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ON THE LOCAL DYNAMICS OF POLYNOMIAL DIFFERENCE EQUATIONS WITH FADING STOCHASTIC PERTURBATIONS

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Abstract. We examine the stability-instability behaviour of a polynomial difference equation with state-independent, asymptotically fading stochastic perturbations. We find that the set of initial values can be partitioned into a stability region, an instability region, and a region of unknown dynamics that is in some sense "small". In the first two cases, the dynamic holds with probability at least $1-\gamma$, a value corresponding to the statistical notion of a confidence level. Aspects of an equation with state-dependent perturbations are also treated.
When the perturbations are Gaussian, the difference equation is the Euler-Maruyama discretisation of an Itô-type stochastic differential equation with solutions displaying global a.s. asymptotic stability. The behaviour of any particular solution of the difference equation can be made consistent with the corresponding solution of the differential equation, with probability $1-\gamma$, by choosing the stepsize parameter sufficiently small. We present examples illustrating the relationship between h, γ and the size of the stability region.

Keywords. Nonlinear stochastic difference equation, stability, instability, numerical simulation.

AMS (MOS) subject classification: 39A10, 39A11, 37H10, 34F05, 93E15, 60E05

1 Introduction

The global a.s. asymptotic stability of solutions of nonlinear stochastic difference equations has been widely discussed in the literature. The most relevant publications are: [1, 2, 3, 4, 5, 7, 14, 15]. However, little attention has been paid to local stability for such equations. An early attempt to address local dynamics in an equation with bounded noise can be found in Fraser et al [9]; general results for equations with fading, state independent noise may be found in [1].

In this paper we consider the local dynamics of

$$\begin{aligned} X_{n+1} &= X_n - h\beta X_n|X_n|^\nu + \sqrt{h}\sigma_n\xi_{n+1}, \quad n \in \mathbb{N}, \\ X_0 &\in \mathbb{R}. \end{aligned} \tag{1}$$

Here, $\{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of independent random variables, each with zero mean, unit variance and distribution function Ψ_n. The constants β and ν are positive real numbers.

In [3], a global stability result was proven for the equation

$$X_{n+1} = X_n - f(X_n) + \sigma_n\xi_{n+1}, \quad n \in \mathbb{N}, \quad X_0 \in \mathbb{R}, \tag{2}$$

with globally Lipshitz f and square summable $\{\sigma_n\}_{n\in\mathbb{N}}$. The analysis was based on a particular semi-martingale decomposition (see [4] and Liptser & Shiryaev [13]). However, by departing from this technique, and treating the stochastic term as a perturbation of the dynamics of a deterministic equation, it became possible in [1] to show that neither the global Lipschitz condition nor the summation condition are necessary for local asymptotic stability. Instead, it is only necessary that f be locally Lipschitz continuous and that $\lim_{n\to\infty}\sigma_n\xi_{n+1} = 0$ a.s.; this latter reduces to a summation condition on the tails of the distribution functions Ψ_n.

Under these weaker conditions, it was proved in particular in [1] that global asymptotic stability is impossible for solutions of (1). [1] concludes that with probability close to one, $\lim_{n\to\infty} X_n = 0$ when the initial value X_0 is small, and $\lim_{n\to\infty} X_n = \infty$ when X_0 is big. In this paper, we seek to improve upon these results, giving a more detailed picture of the dynamics of (1).

When $\{\xi_n\}_{n\in\mathbb{N}}$ is an independent sequence of standard Normal random variables, (1) is the uniform Euler-Maruyama discretisation of the Itô-type stochastic differential equation

$$dX(t) = -\beta X(t)|X(t)|^\nu dt + \sigma(t)dW(t), \quad t \geq 0, \quad X(0) \in \mathbb{R}. \tag{3}$$

Here, $(W(t))_{t\geq 0}$ is a standard Brownian motion and the continuous process $(\sigma(t))_{t\geq 0}$ satisfies $\lim_{t\to\infty}\sigma^2(t)\ln t = 0$. Chan & Williams [8] showed that all solutions of (3) satisfy $\lim_{t\to\infty} X(t) = 0$ a.s. By contrast, the description of the dynamics of (1), given in [3] and refined in [1], is incomplete.

