending the class of functions on which D^α is defined is to work in a distributional setting.

The Banach spaces $L^p_{2\pi}$ were also used by Butzer and Westphal [3] in their investigations into the difference quotient approach to fractional derivatives, where a (strong) Liouville-Grünwald fractional derivative $D^{<\alpha>}$ of order α is defined by

$$D^{<\alpha>}\phi := s - \lim_{t\to 0^+} \frac{\triangle^\alpha_t \phi}{t^\alpha}, \tag{1. 20}$$

provided that the limit exists in $L^p_{2\pi}$. One of the main results obtained in [3] is the following theorem.

Theorem 1.1 *The following statements are equivalent for* $\phi \in L^p_{2\pi}, 1 \le p < \infty$, *and* $\alpha > 0$.

(i) $D^{<\alpha>}\phi \in L^p_{2\pi}$.

(ii) There exists $\psi \in L^p_{2\pi}$ *such that* $\hat{\psi}(l) = (il)^\alpha \hat{\phi}(l)$, $l \in \mathbb{Z}$.

(iii) There exists $\psi \in L^p_{2\pi}$ *such that* $I^\alpha \psi = \phi - \hat{\phi}(0)$ *almost everywhere, where* I^α *is defined by (1. 15).*

If (i) is satisfied, then $D^{<\alpha>}\phi = \psi$, *where* $\psi \in L^p_{2\pi}$ *is the function defined in (ii). Similarly if (ii) is satisfied then* $D^{<\alpha>}\phi = \psi$.

Proof: See [3, Theorem 4.1]. □

Our main aim in this paper is to extend the work of Butzer and Westphal in [3] to the case of periodic distributions. Distributional versions of the Riemann-Liouville and Weyl fractional integrals (and the related Erdélyi-Kober operators) have been produced by Erdélyi [4] and McBride [7]. However, it appears that no attempt has been made to define fractional derivatives of distributions via fractional difference quotients. Here we adopt a strategy that we applied successfully in [6] to produce a distributional theory of the fractional Fourier transform on the space of tempered distributions. Hence we begin with an appropriate symmetric differential operator T with a complete orthonormal system of smooth eigenfunctions $\{\psi_j\}_{j\in J}$ in a Hilbert space H. We then follow the theory developed by Zemanian in [11, Chapter 9] to construct a space $\mathcal{A}'_{2\pi}$ of generalised functions in which all elements can be represented by Fourier expansions in terms of $\{\psi_j\}_{j\in\mathbb{Z}}$, with the series converging in the weak* topology.

In Section 2, we introduce the spaces of test-functions $\mathcal{A}_{2\pi}$ and distributions $\mathcal{A}'_{2\pi}$ and demonstrate that $\mathcal{A}'_{2\pi}$ can be identified with the space $P'_{2\pi}$ of -periodic distributions defined in [10, Chapter 11].

Sections 3 and 4 are concerned with the the Fourier series-based approach and difference quotient approach to fractional calculus on $\mathcal{A}_{2\pi}$ and $\mathcal{A}'_{2\pi}$. By modifying the arguments presented in [3], we establish that both approaches lead to equivalent definitions of the α^{th} fractional derivative of a 2π-periodic distribution f. Finally, in Section 5, we consider fractional diffusion equations involving distributional initial conditions.

2 Spaces of Periodic Test-Functions and Distributions

In [11, Chapter 9], Zemanian discusses the convergence of orthonormal series of eigenfunctions in a complete multinormed space $\mathcal{A}$, constructed around a symmetric differential operator T in the Hilbert space $L^2(I)$, where $I \subseteq \mathbb{R}$ is an interval. We consider the case when

$$I = (0, 2\pi), T = -iD, \psi_l(x) = e^{ilx}, l \in \mathbb{Z}, \text{and } \lambda_l = l, \tag{2.1}$$

given in [11, Example 9.2-1]. Note that in [11, Example 9.2-1], a normalising factor of $1/\sqrt{2\pi}$ appears in the definition of ψ_l. This is unnecessary here as we choose to work with a weighted version, $L^2_{2\pi}(I)$, of the usual $L^2(I)$ space in which the inner product and norm are given by

$$(\phi, \psi)_{L^2_{2\pi}(I)} := \frac{1}{2\pi}\int_0^{2\pi} \phi(x)\overline{\psi(x)}\, dx, \quad \|\phi\|_{L^2_{2\pi}(I)} := \left[\frac{1}{2\pi}\int_0^{2\pi} |\phi(x)|^2\, dx\right]^{1/2}. \tag{2.2}$$

Note that

$$(\phi, \psi)_{L^2_{2\pi}(I)} = (\phi, \psi)_{L^2_{2\pi}} \quad \text{for all periodic functions } \phi\,, \psi \in L^2_{2\pi}.$$

The domain, $D(T)$, of the operator T is given by

$$D(T) := \left\{\phi \in C^\infty(I) : T^k\phi \in L^2_{2\pi}(I),\ (T^k\phi, \psi_l)_{L^2_{2\pi}(I)} = (\phi, T^k\psi_l)_{L^2_{2\pi}(I)}, \forall\ k, l = 0, 1, \dots\right\}. \tag{2.3}$$

A Fréchet space, $\mathcal{A}$, is obtained from $D(T)$ by imposing on $D(T)$ the topology generated by the countable multinorm $\{\beta_k\}_{k=0}^\infty$, where

$$\beta_k(\phi) := \left(\frac{1}{2\pi}\int_0^{2\pi} |(-i)^k D^k\phi(x)|^2 dx\right)^{1/2} = \left(\frac{1}{2\pi}\int_0^{2\pi} |\phi^{(k)}(x)|^2 dx\right)^{1/2}.$$

We now define a space of periodic test-functions $\mathcal{A}_{2\pi}$ and corresponding space of periodic distributions $\mathcal{A}'_{2\pi}$ that are closely related to the spaces $P_{2\pi}$ and $P'_{2\pi}$ defined in [10, Chapter 11]. The space $P_{2\pi}$ consists of 2π-periodic test-functions, and is equipped with the topology generated by multinorms $\{\eta_k\}_{k=0}^\infty$, where

$$\eta_k(\phi) := \sup_{x\in\mathbb{R}} |(D^k\phi)(x)| = \| D^k\phi \|_\infty, \qquad k \in \mathbb{N}_0,\ \phi \in C^\infty_{2\pi},$$

and $C^\infty_{2\pi}$ is the space of infinitely differentiable, 2π-periodic functions. As stated in [11, pp.256 - 257], each $\phi \in \mathcal{A}$ can be extended to a smooth periodic function $\phi^{per} \in P_{2\pi}$, via

$$\phi^{per}(x) := \phi(x), \quad 0 < x < 2\pi, \tag{2.4}$$

$$\phi^{per}(0) := \lim_{x\to 0^+} \phi(x), \quad \phi^{per}(2\pi) := \lim_{x\to 2\pi^-} \phi(x), \tag{2.5}$$

$$\phi^{per}(x + 2\pi) := \phi^{per}(x), \quad -\infty < x < \infty. \tag{2.6}$$

Also $\{\phi_n\}$ converges in $\mathcal{A}$ if and only if $\{\phi_n^{per}\}$ converges in $P_{2\pi}$. Consequently, the test-function space $\mathcal{A}$ can be identified through this extension with $P_{2\pi}$.

Motivated by the fact that each $\phi \in \mathcal{A}$ gives rise to an infinitely differentiable, periodic function on $\mathbb{R}$, we introduce the following "periodic version" of $\mathcal{A}$, denoted by $\mathcal{A}_{2\pi}$.