In this paper we are able to describe the behaviour of X_n for all $X_0 \in \mathbb{R}$, with the exception of two intervals with ‘small’ measure. We define a basin of attraction $\mathcal{A} \subset \mathbb{R}$ for the equilibrium at zero with a probability of $1-\gamma$. This probability corresponds to the statistical notion of *confidence level*, and γ may be assigned the values 0.1, 0.05 or 0.01 in practice. The same method allows us to define an instability region $\mathcal{B} \subset \mathbb{R}$. The absolute values of solutions originating in $\mathcal{B}$ have infinite limits at infinity, with probability $1-\gamma$. Finally, we will show that there exists a region of unknown behaviour, $\mathcal{C} \in \mathbb{R}$, which does not overlap with $\mathcal{A}$ or $\mathcal{B}$. $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ constitute a partition of $\mathbb{R}$.

All three regions depend on the parameter h. By choosing h small enough, the measure of the region of unknown behaviour, $\mathcal{C}$, can be made arbitrarily small. Additionally, for any initial value $X_0 \in \mathbb{R}$, and for any confidence level $1-\gamma$, we can choose h small enough that the resulting solution of (1) is asymptotically stable with probability $1-\gamma$. Thus the value of the step parameter h allows us to control the level of correspondence between the observed behaviour of the difference equation (1) and the known behaviour of the differential equation (3). In fact for a given initial value X_0, we can explicitly calculate an upper bound on h that is sufficient to ensure a stable trajectory with a given confidence level $1-\gamma$. We give examples of such calculations in Section 6, for various distribution types.

The ability to calculate values for h that give asymptotic stability with probability $1-\gamma$ has implications for numerical analysis. Recall that all solutions of (3) are a.s. asymptotically stable. In theory, in order to provide a numerical simulation that reflects the 'true' behaviour of a solution of (3) we simply calculate the appropriate stepsize h and simulate a solution of (1). However, a practical implementation of (1) on a finite-state machine, using a pseudo-random approximation of Gaussian numbers, shows that the theoretical analysis in this paper is in fact highly conservative. We demonstrate this using numerical examples in Section 7, where we also discuss some of the challenges involved in simulating the probability of stability of solutions of (1) when the rate of decay of $\{\sigma_n\}_{n\in\mathbb{N}}$ is slow.

We apply the same techniques to a stochastic difference equation with state-dependent perturbation:

$$\begin{aligned} X_{n+1} &= X_n - h\beta X_n|X_n|^{\nu} + \sqrt{h}\sigma_n|X_n|^{\nu_1}\xi_{n+1}, \quad n\in\mathbb{N}, \\ X_0 &\in \mathbb{R}, \end{aligned} \tag{4}$$

where ν_1 is a positive real number. The resultant analysis is less complete.

As a final comment, we note that, for equations with state independent perturbations like (1), it is incorrect to say '$X_n(\omega)$ asymptotically stable' when $\lim_{n\to\infty} X_n(\omega) = 0$, since (1) does not have an equilibrium solution at zero. However, we persist in using this nomenclature for two reasons. First, we are investigating the effect of noisy input on the dynamics of a deterministic equation that does have an equilibrium solution at zero. So although the inclusion of noise destroys the stable equilibrium, solutions of the perturbed equation can still behave as though it was there. Second, equations with state-dependent perturbations like (4) do have an equilibrium solution at zero; we wish to avoid distracting changes in terminology.

The paper is organized as follows. In Section 2, we set out the assumptions and analysis upon which the remainder of the paper relies. In Subsection 2.1, we impose the requirement that the stochastic perturbation $\{\sigma_n\xi_{n+1}\}_{n\in\mathbb{N}}$, converge to zero a.s. Because of this we can estimate it by some non-random number on an event with probability close to one. In Subsection 2.2 we summarise certain stability and instability results from [1] which are necessary for the proof of Theorem 3.1 in Section 3. In Subsection 2.3 we describe the essential properties of the unperturbed mapping given by the right hand side of (1). Finally, in Subsections 2.4 and 2.5 we use our ability to estimate the noise term to find intervals, centred on zero, which are invariant under perturbed versions of that mapping.