Definition 2.1 *We define $\mathcal{A}_{2\pi}$ to be the vector space $C^{\infty}_{2\pi}$ equipped with the topology generated by the family of seminorms $\{\rho_k\}_{k=0}^{\infty}$, where*

$$\rho_k(\varphi) := \| T^k\varphi \|_{L^2_{2\pi}} = \|D^k\varphi\|_{L^2_{2\pi}}, \ \phi \in \mathcal{A}_{2\pi}, \quad T = -iD. \tag{2.7}$$

Clearly, if $\varphi \in \mathcal{A}_{2\pi}$ then the function φ restricted to the interval $[0, 2\pi]$, denoted by $\varphi_{|_{[0,2\pi]}}$ or φ^{res}, lies in $\mathcal{A}$. It is easy to show that the mapping $\mathcal{R} : \mathcal{A}_{2\pi} \to \mathcal{A}$, defined by $\mathcal{R}\varphi := \varphi^{res}$, is a homeomorphism from $\mathcal{A}_{2\pi}$ onto $\mathcal{A}$ with inverse $\mathcal{R}^{-1} : \mathcal{A} \to \mathcal{A}_{2\pi}$ defined by $\mathcal{R}^{-1}\psi = \psi^{per}$. Consequently, the spaces $\mathcal{A}$ and $\mathcal{A}_{2\pi}$ are homeomorphic and this means that all the properties we require of $\mathcal{A}_{2\pi}$ are inherited from $\mathcal{A}$. Therefore, from the results presented in [11, pp.253-254], $\mathcal{A}_{2\pi}$ is not only a Fréchet space but also a test-function space.

Lemma 2.2 *The operators T^r and D^r are continuous linear operators from $\mathcal{A}_{2\pi}$ into $\mathcal{A}_{2\pi}$ for each $r \in \mathbb{N}$; i.e $T^r \in L(\mathcal{A}_{2\pi})$ and $D^r \in L(\mathcal{A}_{2\pi})$, where $L(\mathcal{A}_{2\pi})$ denotes the vector space of continuous liner operators from $\mathcal{A}_{2\pi}$ into $\mathcal{A}_{2\pi}$.*

Proof: The proof follows from the fact that the restricted operators T^r and D^r are continuous on $\mathcal{A}$, as the topologies defined on $\mathcal{A}$ and $\mathcal{A}_{2\pi}$ are defined via the same expressions. □

Lemma 2.3 *Let K be the convolution integral operator defined on $\mathcal{A}_{2\pi}$ by*

$$(K\phi)(x) = \frac{1}{2\pi}\int_0^{2\pi} k(x-u)\phi(u)\ du = \frac{1}{2\pi}\int_0^{2\pi} k(u)\phi(x-u)\ du. \tag{2.8}$$

If $k \in L^1_{2\pi}$, then $K \in L(\mathcal{A}_{2\pi})$.

Proof: It can be shown that

$$D^rK\phi = KD^r\phi, \forall\phi \in \mathcal{A}_{2\pi} \text{ and } r = 0, 1, 2, \ldots.$$

Hence, by [2, pp.8-10],

$$\rho_r(K\phi) = \|KD^r\phi\|_{L^2_{2\pi}} \le \|k\|_{L^1_{2\pi}} \|D^r\phi\|_{L^2_{2\pi}} = \|k\|_{L^1_{2\pi}} \rho_r(\phi).$$

□

The following results on the summability of functions in $\mathcal{A}_{2\pi}$ can be established by using the same arguments as for the theory presented in [11, pp.254-255].

Lemma 2.4

(i) The functions $\{(\varphi, \psi_l)_{L^2_{2\pi}} \psi_l\}_{l\in\mathbb{Z}}$ are summable to φ in $\mathcal{A}_{2\pi}$ for each $\varphi \in \mathcal{A}_{2\pi}$.

(ii) Let $\{a_l\}_{l\in\mathbb{Z}}$ be a family of scalars. Then the functions $\{a_l\psi_l\}_{l\in\mathbb{Z}}$ are summable to some function $\phi \in \mathcal{A}_{2\pi}$ if and only if $\sum_{l\in\mathbb{Z}} l^{2k}|a_l|^2 < \infty$ for each $k \in \mathbb{N}_0$.

□

We now turn to the space $\mathcal{A}'_{2\pi}$, the dual of $\mathcal{A}_{2\pi}$, and equip $\mathcal{A}'_{2\pi}$ with the weak* topology. Each $f \in \mathcal{A}'_{2\pi}$ assigns a unique complex number $< f, \phi >$ to each $\phi \in \mathcal{A}_{2\pi}$. In the following, it is convenient to use the notation

$$2\pi(f, \phi) :=< f, \bar{\phi} >, \qquad f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}.$$

Note that each $\varphi \in L^2_{2\pi}$ generates a regular generalised function $\widetilde{\varphi} \in \mathcal{A}'_{2\pi}$ defined by

$$< \widetilde{\varphi}, \overline{\phi} >= 2\pi(\widetilde{\varphi}, \phi) := 2\pi(\varphi, \phi)_{L^2_{2\pi}} = \int_0^{2\pi} \varphi(x)\overline{\phi(x)}\, dx. \qquad (2.9)$$

Also $L^2_{2\pi}$ is continuously imbedded in $\mathcal{A}'_{2\pi}$. Since $\psi_l \in \mathcal{A}_{2\pi}$, $\{\widetilde{\psi}_l\}_{l\in\mathbb{Z}}$ is a sequence of regular generalised functions in $\mathcal{A}'_{2\pi}$. Consequently we can discuss the convergence in $\mathcal{A}'_{2\pi}$ of infinite series of the form $\sum_{l=-\infty}^{\infty} b_l \widetilde{\psi}_l$.

Theorem 2.5

(i) Each $f \in \mathcal{A}'_{2\pi}$ can be expressed as $f = \sum_{l\in\mathbb{Z}}(f, \psi_l)\widetilde{\psi}_l = \lim_{N\to\infty} \sum_{l=-N}^{N}(f, \psi_l)\widetilde{\psi}_l$ where the series converges in $\mathcal{A}'_{2\pi}$.

(ii) Let $\{b_l\}_{l\in\mathbb{Z}} \subset \mathbb{C}$. Then $\sum_{n\in\mathbb{Z}} b_l\widetilde{\psi}_l = f$ in $\mathcal{A}'_{2\pi}$ if there exists $q \in \mathbb{N}_0$ such that $\sum_{l\neq 0} l^{-2q}|b_l|^2$ is convergent in $\mathbb{R}$.

Proof: The proof is analogous to that of [11, Theorem 9.6-1]. □

For $\widetilde{T}$ to be an extension of T defined on $\mathcal{A}'_{2\pi}$, we require $\widetilde{T}\widetilde{\varphi} = \widetilde{T\varphi}$, for all $\varphi \in D(T)$. Then

$$(\widetilde{T}\widetilde{\varphi}, \phi) = (\widetilde{T\varphi}, \phi) = (T\varphi, \phi)_{L^2_{2\pi}} = (\varphi, T\phi)_{L^2_{2\pi}} = (\widetilde{\varphi}, T\phi).$$

Therefore the operator $\widetilde{T}$ on $\mathcal{A}'_{2\pi}$ is defined by

$$(\widetilde{T}f, \phi) := (f, T\phi) \quad f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}. \qquad (2.10)$$

Similarly, the generalised differential operator $\widetilde{D}$ on $\mathcal{A}'_{2\pi}$ is defined via

$$(\widetilde{D}f, \phi) := (f, -D\phi) \quad f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}. \qquad (2.11)$$

Theorem 2.6 *For each $r \in \mathbb{N}$, $\widetilde{T}^r, \widetilde{D}^r \in L(\mathcal{A}'_{2\pi})$.*

Proof: For each $r \in \mathbb{N}$, $\widetilde{T^r} = \widetilde{T}^r$ and $\widetilde{D^r} = \widetilde{D}^r$. It follows that $\widetilde{T}^r$ and $\widetilde{D}^r$ are the adjoints of T^r and $(-D)^r$ respectively. As T^r and $(-D)^r$ are in $L(\mathcal{A}_{2\pi})$, the result follows from [11, Theorem 1.10-1]. □

The above discussion enables us to develop a theory of fractional calculus on the spaces of test-functions $\mathcal{A}_{2\pi}$ and of periodic distributions $\mathcal{A}'_{2\pi}$.