In Section 3 we present and prove the main result of the paper, describing the stability and instability behaviour of solutions of (1). A similar analysis is performed in Section 4 for solutions of (4).

In Section 5 we show how, given $\gamma \in (0,1)$, a non-random number $j(\gamma)$ that estimates the noise term $\sigma_n \xi_{n+1}$, with probability $1-\gamma$, may be calculated. We perform these calculations for perturbations with Normal distributions, and for perturbations with polynomial-tailed distributions.

In Section 6 we find explicit formulae that allow us to calculate the maximum stepsize $\bar{h}$ for which we can give a complete description of the stability-instability dynamics of the solutions of (1) with a given level of confidence, according to the statement of Theorem 3.1. We illustrate these formulae with examples. Additionally, for any particular $X_0 \in \mathbf{R}$, we estimate the maximum stepsize h that will give asymptotic stability of a solution of (1) for any given X_0.

In Section 7 we bring together our main results and the calculations performed in Sections 5 and 6 to write down an explicit description of the dynamics of an Euler-Maruyama polynomial difference equation with fading, state-independent perturbations. We show that this description is consistent with the known dynamics of the corresponding Itô differential equation. We then use numerical simulation to investigate the completeness of this description.

Finally, in Section 8 we present the proofs of three lemmas deferred from Section 2 for narrative reasons.

2 Mathematical Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a complete, filtered probability space. All stochastic perturbations in the paper will be driven by a sequence of independent random variables $\{\xi_n\}_{n\in\mathbb{N}}$ with distribution functions Ψ_n and with $\mathbb{E}\xi_n = 0$, $\mathbb{E}\xi_n^2 = 1$. The filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is naturally generated by this sequence: $\mathcal{F}_n = \sigma\{\xi_i : 1 \le i \le n\}$, for $n \in \mathbb{N}$. Among all sequences $\{X_n\}_{n\in N}$ of random variables we distinguish those for which X_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$. We use the standard abbreviation "a.s." for the wordings "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbb{P}$ throughout the text. A more detailed discussion of stochastic concepts and notation may be found in, for example, Shiryaev [17].

2.1 Conditions on the stochastic perturbation

Everywhere in this paper we make the following assumption.

Assumption 2.1 *The sequences* $\{\sigma_n\}_{n\in\mathbb{N}}$ *and* $\{\xi_n\}_{n\in\mathbb{N}}$ *satisfy*

$$\lim_{n\to\infty} \sigma_n \xi_{n+1} = 0, \quad a.s. \tag{5}$$

Assumption 2.1 leads to the following result.

Lemma 2.2 *Let (5) hold. Then for all* $\gamma \in (0,1)$ *there exist* Ω_γ *and* $j(\gamma)$ *such that*

$$\max_{n\in\mathbf{N}} |\sigma_n \xi_{n+1}(\omega)| < j(\gamma), \quad \omega \in \Omega_\gamma, \quad \mathbb{P}(\Omega_\gamma) > 1-\gamma. \tag{6}$$

Conditions which guarantee (5) in Assumption 2.1 can be found in [1], along with the proof of Lemma 2.2.

2.2 Stability for a general equation under deterministic perturbation

We present here some lemmas from [1] which will be widely used in this paper, and which address the stability of solutions of a deterministic inhomogeneous difference equation of the form

$$x_{n+1} = x_n - f(x_n) + S_n, \quad n \in \mathbb{N}, \quad x_0 \in \mathbb{R}, \tag{7}$$

where $f : \mathbb{R} \to \mathbb{R}$, $f(0) = 0$ is a continuous function with the properties

$$u f(u) \quad > \quad 0, \text{ for all } u \neq 0, \tag{8}$$

$$\inf_{u>c} |f(u)| \quad > \quad 0, \text{ for all } c > 0. \tag{9}$$

The first lemma provides conditions on f under which certain solutions of (7) are asymptotically stable.