3 Fractional Calculus of Periodic Test Functions

In this section, we produce a general theory of fractional calculus operators defined on the space of test-functions $\mathcal{A}_{2\pi}$. By continuity of D^r on $\mathcal{A}_{2\pi}$ for each $r \in \mathbb{N}$, we have

$$D^r\phi = \sum_{l\in\mathbb{Z}}(\phi, \psi_l)_{L^2_{2\pi}} D^r\psi_l = \sum_{l\in\mathbb{Z}}(\phi, \psi_l)_{L^2_{2\pi}} (il)^r\psi_l, \quad \forall\, \phi \in \mathcal{A}_{2\pi}. \qquad (3.1)$$

Motivated by (3. 1), we define the fractional derivative D^α of order $\alpha > 0$ on $\mathcal{A}_{2\pi}$ by

$$D^\alpha\phi = \sum_{l\in\mathbb{Z}}(\phi, \psi_l)_{L^2_{2\pi}} (il)^\alpha\psi_l. \qquad (3.2)$$

This definition is identical to the Weyl formula (1. 18) for D^α on $L^2_{2\pi}$. In the following lemmas, we discuss the properties of D^α on the space $\mathcal{A}_{2\pi}$.

Theorem 3.1 *For each $\alpha > 0$, $D^\alpha \in L(\mathcal{A}_{2\pi})$.*

Proof: We can write $D^\alpha \phi = \sum_{l\in\mathbb{Z}} a_l \psi_l$, where $a_l = (\phi, \psi_l)_{L^2_{2\pi}} (il)^\alpha$. From Theorem 2.4, $\sum_{l\in\mathbb{Z}} a_l \psi_l$ converges in $\mathcal{A}_{2\pi}$ if and only if $\sum_{l\in\mathbb{Z}} |a_l|^2 |l|^{2k}$ converges for each $k \in \mathbb{N}_0$. Now, for each $\phi \in \mathcal{A}_{2\pi}$, $\sum_{l\in\mathbb{Z}} |(\phi, \psi_l)|^2 l^{2N}$ converges for any $N \in \mathbb{N}_0$. Suppose for any given $\alpha > 0$ and $k \in \mathbb{N}_0$, we choose N such that $(\alpha + k) < N$. Then

$$\sum_{l\in\mathbb{Z}} |(\phi, \psi_l)_{L^2_{2\pi}}|^2 \, l^{2(\alpha+k)} \leq \sum_{l\in\mathbb{Z}} |(\phi, \psi_l)_{L^2_{2\pi}}|^2 \, l^{2N} < \infty.$$

Linearity of D^α on $\mathcal{A}_{2\pi}$ is clear from the definition. To prove continuity of D^α, we proceed as follows. Given $\alpha > 0$, there exists $N \in \mathbb{N}_0$ such that $\alpha \in (N, N+1]$, and so

$$[\rho_k(D^\alpha \phi)]^2 = \| \sum_{l\in\mathbb{Z}} (\phi, \psi_l)_{L^2_{2\pi}} (il)^{\alpha+k} \psi_l \|^2_{L^2_{2\pi}} \leq \sum_{l\in\mathbb{Z}} |(\phi, \psi_l)_{L^2_{2\pi}}|^2 \, |l|^{2(N+1)+2k} = \|D^{k+N+1}\phi\|^2_{L^2_{2\pi}}.$$

Therefore

$$\rho_k(D^\alpha \phi) \leq \rho_{k+N+1}(\phi),$$

and the result follows. □

We now express I^α in the form of a Fourier series as

$$I^\alpha \phi := \sum_{l\neq 0} (il)^{-\alpha} (\phi, \psi_l)_{L^2_{2\pi}} \psi_l. \tag{3.3}$$

It is clear from Theorem 2.4 that for each $\phi \in \mathcal{A}_{2\pi}$, the series in (3. 3) converges in $\mathcal{A}_{2\pi}$. The operator I^α can also be expressed as the convolution integral operator (1. 15). From Lemma 2.3, we deduce that $I^\alpha \in L(\mathcal{A}_{2\pi})$. The following theorem shows that I^α and D^α satisfy the expected properties of fractional integral and differential operators on the space of test-functions $\mathcal{A}_{2\pi}$.

Theorem 3.2 *For all α, $\beta > 0$ and $\phi \in \mathcal{A}_{2\pi}$*

(i) $I^\alpha I^\beta \phi = I^{\alpha+\beta}\phi$, $D^\alpha D^\beta \phi = D^{\alpha+\beta}\phi$,

(ii) $D^\alpha I^\alpha \phi = I^\alpha D^\alpha \phi = \phi - \hat{\phi}(0)$.

Proof: The proof is similar to that of [3, Proposition 4.1]. □

The fact that D^α is defined as a continuous linear operator on the entire space $\mathcal{A}_{2\pi}$ is one advantage of working within this Fréchet space framework. Recall that in the $L^2_{2\pi}$ setting, D^α is not continuous and is not defined on all of $L^2_{2\pi}$. Unfortunately, to achieve these preferred properties of D^α, we have severely restricted the class of functions on which D^α acts. However, this drawback can be overcome by extending the theory to the generalised functions in $\mathcal{A}'_{2\pi}$. As we shall again use the adjoint method to define the distributional versions of the operators, we must first introduce the fractional differential operator $(-D)^\alpha$, given by

$$(-D)^\alpha \phi = \sum_{l\in\mathbb{Z}} (-il)^\alpha (\phi, \psi_l)_{L^2_{2\pi}} \psi_l, \quad \phi \in \mathcal{A}_{2\pi}. \tag{3.4}$$

An argument similar to that used in the proof of Theorem 3.1 shows that $(-D)^\alpha \in L(\mathcal{A}_{2\pi})$ for each $\alpha > 0$.

Similarly, we can define a fractional integral operator $(-I)^\alpha$ by

$$(-I)^\alpha \phi = \sum_{l\neq 0} (-il)^{-\alpha} (\phi, \psi_l)_{L^2_{2\pi}} \psi_l, \quad \phi \in \mathcal{A}_{2\pi}. \tag{3.5}$$

It is straightforward to show that $(-I)^\alpha$ and $(-D)^\alpha$ satisfy the usual properties (analogous to those listed in Theorem 3.2) of fractional integral and differential operators on the space of test-functions $\mathcal{A}_{2\pi}$.

One of the key results obtained by Butzer and Westphal in [3] is a connection between (1. 18) and (1. 20). The first step towards this is an explicit formula for the each Fourier coefficient of the fractional difference $\triangle_t^\alpha \phi$, namely

$$(\triangle_t^\alpha \phi)\hat{}(l) := \frac{1}{2\pi} \int_0^{2\pi} \phi(x) e^{-ilx} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} e^{-iltj} dx = (\frac{1-e^{-ilt}}{t})^\alpha \hat{\phi}(l), \;\; \phi \in L^p_{2\pi}, l \in \mathbb{Z}. \tag{3.6}$$

Consequently,

$$\lim_{t\to 0^+} (\frac{\triangle_t^\alpha \phi}{t^\alpha})\hat{}(l) = (il)^\alpha \hat{\phi}(l). \tag{3.7}$$

To establish the results stated in Theorem 1.1 for the strong Liouville-Grünwald fractional derivative defined on $L^p_{2\pi}(1 \le p < \infty)$ by (1. 20), Butzer and Westphal made use of the function

$$p_\alpha(x) := \begin{cases} \frac{1}{\Gamma(\alpha)} \sum\limits_{0\le j<x} (-1)^j \binom{\alpha}{j} (x-j)^{\alpha-1}, & (0<x<\infty) \\ 0 & (-\infty < x < 0) \end{cases} \tag{3.8}$$

where $\alpha > 0$. This function has the following properties for each $\alpha > 0$ (see [3, Proposition 3.1]):

$$p_\alpha \in L^1(-\infty, \infty), \quad \int_{-\infty}^{\infty} p_\alpha(u)\, du = 1, \tag{3.9}$$

and

$$(\mathcal{F}p_\alpha)(v) = \begin{cases} (iv)^{-\alpha}(1-e^{-iv})^\alpha, & v \neq 0 \\ 1 & v = 0, \end{cases} \tag{3.10}$$

where $\mathcal{F}p_\alpha$ denotes the Fourier transform of p_α. In addition, p_α can be used to construct a related function χ_α defined by

$$\chi_\alpha(x;t) \;\; = \;\; 2\pi \sum_{\frac{-x}{2\pi}<j<\infty} \frac{p_\alpha(\frac{x+2\pi j}{t})}{t} \qquad (t>0).$$

Then, from [2] and [3, Lemma 3.1], for each $\alpha > 0$

$$\chi_\alpha(.;t) \in L^1_{2\pi}, \quad \int_0^{2\pi} \chi_\alpha(u,t)\, du = 2\pi, \tag{3.11}$$

$$\widehat{\chi_\alpha(.;t)}(k) = \begin{cases} (ikt)^{-\alpha}(1-e^{-ikt})^\alpha, & k\neq 0 \\ 1 & k=0, \end{cases}$$

and

$$\chi_\alpha(x;t) = t^{-\alpha}(\triangle_t^\alpha \eta_\alpha)(x) + 1 = t^{-\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} \eta_\alpha(x-tj) + 1, \tag{3.12}$$

where η_α is defined by (1. 14). Moreover, on setting

$$(J_{t,\alpha}\phi)(x) := \frac{1}{2\pi}\int_0^{2\pi} \chi_\alpha(x-u;t)\phi(u)\, du,$$

we have

$$\lim_{t\to 0^+} \|J_{t,\alpha}\phi - \phi\|_{L^p_{2\pi}} = 0, \quad \forall \phi \in L^p_{2\pi}. \tag{3. 13}$$

We now take a modified version of the Liouville-Grünwald fractional derivative $D^{<\alpha>}$ in $\mathcal{A}_{2\pi}$, where the limit in (1. 20) is with respect to the topology on $\mathcal{A}_{2\pi}$. Note that if $\triangle_t^\alpha\phi/t^\alpha \to \phi$ in $\mathcal{A}_{2\pi}$ as $t \to 0^+$ then $\triangle_t^\alpha\phi/t^\alpha \to \phi$ in $L^2_{2\pi}$ as $t \to 0^+$. Therefore, we can write

$$\lim_{t\to 0^+} \frac{\triangle_t^\alpha \phi}{t^\alpha} = D^{<\alpha>}\phi, \qquad \alpha > 0,\, \phi \in \mathcal{A}_{2\pi}$$

whenever the limit on the left-hand side exists in $\mathcal{A}_{2\pi}$.
In the subsequent discussion, we shall require the following extension of equation (3. 13).

Lemma 3.3 *If $\phi \in \mathcal{A}_{2\pi}$, then $\lim_{t\to 0^+} J_{t,\alpha}\phi = \phi$ in $\mathcal{A}_{2\pi}$.*

Proof: Let $\phi \in \mathcal{A}_{2\pi}$. Then, from the definition of ρ_k, we obtain

$$\rho_k(J_{t,\alpha}\phi - \phi) = \|D^k(J_{t,\alpha}\phi) - D^k\phi\|_{L^2_{2\pi}} = \|J_{t,\alpha}(D^k\phi) - D^k\phi\|_{L^2_{2\pi}}.$$

Since $D^k\phi \in L^2_{2\pi}$ for all $k \in \mathbb{N}_0$, it follows from (3. 13) that

$$\rho_k(J_{t,\alpha}\phi - \phi) \to 0 \quad \text{as } t \to 0^+.$$

□

The following theorem is analogous to Theorem 1.1.

Theorem 3.4 *Let $\phi \in \mathcal{A}_{2\pi}$, $\alpha > 0$. Then the following statements are equivalent.*

(i) $D^{<\alpha>}\phi \in \mathcal{A}_{2\pi}$.

(ii) There exists $\psi \in \mathcal{A}_{2\pi}$ such that $\hat{\psi}(l) = (il)^\alpha\hat{\phi}(l)$, $l \in \mathbb{Z}$.

(iii) There exists $\psi \in \mathcal{A}_{2\pi}$ such that $(I^\alpha\psi)(x) = \phi(x) - \hat{\phi}(0)$.

If (i) is satisfied, then $D^{<\alpha>}\phi = \psi$, where $\psi \in \mathcal{A}_{2\pi}$ is the function defined in (ii). Similarly if (ii) is satisfied then $D^{<\alpha>}\phi = \psi$.

Proof: Let $\phi \in \mathcal{A}_{2\pi}$ be such that $D^{<\alpha>}\phi = \psi$ in $\mathcal{A}_{2\pi}$. Then $D^{<\alpha>}\phi = \psi$ in $L^2_{2\pi}$ and so, as in Theorem 1.1,

$$\hat{\psi}(l) = (D^{<\alpha>}\phi)\hat{}(l) = (il)^\alpha\hat{\phi}(l),\, l \in \mathbb{Z}.$$

Now we suppose that (ii) holds. Then $\psi \in \mathcal{A}_{2\pi}$ is such that

$$\hat{\psi}(l) = (il)^\alpha\hat{\phi}(l), \quad l \in \mathbb{Z}.$$

Since $\psi, \phi \in L^2_{2\pi}$, it follows from Theorem 1.1 that $(I^\alpha\psi)(x) = \phi(x) - \hat{\phi}(0)$ almost everywhere.
Let (iii) be true so that

$$(I^\alpha\psi)(x) = \phi(x) - \hat{\phi}(0) \quad \text{a.e.}$$

Since $(I^\alpha\psi)\hat{}(0) = 0$, we can assume that $\hat{\psi}(0) = 0$. Then, as in the proof of Theorem 1.1, given in [3, Theorem 4.1],

$$\frac{\triangle_t^\alpha \phi}{t^\alpha} = J_{t,\alpha}\psi.$$

Hence from Lemma 3.3, it follows that

$$\lim_{t\to 0^+} \frac{\triangle_t^\alpha \phi}{t^\alpha} = \lim_{t\to 0^+} J_{t,\alpha}\psi = \psi$$

in $\mathcal{A}_{2\pi}$. □

We now obtain the following result.

Theorem 3.5 *Let $\phi \in \mathcal{A}_{2\pi}$. Then $D^{<\alpha>}\phi \in \mathcal{A}_{2\pi}$ for all $\alpha > 0$. Also $D^\alpha\phi = D^{<\alpha>}\phi$ and so $D^{<\alpha>} \in L(\mathcal{A}_{2\pi})$.*

Proof: Let $\phi \in \mathcal{A}_{2\pi}$. Then

$$\sum_{l\in\mathbb{Z}} |\hat{\phi}(l)|^2\, l^{2k} < \infty,\ \ k \in \mathbb{N}_0.$$

Therefore, for each $\alpha > 0$ and $k \in \mathbb{N}_0$,

$$\sum_{l\in\mathbb{Z}} |\hat{\phi}(l)|^2\, l^{2\alpha+2k} < \infty.$$

Hence, by Theorem 2.4, $\sum_{l\in\mathbb{Z}}(il)^\alpha\hat{\phi}(l)\psi_l$ exists in $\mathcal{A}_{2\pi}$. Thus, it follows from Theorem 3.4 that $D^{<\alpha>}\phi$ exists in $\mathcal{A}_{2\pi}$ and

$$D^{<\alpha>}\phi = \sum_{l\in\mathbb{Z}}(il)^\alpha\hat{\phi}(l)\psi_l = D^\alpha\phi \qquad \text{(from} \quad (3.2)).$$

□

For the distributional treatment of the Liouville-Grünwald fractional derivative, we also require properties of an operator that is closely related to $D^{<\alpha>}$, namely $(-D)^{<\alpha>}$. The latter is defined by

$$(-D)^{<\alpha>}\phi := \lim_{t\to 0^+} \frac{\triangle_{-t}^\alpha \phi}{t^\alpha}, \quad \phi \in \mathcal{A}_{2\pi}, \tag{3.14}$$

where the limit is with respect to the topology on $\mathcal{A}_{2\pi}$ and

$$(\triangle_{-t}^\alpha\phi)(x) = \sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}\phi(x+tj). \tag{3.15}$$

Then, following the analysis of $D^{<\alpha>}$ given in [3], we obtain

$$(\triangle_{-t}^\alpha\phi)\hat{}(l) = (1-e^{ilt})^\alpha\hat{\phi}(l), \quad \phi \in \mathcal{A}_{2\pi}, l \in \mathbb{Z},$$

which gives

$$\lim_{t\to 0^+}\left(\frac{\triangle_{-t}^\alpha \phi}{t^\alpha}\right)\hat{}(l) = \lim_{t\to 0^+}\left(\frac{1-e^{ilt}}{t}\right)^\alpha\hat{\phi}(l) = (-il)^\alpha\hat{\phi}(l). \tag{3.16}$$

Let p_α^* be the function defined by

$$p_\alpha^*(u) := p_\alpha(-u),$$

where p_α is given by (3. 8), and define Λ_α by

$$\Lambda_\alpha(x;t) := 2\pi \sum_{j=-\infty}^{\infty} \frac{p_\alpha^*(\frac{x+2\pi j}{t})}{t}, \quad t > 0.$$

Then

$$\widehat{\Lambda_\alpha(.;t)}(l) = (\mathcal{F}p_\alpha^*)(lt) = \begin{cases} (-ilt)^{-\alpha}(1-e^{ilt})^\alpha, & l \neq 0 \\ 1 & l = 0, \end{cases}$$

and therefore

$$\Lambda_\alpha(x;t) = t^{-\alpha}(\triangle_{-t}^\alpha \eta_\alpha^*)(x) + 1, \tag{3. 17}$$

where

$$\eta_\alpha^*(x) = \overline{\eta_\alpha(-x)} = \sum_{l\neq 0}(-il)^{-\alpha}e^{ilx}.$$

We can now present the following result.