Lemma 2.3 *Suppose the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is a solution of (7), where* f *obeys* (8) *and* (9)*, and* $S_n \to 0$*, as* $n \to \infty$*. Define the mapping*

$$G_B(u) = u - f(u) + B, \quad u, B \in \mathbb{R}. \tag{10}$$

Suppose that there exists an interval $(-a, a)$*, and a number* $\lambda \in (0, 2)$*, such that* $G_B : (-a, a) \to (-a, a)$*, when* $|B| \le \max_{n\in\mathbf{N}} |S_n|$*, and*

$$|f(u)| \le (2-\lambda)|u|, \quad u \in (-a, a). \tag{11}$$

Then $\lim_{n\to\infty} x_n = 0$ *when* $x_0 \in (-a, a)$*.*

The next lemma provides conditions on f outside of $(-a, a)$ under which oscillatory instability of solutions of (7) is observed.

Lemma 2.4 *Suppose the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a solution of (7), where f satisfies (8). Suppose also that there exist constants $a, \lambda > 0$ such that*

$$|f(u)| \geq (2+\lambda)|u|, \quad u \in (-\infty, -a) \cup (a, \infty). \tag{12}$$

Suppose also that $|S_n| < \frac{\lambda a}{2}$ for all $n \in \mathbb{N}$. Then

$$\limsup_{n\to\infty} x_n = \infty \quad \text{and} \quad \liminf_{n\to\infty} x_n = -\infty$$

when $x_0 \in (-\infty, -a) \cup (a, \infty)$.

Finally, we generalise Lemma 2.4 so that it applies to deterministic difference equations with a state-dependent perturbation.

$$x_{n+1} = x_n - f(x_n) + S_n g(x_n), \quad n \in \mathbb{N}. \tag{13}$$

Lemma 2.5 *Suppose the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a solution of (7), where f satisfies (8). Suppose also that there exist constants $a, \lambda, \nu_1, H > 0$ such that*

$$|f(u)| \geq (2+\lambda)|u| + H|u|^{\nu_1}, \quad u \in (-\infty, -a) \cup (a, \infty). \tag{14}$$

Suppose also that $|S_n g(x_n)| < H|x_n|^{\nu_1}$ for all $n \in \mathbb{N}$. Then

$$\limsup_{n\to\infty} x_n = \infty \quad \text{and} \quad \liminf_{n\to\infty} x_n = -\infty,$$

when $x_0 \in (-\infty, -a) \cup (a, \infty)$

The proof of Lemma 2.5 is similar to that of Lemma 2.4.

2.3 Properties of the unperturbed mapping

The deterministic mapping from $\mathbb{R}$ to $\mathbb{R}$ corresponding to an unperturbed version of (1), is given by the function

$$F_h(x) = x - h\beta x|x|^{\nu}, \quad x \in \mathbb{R}. \tag{15}$$

In this subsection we describe the properties of this function as fully as possible, before moving to to examine the effect of perturbations on its action in Subsections 2.4 and 2.5. Since $F_h(-x) = -F_h(x)$ and $F_h(0) = 0$, we may restrict our attention to $(0, \infty)$ without loss of generality. First, we pick out the key features of F_h:

Definition 2.6 *Let F_h be the mapping from $\mathbb{R}$ to $\mathbb{R}$ defined in (15). Then*

$$x_{max}: \quad F_h(x_{max}) = \max_{x\in[0,\infty)} F_h(x), \qquad [F_h]_{max} := F_h(x_{max}),$$

and

$$\begin{aligned} x_1 &:= \text{the unique value of } x \in (0,\infty) \text{ where } F_h(x) = 0, \\ x_2 &:= \text{the unique value of } x \in (0,\infty) \text{ where } F_h(x) = -[F_h]_{max}, \\ x_3 &:= \text{the unique value of } x \in (0,\infty) \text{ where } F_h(x) = -x. \end{aligned}$$

The properties of the mapping F_h are contained in the following lemma.