Theorem 3.6 *The following statements are equivalent for $\phi \in \mathcal{A}_{2\pi}, \alpha > 0$.*

(i) $(-D)^{<\alpha>}\phi \in \mathcal{A}_{2\pi}$.

(ii) There exists $\psi \in \mathcal{A}_{2\pi}$ such that $\hat{\psi}(l) = (-il)^\alpha \hat{\phi}(l)$, $\forall l \in \mathbb{Z}$.

(iii) There exists $\psi \in \mathcal{A}_{2\pi}$ such that $((-I)^\alpha \psi)(x) = \phi(x) - \hat{\phi}(0)$ almost everywhere, where $(-I)^\alpha$ is defined by (3. 5).

If (i) is satisfied, then $(-D)^{<\alpha>}\phi = \psi$, where $\psi \in \mathcal{A}_{2\pi}$ is the function defined in (ii). Similarly if (ii) is satisfied then $(-D)^{<\alpha>}\phi = \psi$.

Proof: The proof is similar to that of Theorem 3.4. □

Lemma 3.7 *Let $\phi \in \mathcal{A}_{2\pi}$. Then, $(-D)^{<\alpha>}\phi \in \mathcal{A}_{2\pi}$ for all $\alpha > 0$. Also $(-D)^\alpha\phi = (-D)^{<\alpha>}\phi$ and so $(-D)^{<\alpha>} \in L(\mathcal{A}_{2\pi})$.*

Proof: The proof is a consequence of Theorem 3.6 and similar to that of Theorem 3.5 and hence is omitted. □

4 Fractional Calculus of Periodic Distributions

We now define an operator $\widetilde{D^\alpha}$ on $\mathcal{A}'_{2\pi}$ which can be interpreted as an extension of $D^\alpha \in L(\mathcal{A}_{2\pi})$ to the space of periodic distributions $\mathcal{A}'_{2\pi}$. For this purpose, we apply (2. 11) to obtain

$$(\widetilde{D^r} f, \phi) = (f, (-D)^r\phi) \quad \forall r \in \mathbb{N}, f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}.$$

This motivates us to define $\widetilde{D^\alpha}$, for $\alpha > 0$, as

$$(\widetilde{D^\alpha} f, \phi) := (f, (-D)^\alpha\phi), \tag{4. 1}$$

where $(-D)^\alpha$ is defined by (3. 4). Therefore, for $\phi \in \mathcal{A}_{2\pi}$, we have

$$\begin{aligned}(\widetilde{D^\alpha} f, \phi) &:= (f, (-D)^\alpha \phi) = (f, \sum_{l\in\mathbb{Z}} (-il)^\alpha (\phi, \psi_l)_{L^2_{2\pi}} \psi_l) \\ &= \sum_{l\in\mathbb{Z}} (il)^\alpha (\widetilde{\psi_l}, \phi)(f, \psi_l) = \left(\sum_{l\in\mathbb{Z}} (il)^\alpha (f, \psi_l) \widetilde{\psi_l}, \phi \right).\end{aligned}$$

Hence the generalised differential operator $\widetilde{D^\alpha}$ may also be represented by

$$\widetilde{D^\alpha} f := \sum_{l\in\mathbb{Z}} (il)^\alpha (f, \psi_l) \widetilde{\psi_l}, \qquad \text{for all } f \in \mathcal{A}'_{2\pi}, \tag{4. 2}$$

where the infinite series converges in $\mathcal{A}'_{2\pi}$.

Lemma 4.1 *The operator $\widetilde{D^\alpha}$ is a continuous linear mapping from $\mathcal{A}'_{2\pi}$ into $\mathcal{A}'_{2\pi}$.*

Proof: It follows from (4. 1) that $\widetilde{D^\alpha}$ is the adjoint operator of $(-D)^\alpha$. Since $(-D)^\alpha$ is a continuous linear operator from $\mathcal{A}_{2\pi}$ into $\mathcal{A}_{2\pi}$, $\widetilde{D^\alpha}$ is a continuous linear mapping from $\mathcal{A}'_{2\pi}$ to $\mathcal{A}'_{2\pi}$.

□

A generalised fractional integral operator $\widetilde{I^\alpha}$, an extension of I^α, can also be defined via

$$(\widetilde{I^\alpha} f, \phi) := (f, (I^\alpha)^* \phi) \tag{4. 3}$$

where $(I^\alpha)^*$ is given by

$$((I^\alpha)^* \phi)(x) = (\eta^*_\alpha * \phi)(x) = \frac{1}{2\pi} \int_0^{2\pi} \eta^*_\alpha (x-u)\, \phi(u)\, du, \tag{4. 4}$$

with

$$\eta^*_\alpha (x) := \sum_{l\neq 0} (-il)^{-\alpha} e^{ilx}.$$

Hence $(I^\alpha)^* = (-I)^\alpha$. Therefore, we obtain

$$(\widetilde{I^\alpha} f, \phi) \;:=\; (f, (-I)^\alpha \phi) = (f, \sum_{l\neq 0} (-il)^{-\alpha} (\phi, \psi_l)_{L^2_{2\pi}} \psi_l) = \left(\sum_{l\neq 0} (il)^{-\alpha} (f, \psi_l) \widetilde{\psi_l}, \phi \right).$$

It follows that $\widetilde{I^\alpha}$ may also be defined by

$$\widetilde{I^\alpha} f := \sum_{l\neq 0} (il)^{-\alpha} (f, \psi_l) \widetilde{\psi_l} \qquad \text{for all } f \in \mathcal{A}'_{2\pi}. \tag{4. 5}$$

Lemma 4.2 *The operator $\widetilde{I^\alpha}$ is a continuous linear mapping from $\mathcal{A}'_{2\pi}$ into $\mathcal{A}'_{2\pi}$.*

Proof: The integral operator $\widetilde{I^\alpha}$, being the adjoint of $(-I)^\alpha \in L(\mathcal{A}_{2\pi})$, is a continuous linear operator on $\mathcal{A}'_{2\pi}$. □

Properties of these fractional differential and integral operators on $\mathcal{A}'_{2\pi}$ are summarised in the following theorem.

Theorem 4.3

(i) $\widetilde{D^\alpha}\widetilde{D^\beta} f = \widetilde{D^{\alpha+\beta}} f$, $\widetilde{I^\alpha}\widetilde{I^\beta} f = \widetilde{I^{\alpha+\beta}} f$ *for all* $\alpha\,,\,\beta > 0$ *and* $f \in \mathcal{A}'_{2\pi}$.

(ii) $\widetilde{D^\alpha}\widetilde{I^\alpha} f = \widetilde{I^\alpha}\widetilde{D^\alpha} f = f - \hat{f}(0)$ *for all* $f \in \mathcal{A}'_{2\pi}$.

Note that $\hat{f}(l) = (f, \psi_l)$, $l \in \mathbb{Z}$.

Proof: The proof for (i) follows directly from (4. 1) and (4. 3).

(ii) Let $f \in \mathcal{A}'_{2\pi}$. Then f can be expressed uniquely as $f = \sum_{l\in\mathbb{Z}}(f,\psi_l)\widetilde{\psi_l}$, where the series converges in $\mathcal{A}'_{2\pi}$. Similarly, since for all $\alpha > 0$, $g = \widetilde{D^\alpha}\widetilde{I^\alpha} f$ is also in $\mathcal{A}'_{2\pi}$, we can write

$$g = \sum_{l\in\mathbb{Z}} (\widetilde{D^\alpha}\widetilde{I^\alpha} f, \psi_l)\widetilde{\psi_l}.$$

Now, for each $n \in \mathbb{Z}$,

$$(\widetilde{D^\alpha}\widetilde{I^\alpha} f, \psi_l) = (f, (-I)^\alpha(-D)^\alpha\psi_l) = \begin{cases} (f,\psi_l) & l \neq 0 \\ 0 & l = 0. \end{cases}$$

Hence

$$\widetilde{D^\alpha}\widetilde{I^\alpha} f = \sum_{l\in\mathbb{Z}} (f,\psi_l)\widetilde{\psi_l} - (f,\psi_0)\widetilde{\psi_0} = f - \hat{f}(0).$$

Similarly, it can be shown that

$$\widetilde{I^\alpha}\widetilde{D^\alpha} f = f - \hat{f}(0).$$

$\square$

Now we extend $D^{<\alpha>}$ to a generalised operator $\widetilde{D^{<\alpha>}}$ on $\mathcal{A}'_{2\pi}$ by defining

$$(\widetilde{D^{<\alpha>}} f, \phi) := (f, (-D)^{<\alpha>}\phi) \quad \forall f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}, \tag{4. 6}$$

where the operator $(-D)^{<\alpha>}$ is given by (3. 14). Therefore, for $f \in \mathcal{A}'_{2\pi}$ and $\alpha > 0$, we obtain

$$\begin{aligned} (\widetilde{D^{<\alpha>}} f, \phi) &:= (f, (-D)^{<\alpha>}\phi) = (f, \lim_{t\to 0^+} \frac{\triangle^\alpha_{-t}\phi}{t^\alpha}) \\ &= (f, \sum_{l\in\mathbb{Z}} (-il)^\alpha (\phi,\psi_l)_2\psi_l) \quad \text{(from (3. 16))} \\ &= \sum_{l\in\mathbb{Z}} (il)^\alpha(\psi_l,\phi)_2(f,\psi_l) = \sum_{l\in\mathbb{Z}} (il)^\alpha(\tilde{\psi}_l,\phi)(f,\psi_l) \\ &= (\sum_{l\in\mathbb{Z}} (il)^\alpha (f,\psi_l)\tilde{\psi}_l, \phi) = (\widetilde{D^\alpha} f, \phi) \qquad \forall \phi \in \mathcal{A}_{2\pi}. \end{aligned}$$

Note that an equivalent definition of $\widetilde{D^{<\alpha>}} f$ is given by the weak* limit

$$\widetilde{D^{<\alpha>}} f := \lim_{t\to 0^+} \frac{\widetilde{\triangle^\alpha_t} f}{t^\alpha}, \ f \in \mathcal{A}'_{2\pi},\ \alpha > 0, \tag{4. 7}$$

where

$$\widetilde{\triangle_t^\alpha} f := \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - tj)$$

and $f(x - tj)$ represents a translated version of the distribution f. This is a direct consequence of the fact that

$$(\widetilde{\triangle_t^\alpha} f, \phi) = (f, \triangle_{-t}^\alpha \phi), \ \forall f \in \mathcal{A}'_{2\pi}, \ \phi \in \mathcal{A}_{2\pi}.$$

Since the generalised operators $\widetilde{D^{<\alpha>}}$ and $\widetilde{D^\alpha}$ coincide on $\mathcal{A}'_{2\pi}$, the following results can be deduced immediately from Theorem 4.3.

Theorem 4.4

(i) $\widetilde{D^{<\alpha>}}\widetilde{D^{<\beta>}} f = \widetilde{D^{<\alpha+\beta>}} f$, *for all* α, $\beta > 0$ *and* $f \in \mathcal{A}'_{2\pi}$.

(ii) $\widetilde{D^{<\alpha>}}\widetilde{I^{<\alpha>}} f = \widetilde{I^{<\alpha>}}\widetilde{D^{<\alpha>}} f = f - \hat{f}(0)$ *for all* $f \in \mathcal{A}'_{2\pi}$.

□

5 Fractional Diffusion-Type Equations

As an application of the distributional theory of fractional differential operators developed above, we investigate a distributional version of a fractional diffusion-type equation. The classical formulation of this equation is

$$\frac{\partial u(x,t)}{\partial t} = (\frac{\partial}{\partial x})^{(\alpha)} u(x,t), \quad 0 \le x \le 2\pi, \ t > 0, \ 1 < \alpha < 3, \tag{5. 1}$$

with u also required to satisfy the periodic boundary conditions

$$(\frac{\partial}{\partial x})^{(\alpha-2)} u(0,t) = (\frac{\partial}{\partial x})^{(\alpha-2)} u(2\pi,t), \ (\frac{\partial}{\partial x})^{(\alpha-1)} u(0,t) = (\frac{\partial}{\partial x})^{(\alpha-1)} u(2\pi,t), \ t > 0, \tag{5. 2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad 0 \le x \le 2\pi. \tag{5. 3}$$

In [3, Section 9], (5. 1) - (5. 3) is interpreted as the abstract Cauchy problem (ACP)

$$\frac{du(t)}{dt} = D^\alpha u(t), \ 1 < \alpha < 3, \ t > 0; \quad u(0) = u_0, \tag{5. 4}$$

posed in the space $L^p_{2\pi}$, where $D^\alpha = D^{<\alpha>}$ is defined via (1. 20). The existence and uniqueness of a solution $u(x,t)$ to the ACP (5. 4) is proved in [3], with u obtained explicitly in the form

$$u(x,t) = \sum_{l \in \mathbb{Z}} e^{(il)^\alpha t} \widehat{u_0}(l) e^{ilx} = \frac{1}{2\pi} \int_0^{2\pi} q_\alpha(y,t) u_0(x-y)\, dy, \tag{5. 5}$$

where

$$q_\alpha(x,t) := \sum_{l \in \mathbb{Z}} e^{(il)^\alpha t} e^{ilx}.$$

The function u defined by (5. 5) is actually a continuous 2π-periodic function of x for each fixed $t > 0$ and any $u_0 \in L^p_{2\pi}$, and is a solution of (5. 4), not only for $1 < \alpha < 3$, but also for $4j + 1 < \alpha < 4j + 3, \ j = 0, 1, 2, \ldots$.

Formula (5. 5) can also be used to define a family of operators $\{G_\alpha(t)\}_{t\geq 0}$ on $L^p_{2\pi}$ for each $p \in [1,\infty)$ and $4j+1 < \alpha < 4j+3,\ j = 0,1,2,\ldots$. As before, we shall concentrate on the case when $p = 2$ and, for simplicity, will take $1 < \alpha < 3$. For each such α and $t \geq 0$, we define

$$G_\alpha(t)\phi := \sum_{l\in\mathbb{Z}} e^{(il)^\alpha t}(\phi,\psi_l)_{L^2_{2\pi}}\psi_l, \quad \phi \in L^2_{2\pi}. \tag{5. 6}$$

Theorem 5.1 *Let $G_\alpha(t)$ be defined by (5. 6). Then, for each $\alpha \in (1,3)$, $\{G_\alpha(t)\}_{t\geq 0}$ is a strongly continuous semigroup of operators on $L^2_{2\pi}$. Moreover, the infinitesimal generator of $\{G_\alpha(t)\}_{t\geq 0}$ is D^α defined on the domain given by (1. 19).*

Proof: For each $\phi \in L^2_{2\pi}$, we have

$$\sum_{l\in\mathbb{Z}} |e^{(il)^\alpha t}(\phi,\psi_l)_{L^2_{2\pi}}|^2 = \sum_{l\in\mathbb{Z}} \left(e^{|l|^\alpha t\cos(\alpha\pi/2)}\right)^2 \left|(\phi,\psi_l)_{L^2_{2\pi}}\right|^2, \quad t \geq 0.$$

Consequently

$$\|G_\alpha(t)\phi\|_{L^2_{2\pi}} \leq \|\phi\|_{L^2_{2\pi}}, \quad 1 < \alpha < 3,$$

showing that $G_\alpha(t)$ is a bounded linear operator on $L^2_{2\pi}$ for each $t \geq 0$.