Lemma 2.7 *For the mapping* F_h*,*

(i) x_1 *and* x_3 *satisfy*

$$x_1 = \sqrt[\nu]{\frac{1}{h\beta}}, \quad x_3 = \sqrt[\nu]{\frac{2}{h\beta}};$$

(ii) there exists a unique $c(\nu) \in \left(\sqrt[\nu]{1+\nu}, \sqrt[\nu]{2(1+\nu)}\right)$ *such that*

$$x_2 = \frac{c(\nu)}{\sqrt[\nu]{h(1+\nu)\beta}},$$

and therefore x_2 *is unique, and* $x_2 \in \left(1/\sqrt[\nu]{h(1+\nu)}, \infty\right)$*;*

(iii) $x_1 < x_2 < x_3$*.*

The proof of Lemma 2.7 is deferred until Section 8.

Definition 2.8 *By Part (ii) of Lemma 2.7 we can define* $q^* \in (0,1)$ *by*

$$q^* = \frac{c(\nu)}{\sqrt[\nu]{2(1+\nu)}}. \tag{16}$$

Note that q^* *does not depend on* h*.*

Definition 2.9 *For any* $q \in (q^*, 1)$*, we construct the sets*

$$I_q^M = \left(-q\sqrt[\nu]{\frac{2}{h\beta}}, \quad q\sqrt[\nu]{\frac{2}{h\beta}}\right), \tag{17}$$

and

$$I_q^L = \left(-\infty, -(2-q)\sqrt[\nu]{\frac{2}{h\beta}}\right), \quad I_q^R = \left((2-q)\sqrt[\nu]{\frac{2}{h\beta}}, \infty\right). \tag{18}$$

Finally, denote

$$I_q^{LR} = I_q^L \cup I_q^R. \tag{19}$$

2.4 Invariance of I_q^M under F_h with a state-independent perturbation

The next stage is to apply a deterministic perturbation to the mapping (15) and construct an interval invariant under the perturbed map. First we examine a state-independent perturbation. In Subsection 2.5 we consider a state-dependent perturbation.

For $C \in \mathbb{R}$, define

$$G_{C,h}(x) = x - h\beta x|x|^\nu + \sqrt{h}C, \quad x \in \mathbb{R}. \tag{20}$$

We note that $G_{C,h}(x) = F_h(x) + \sqrt{h}C$, where F_h is defined as in (15).

Lemma 2.10 *Let $G_{C,h}$ be the mapping from $\mathbb{R}$ to $\mathbb{R}$ defined in (20). For any constants $H > 0$, $\beta > 0$ and $\nu > 0$, there exists $\bar{h}(H, \nu, \beta) > 0$ sufficiently small that the following is true:*

For any $h \in (0, \bar{h})$ there exists $q = q(h) \in (q^*, 1)$ such that

$$G_{C,h} : I_q^M \to I_q^M \tag{21}$$

holds for any $C \in (-H, H)$.

Corollary 2.1 *Define the mapping*

$$G_B(u) = F_h(u) + B, \tag{22}$$

where $|B| \le \sqrt{h}|j(\gamma)|$, $h < \bar{h}(\gamma, \nu, \beta)$, and F_h is defined by (15).

Under the conditions of Lemma 2.10 we have

$$G_B : I_q^M \to I_q^M,$$

with $q = 1 - \varepsilon(h)$, where

$$\varepsilon(h) = 2^{1-\frac{1}{\nu}} h^{\frac{1}{2}+\frac{1}{\nu}} \nu^{-1} \beta^{\frac{1}{\nu}} H. \tag{23}$$

2.5 Invariance of I_q^M under F_h with a state-dependent perturbation

Now we take exactly the same approach for a state-dependent perturbation of (15). Define for $C \in \mathbb{R}$ the mapping

$$\widetilde{G}_{C,h}(x) = x - h\beta x|x|^\nu + \sqrt{h}|x|^{\nu_1} C, \quad x < 0. \tag{24}$$

We note that

$$\widetilde{G}_{C,h}(x) = F_h(x) + \sqrt{h}|x|^{\nu_1} C,$$

where F_h is defined as in (15).