It follows from (5. 6), that the operators $\{G_\alpha(t)\}_{t\geq 0}$ satisfy the algebraic properties of a semigroup, that is $G_\alpha(0) = I$, and $G_\alpha(s)G_\alpha(t) = G_\alpha(s+t) = G_\alpha(t)G_\alpha(s)$, for all $s,t \geq 0$, $1 < \alpha < 3$. Also, for each $\phi \in L^2_{2\pi}$,

$$\|G_\alpha(t)\phi - \phi\|^2_{L^2_{2\pi}} = \sum_{l\in\mathbb{Z}} |(e^{(il)^\alpha t} - 1)|^2 |(\phi,\psi_l)_{L^2_{2\pi}}|^2.$$

Clearly $|(e^{(il)^\alpha t} - 1)|^2|(\phi,\psi_l)_{L^2_{2\pi}}|^2 \to 0$ as $t \to 0^+$ for all l. Moreover, the restriction that $\alpha \in (1,3)$ means that

$$\begin{aligned}(Ae^{i|l|^\alpha t\sin(\alpha\pi/2)} - 1)(Ae^{-i|l|^\alpha t\sin(\alpha\pi/2)} - 1) &= A^2 + 1 - 2A\cos\left(|l|^\alpha t\sin(\alpha\pi/2)\right)\\ &\leq 4,\end{aligned}$$

where $A = e^{|l|^\alpha t\cos(\alpha\pi/2)}$, and so we have

$$|e^{(il)^\alpha} - 1|^2|(\phi,\psi_l)_{L^2_{2\pi}}|^2 \leq 4|(\phi,\psi_l)_{L^2_{2\pi}}|^2.$$

Therefore it follows from Weierstrass M-test [1, p.438] that

$$\|G_\alpha(t)\phi - \phi\|_{L^2_{2\pi}} \to 0 \text{ as } t \to 0^+.$$

Hence $\{G_\alpha(t)\}_{t\geq 0}$ is a strongly continuous semigroup of operators on $L^2_{2\pi}$. Likewise, it can be verified using the same arguments as in [6, Theorem 2.2], that D^α is the infinitesimal generator of $\{G_\alpha(t)\}_{t\geq 0}$ since, for all $\phi \in Dom(D^\alpha)$, where the latter is given by (1. 19), we have

$$\lim_{t\to 0^+} \left\| \frac{G_\alpha(t)\phi - \phi}{t} - D^\alpha\phi \right\|_{L^2_{2\pi}} = 0.$$

□

It follows that, when $p = 2$, the strong solution of (5. 4) is given by $u(t) = G_\alpha(t)u_0$ provided that $u_0 \in Dom(D^\alpha)$.

To enable us to cater for a larger class of initial data, including cases when u_0 is a non-classical function, we now examine a generalised version of the fractional diffusion ACP (5. 4) posed in the space $\mathcal{A}'_{2\pi}$. Our aim is to establish that a unique solution $u : [0,\infty) \to \mathcal{A}'_{2\pi}$ can always be found in the form $u(t) = \widetilde{G_\alpha(t)} u_0$ for any $u_0 \in \mathcal{A}'_{2\pi}$, where $\widetilde{G_\alpha(t)}$ is an appropriately defined extension of the operator $G_\alpha(t)$ to $\mathcal{A}'_{2\pi}$. As a first step towards this, we consider the operators $\{G_\alpha(t)\}_{t\geq 0}$ defined via (5.6), but now restricted to the space of test-functions $\mathcal{A}_{2\pi}$.

Theorem 5.2 *The family of operators $\{G_\alpha(t)\}_{t\geq 0}$ is an equicontinuous semigroup on $\mathcal{A}_{2\pi}$ for each fixed $\alpha \in (1,3)$.*

Proof: We first establish that $G_\alpha(t) \in L(\mathcal{A}_{2\pi})$. For each $\phi \in \mathcal{A}_{2\pi}$,

$$\begin{aligned}\sum_l |l|^{2k} |e^{(il)^\alpha} (\phi, \psi_l)_{L^2_{2\pi}}|^2 &= \sum_l |l|^{2k} \left(e^{|l|^\alpha t \cos(\alpha\pi/2)}\right)^2 \left|(\phi,\psi_l)_{L^2_{2\pi}}\right|^2 \\ &\leq \sum_l |l|^{2k} |(\phi,\psi_l)_{L^2_{2\pi}}|^2 < \infty \quad \forall\, k = 0,1,2,\ldots \text{ (since } 1<\alpha<3),\end{aligned}$$

and therefore, from Lemma 2.4, $G_\alpha(t)\phi \in \mathcal{A}_{2\pi}$. Also, for each $\phi \in \mathcal{A}_{2\pi}$ and $k = 0,1,\ldots,$

$$\begin{aligned}\rho_k(G_\alpha(t)\phi) &= \|D^k G_\alpha(t)\phi\|_{L^2_{2\pi}} = \|G_\alpha(t) D^k \phi\|_{L^2_{2\pi}} \quad \text{(from the linearity and continuity of } D^k \text{ on } \mathcal{A}_{2\pi}) \\ &\leq \|D^k\phi\|_{L^2_{2\pi}} \quad \text{(from Theorem 5.1, since } D^k\phi \in L^2_{2\pi}\text{)} \\ &= \rho_k(\phi).\end{aligned}$$

This shows that $G_\alpha(t) \in L(\mathcal{A}_{2\pi})$. Next, it is clear from Theorem 5.1 that $\{G_\alpha(t)\}_{t\geq 0}$ satisfies the algebraic properties of a semigroup because $\mathcal{A}_{2\pi} \subseteq L^2_{2\pi}$. Also, by arguing as above, we obtain

$$\rho_k(G_\alpha(t)\phi - \phi) = \|G_\alpha(t)(D^k\phi) - D^k\phi\|_{L^2_{2\pi}} \ \to 0 \ \text{ as } \ t \to 0^+ \ \text{ (from Theorem 5.1).}$$

Finally, equicontinuity follows directly from the fact that

$$\rho_k(G_\alpha(t)\phi) \leq \rho_k(\phi), \qquad \text{for all } \phi \in \mathcal{A}_{2\pi},\ t \geq 0 \text{ and } k = 0,1,\ldots.$$

□

The infinitesimal generator of the semigroup $\{G_\alpha(t)\}_{t\geq 0}$ on $\mathcal{A}_{2\pi}$ is the operator $D^\alpha \in L(\mathcal{A}_{2\pi})$ defined by (3.2) (or, equivalently, from Theorem 3.5, $D^{<\alpha>} \in L(\mathcal{A}_{2\pi})$). To establish this, let $\phi \in \mathcal{A}_{2\pi}$ so that $\sum_l |\lambda_l|^{2k} |(\phi,\psi_l)_{L^2_{2\pi}}|^2 < \infty$ for each $k = 0,1,2,\ldots$. Then

$$\begin{aligned}[\rho_k(\frac{G_\alpha(t)\phi - \phi}{t} - D^\alpha\phi)]^2 &= \left\|D^k\left(\frac{G_\alpha(t)\phi-\phi}{t}\right) - D^k D^\alpha \phi\right\|^2_{L^2_{2\pi}} \\ &= \left\|\frac{G_\alpha(t)(D^k\phi) - D^k\phi}{t} - D^\alpha(D^k\phi)\right\|^2_{L^2_{2\pi}} \to 0 \quad \text{as} \quad t \to 0^+.\end{aligned}$$