Lemma 2.11 *For all $H > 0$, $\beta > 0$, $\nu > 0$, and*

$$\nu_1 < 1 + \frac{\nu}{2},$$

we can find $\bar{h}(H, \nu, \nu_1, \beta) > 0$ so small that for each $h \in (0, \bar{h})$ we can define $q = q(h) \in (q^*, 1)$ such that for all $C \in (-H, H)$ we have

$$\widetilde{G}_{C,h} : I_q^M \to I_q^M. \tag{25}$$

Corollary 2.2 *Define G_B as in (22), with*

$$|B| \le \sqrt{h} q^{\nu_1} \left(\sqrt[\nu]{\frac{2}{h\beta}} \right)^{\nu_1} |j(\gamma)|,$$

and $h < \bar{h}(\gamma, \nu, \nu_1, \beta)$. *Under the conditions of Lemma 2.11 we have*

$$G_B : I_q^M \to I_q^M,$$

where $q = 1 - \varepsilon(h)$, *with*

$$\varepsilon(h) = K_1^{-1}\nu^{-1}2^{-\frac{1}{\nu}+\frac{\nu_1}{\nu}}\beta^{\frac{1}{\nu}-\frac{\nu_1}{\nu}}h^{\frac{1}{2}+\frac{1}{\nu}-\frac{\nu_1}{\nu}}H. \tag{26}$$

Remark 1 *The proofs of Lemmas 2.10 and 2.11 require us to ensure that*

$$\varepsilon(h) < \min\left\{\frac{1}{2}, 1 - q^*\right\}, \tag{27}$$

where q^* *is defined in (16) of Definition 2.8. A consequence of this is that* $q = 1 - \varepsilon(h) \in (q^*, 1)$.

3 Stability and instability: state-independent noise

In this section we state and prove the central result of the paper, which describes regions of stability and instability for solutions of (1).

Theorem 3.1 *Assume that the sequences* $\{\sigma_n\}_{n\in\mathbb{N}}$ *and* $\{\xi_n\}_{n\in\mathbb{N}}$ *satisfy condition (5) in Assumption 2.1. Let the sequence* $\{X_n\}_{n\in\mathbb{N}}$ *be a solution of (1).*

Then, for every $\gamma \in (0, 1)$, *there exists* $\bar{h}(\gamma, \nu, \beta) > 0$ *and* $\Omega_\gamma \in \Omega$, *with* $\mathbb{P}\{\Omega_\gamma\} > 1 - \gamma$, *such that for all* $h < \bar{h}(\gamma, \nu, \beta)$ *and* $\omega \in \Omega_\gamma$ *there exists* $\bar{\varepsilon}(h) > 0$ *with*

$$\lim_{h\to 0} \bar{\varepsilon}(h) = 0, \tag{28}$$

such that

(i) $\lim_{n\to\infty} X_n(\omega) = 0$, *when*

$$X_0 \in \left(-\sqrt[\nu]{\frac{2}{h\beta}} + \bar{\varepsilon}(h), \quad \sqrt[\nu]{\frac{2}{h\beta}} - \bar{\varepsilon}(h)\right); \tag{29}$$

(ii) $\limsup_{n\to\infty} X_n = \infty$ *and* $\liminf_{n\to\infty} X_n = -\infty$, *when*

$$X_0 \in \left(-\infty, -\sqrt[\nu]{\frac{2}{h\beta}} - \bar{\varepsilon}(h)\right) \bigcup \left(\sqrt[\nu]{\frac{2}{h\beta}} + \bar{\varepsilon}(h), \infty\right). \tag{30}$$

Remark 2 *Notice from* (28) *and the form of the regions defined in* (29) *and* (30) *that, as* h *decreases to zero, the region of stability expands to encompass all of* $\mathbb{R}$. *Thus the dynamics of solutions of* (1) *can be made consistent with those of* (3) *by choosing* h *small enough. In Subsection 6.2 we show how, given* X_0, *values of* h *sufficient for stability with confidence at least* $1-\gamma$ *may be computed.*

Proof We begin by relating the interval I_q^M to the statement of the theorem. For a fixed $\gamma > 0$, let Ω_γ and $j(\gamma)$ be defined as in Lemma 2.2, and let $\varepsilon(h)$ be defined as in (23) with $H = j(\gamma)$. We define

$$\bar{\varepsilon}(h) = \varepsilon(h) \sqrt[\nu]{\frac{2}{h\beta}} \tag{31}$$

and note that for fixed γ and ν,

$$\lim_{h\to 0} \bar{\varepsilon}(h) = \lim_{h\to 0} \left\{ 2h^{\frac{1}{2}} j(\gamma) \frac{1}{\nu} \right\} = 0.$$

Fix $h \leq \bar{h}(\gamma, \nu, \beta)$ and set

$$q = 1 - \varepsilon(h). \tag{32}$$

By (27) in Remark 1 we have $q \in (q^*, 1)$. From (31) and (32), we conclude that

$$q \sqrt[\nu]{\frac{2}{h\beta}} = (1 - \varepsilon(h)) \sqrt[\nu]{\frac{2}{h\beta}} = \sqrt[\nu]{\frac{2}{h\beta}} - \bar{\varepsilon}(h),$$

thus the interval $\left(-\sqrt[\nu]{\frac{2}{h\beta}} + \bar{\varepsilon}(h),\ \sqrt[\nu]{\frac{2}{h\beta}} - \bar{\varepsilon}(h) \right)$ coincides with the interval I_q^M, as defined in (17).

Now we turn to the proof of Part (i). The function $f : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$f(u) = h\beta u|u|^\nu$$

satisfies (8) and (9). For $u \in I_q^M$ we have

$$|f(u)| = h\beta|u||u|^\nu \leq h\beta|u| \left| q \sqrt[\nu]{\frac{2}{h\beta}} \right|^\nu = 2q^\nu |u|.$$

Then, for $\lambda = 2(1 - q^\nu)$, condition (11) holds.

Let $X_0 \in I_q^M$, where q satisfies (32). Fix a path $\omega \in \Omega_\gamma$ and set

$$S_n = \sqrt{h}\sigma_n \xi_{n+1}(\omega), \quad \text{and} \quad x_n = X_n(\omega).$$

By (5) in Assumption 2.1, $\lim_{n\to\infty} S_n = 0$, and by Lemma 2.2, $|S_n| = |\sqrt{h}\sigma_n\xi_{n+1}(\omega)| < \sqrt{h} j(\gamma)$. Also, we find that G_B, as defined in (22), maps I_q^M into itself, by Corollary 2.1. Thus we can apply Lemma 2.3 and conclude that $\lim_{n\to\infty} x_n = 0$.

To prove Part (ii), consider $X_0 \in I_q^{LR}$. For $u \in I_q^R$ we have

$$\begin{aligned} |f(u)| = h\beta|u||u|^\nu &\geq h\beta|u| \left| \sqrt[\nu]{\frac{2}{h\beta}} (1 + \varepsilon(h)) \right|^\nu \\ &= 2(1 + \varepsilon(h))^\nu |u|. \end{aligned}$$

Define $\lambda = 2(1+\varepsilon(h))^\nu - 2$ and $a = \sqrt[\nu]{\frac{2}{h\beta}} + \bar{\varepsilon}(h)$. Since for $h \le \bar{h}(\gamma,\nu,\beta)$ the second inequality in (74) holds, we have

$$\begin{aligned}\frac{\lambda a}{2} &= ((1+\varepsilon(h))^\nu - 1)\sqrt[\nu]{\frac{2}{h\beta}}(\varepsilon(h)+1)\\ &> \sqrt[\nu]{\frac{2}{h\beta}}\frac{\nu\varepsilon(h)}{2} = j(h)\sqrt{h}\\ &\ge |\sqrt{h}\sigma_n\xi_{n+1}(\omega)| = |S_n|,\end{aligned}$$

for all $\omega \in \Omega_\gamma$. Now we can apply Lemma 2.4 and obtain the desired result.

4 Stability-instability: state-dependent noise

4.1 Statement of results

In this section we state and prove two theorems for the difference equation with state-dependent noise (4). Following the approach described in Section 3 for the case of state-independent noise, we can define a basin of attraction $\mathcal{A}$ for the zero solution of (4) with an associated confidence level $1-\gamma$. The region $\mathcal{A}$ can be manipulated to include any particular initial value X_0 for a given value of γ by choosing h appropriately small. However, the regions of instability ($\mathcal{B}$) and unknown behaviour ($\mathcal{C}$) are not as clearly defined.