Since this holds for each $k = 0,1,2,\ldots,$, it follows that

$$D^\alpha\phi = \lim_{t\to 0^+} \frac{G_\alpha(t)\phi - \phi}{t}.$$

Our next task is to extend the operators $\{G_\alpha(t)\}_{t\geq 0}$ to a family of generalised operators $\{\widetilde{G_\alpha(t)}\}_{t\geq 0}$ defined on $\mathcal{A}'_{2\pi}$. This will be achieved through the adjoint-based approach that was used in [6]. Let $\varphi \in L^2_{2\pi}$ and $\phi \in \mathcal{A}_{2\pi}$. Then both φ and $G_\alpha(t)\varphi$ generate regular generalised functions $\widetilde{\varphi}$ and $\widetilde{G_\alpha(t)\varphi}$

in $\mathcal{A}'_{2\pi}$ via formula (2. 9). For $\widetilde{G_\alpha(t)}$ to be an extension of $G_\alpha(t)$, we require $\widetilde{G_\alpha(t)}\widetilde{\varphi} = \widetilde{G_\alpha(t)\varphi}$. This will be true provided that

$$< \widetilde{G_\alpha(t)}\widetilde{\varphi}, \overline{\phi} >= 2\pi(\widetilde{G_\alpha(t)}\widetilde{\varphi}, \phi) = 2\pi(G_\alpha(t)\varphi, \phi)_{L^2_{2\pi}} = 2\pi(\widetilde{\varphi}, G^*_\alpha(t)\phi) =< \widetilde{\varphi}, \overline{G^*_\alpha(t)\phi} >,$$

where $G^*_\alpha(t)$, the Hilbert space adjoint of $G_\alpha(t)$, is given by

$$G^*_\alpha(t)\phi = \sum_{l\in\mathbb{Z}} e^{(-il)^\alpha t}(\phi, \psi_l)_{L^2_{2\pi}}\psi_l. \tag{5. 7}$$

Motivated by this, we define the generalised operator $\widetilde{G_\alpha(t)}$ by

$$(\widetilde{G_\alpha(t)}f, \phi) := (f, G^*_\alpha(t)\phi) \quad f \in \mathcal{A}'_{2\pi}, \phi \in \mathcal{A}_{2\pi}. \tag{5. 8}$$

Then $\widetilde{G_\alpha(t)} = (G^*_\alpha(t))'$, the $\mathcal{A}_{2\pi} - \mathcal{A}'_{2\pi}$ adjoint of $G^*_\alpha(t)$. Now suppose that $f \in \mathcal{A}'_{2\pi}$. Then

$$(\widetilde{G_\alpha(t)}f, \phi) \;=\; (f, G^*_\alpha(t)\phi) = (f, \sum_l e^{(-il)^\alpha t}(\phi, \psi_l)_{L^2_{2\pi}}\psi_l) = \sum_l e^{(il)^\alpha t}(f, \psi_l)(\widetilde{\psi_l}, \phi).$$

Since the series on the right-hand side converges for each $\phi \in \mathcal{A}_{2\pi}$, we obtain

$$\widetilde{G_\alpha(t)}f = \sum_l e^{(il)^\alpha t}(f, \psi_l)\widetilde{\psi_l}, \; f \in \mathcal{A}'_{2\pi},$$

where the series is weak*-convergent in $\mathcal{A}'_{2\pi}$.

Theorem 5.3 *The family of operators $\{\widetilde{G_\alpha(t)}\}_{t\geq 0}$, defined on $\mathcal{A}'_{2\pi}$ by (5. 8), is a weak*-continuous semigroup of linear operators on $\mathcal{A}'_{2\pi}$ for each $\alpha \in (1, 3)$. Moreover the infinitesimal generator of $\{\widetilde{G_\alpha(t)}\}_{t\geq 0}$ is the operator $\widetilde{D^\alpha} \in L(\mathcal{A}'_{2\pi})$ defined by (4. 2) (or equivalently, $\widetilde{D^{<\alpha>}} \in L(\mathcal{A}'_{2\pi})$ defined by (4. 6)).*

Proof: It follows from (5. 7) and Theorem 5.2 that, for each $\alpha \in (1, 3)$, $\{G^*_\alpha(t)\}_{t\geq 0}$ is an equicontinuous semigroup on $\mathcal{A}_{2\pi}$ with infinitesimal generator $(-D)^\alpha$, where $(-D)^\alpha$ is defined by (3. 4). From, [5, pp.262-263], it follows that $\{\widetilde{G_\alpha(t)}\}_{t\geq 0} = \{[G^*_\alpha(t)]'\}_{t\geq 0}$ is a weak*-continuous semigroup on $\mathcal{A}'_{2\pi}$ with infinitesimal generator $[(-D)^\alpha]'$. Since

$$(\widetilde{D^\alpha}f, \phi) = \sum_l (il)^\alpha(f, \psi_l)(\widetilde{\psi_l}, \phi) = (f, \sum_l (-il)^\alpha(\phi, \psi_l)_{L^2_{2\pi}}\psi_l) = (f, (-D)^\alpha\phi),$$

for all $f \in \mathcal{A}'_{2\pi}$ and $\phi \in \mathcal{A}_{2\pi}$, we deduce that the generator is $\widetilde{D^\alpha}$. □

An immediate consequence of the above theorem is that the abstract Cauchy problem associated with $\widetilde{D^\alpha}$, namely

$$\frac{du(t)}{dt} = \widetilde{D^\alpha}u(t), \; 1 < \alpha < 3, t > 0, \quad u(0) = f \in \mathcal{A}'_{2\pi}, \tag{5. 9}$$

has a unique weak*-solution given by,

$$u(t) = \widetilde{G_\alpha(t)}f = \sum_{l\in\mathbb{Z}} e^{(il)^\alpha t}(f, \psi_l)\widetilde{\psi_l}, \; t \geq 0, \tag{5. 10}$$

for each $\alpha \in (1, 3)$. We conclude this section with the remark that the series (5. 10) converges with respect to the weak*-topology in $\mathcal{A}'_{2\pi}$ for any α with $4k + 1 < \alpha < 4k + 3, k = 0, 1, \ldots$. Hence the distributional diffusion equation has the solution (5. 10), not only for the fractional order $1 < \alpha < 3$, but also for $\alpha \in (4k + 1, 4k + 3)$ for any $k = 0, 1, 2, \ldots$.

6 Conclusion

Functionals such as the Dirac delta cannot be treated in a mathematically rigorous manner as classical functions. Consequently, to study problems involving such entities, generalised functions and distributions are required. In this paper we have constructed $\mathcal{A}'_{2\pi}$, the dual of the Fréchet space $\mathcal{A}_{2\pi}$, to study distributional versions of fractional differential and integral operators. This theory allows us to work with a larger class of "functions." The fractional differential operator is well defined on these distributions and can be considered either as Weyl's fractional derivative or as Liouville-Grünwald fractional derivative. As an application, a fractional diffusion equation is examined in the space of periodic distributions $\mathcal{A}'_{2\pi}$ where we obtain a weak*-solution of this equation.

References

[1] T.M. Apostol, *Mathematical Analysis,* Addison Wesley, Massachusetts, 1958.

[2] P.L. Butzer and R.J. Nessel, *Fourier Analysis and Approximation,* Vol. I, Academic Press, New York and Birkhauser, Basel, 1971.

[3] P.L. Butzer and U. Westphal," An Access to Fractional Differentiation via Fractional Difference Quotients," *Fractional Calculus and Its Applications,* B. Ross (editor), Springer-Verlag, Berlin, pp.116-146, 1975.

[4] A. Erdélyi, *Operational Calculus and Generalised Functions*, Holt, Rinehart and Winston, New York, 1962.

[5] T. Kōmura, "Semigroups of Operators in Convex Spaces," *J.Funct. Anal,* 2, pp. 258-296, 1968.

[6] K. N. Khan, W. Lamb, A. C. McBride,"Fractional Transformations of Generalised Functions" in *Integral Transforms and Special Functions*, 2009.

[7] A.C. McBride, *Fractional Calculus and Integral Transforms of Generalised Functions*, Pitman Publishing, London, 1979.

[8] J.D. Pryce, *Basic Methods of Linear Functional Analysis,* Hutchinson, London, 1973.

[9] K.R. Stromberg, *An Introduction to Classical Real Analysis,* Wadsworth, Belmont, California, 1981.

[10] A.H. Zemanian, *Distributional Theory and Transform Analysis,* McGraw-Hill, New York, 1965.

[11] A.H. Zemanian, *Generalized Integral Transformations,* Interscience, New York, 1968.