There are two cases, distinguished by the relative sizes of ν_1 and ν.

Assumption 4.1 *Assume that ν_1 and ν satisfy*

$$\nu_1 < \frac{\nu}{2}. \tag{33}$$

Assumption 4.2 *Assume that ν_1 and ν satisfy*

$$\nu_1 < \nu + 1. \tag{34}$$

Theorem 4.3, addressing the stability region of (4), will be proved under Assumption 4.1.

Theorem 4.3 *Let the sequence $\{X_n\}_{n\in\mathbb{N}}$ be a solution of (4), where we assume that the sequences $\{\sigma_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ satisfy condition (5) in Assumption 2.1. Assume also that ν_1 and ν satisfy (33) in Assumption 4.1.*

For every $\gamma \in (0,1)$ there exists $\bar{h}(\gamma,\nu,\nu_1,\beta) > 0$ and $\Omega_\gamma \in \Omega$, with $\mathbb{P}\{\Omega_\gamma\} > 1-\gamma$, such that for all $h < \bar{h}(\gamma,\nu,\nu_1,\beta)$ and $\omega \in \Omega_\gamma$ there exists $\bar{\varepsilon}(h) > 0$ with

$$\lim_{h\to 0}\bar{\varepsilon}(h) = 0, \tag{35}$$

such that $\lim_{n\to\infty} X_n(\omega) = 0$, *when*

$$X_0 \in \left(-\sqrt[\nu]{\frac{2}{h\beta}} + \bar{\varepsilon}(h), \quad \sqrt[\nu]{\frac{2}{h\beta}} - \bar{\varepsilon}(h)\right).$$

Theorem 4.4, addressing the instability region of (4), will be proved under Assumption 4.2.

Theorem 4.4 *Let the sequence* $\{X_n\}_{n\in\mathbb{N}}$ *be a solution to (4), where we assume that the sequences* $\{\sigma_n\}_{n\in\mathbb{N}}$ *and* $\{\xi_n\}_{n\in\mathbb{N}}$ *satisfy condition (5) in Assumption 2.1. Assume also that* ν_1 *and* ν *satisfy (34) in Assumption 4.2.*

For every $\gamma \in (0,1)$ *there exists* $\Omega_\gamma \in \Omega$, *with* $\mathbb{P}\{\Omega_\gamma\} > 1-\gamma$, *such that for each* $h > 0$ *there exists*

$$a = a(\gamma, h) > \sqrt[\nu]{\frac{2}{h\beta}},$$

such that for each $\omega \in \Omega_\gamma$,

$$X_0 \in (-\infty, -a) \cap (a, \infty)$$

implies $\limsup_{n\to\infty} X_n(\omega) = \infty$ *and* $\liminf_{n\to\infty} X_n(\omega) = -\infty$.

Remark 3 *When* $\{\xi_n\}_{n\in\mathbb{N}}$ *is a sequence of independent standard Normal random variables, (4) is the Euler-Maruyama discretisation, over a uniform mesh of length* $h > 0$, *of the Itô-type stochastic differential equation*

$$\begin{aligned} dX(t) &= -\beta|X(t)|^{\nu+1}dt + \sigma(t)|X(t)|^{\nu_1}dW(t), \quad t \geq 0, \\ X(0) &\in \mathbb{R}. \end{aligned} \tag{36}$$

When $\nu_1 \geq 1$, *the solution* $(X_n)_{t\geq 0}$ *with arbitrary initial condition* $X_0 \in \mathbb{R}$, *is well defined, unique and positive on* $[0,\infty)$, *see [5], and most importantly,* $\lim_{t\to\infty} X(t) = 0$, *a.s. This can be shown by applying Itô's formula to the Liapunov function* $V(x) = x^\alpha$ *with* $\alpha \in (0,1)$. *Note that the proof does not require that* $\lim_{t\to\infty}\sigma(t) = 0$.

When $\nu_1 < 1$, *the diffusion coefficient is not Lipschitz continuous. But if* $\nu_1 > \frac{1}{2}$ *we still can prove uniqueness and, therefore, existence on* $[0,\infty)$, *by applying the result by Yamada-Watanabe (see, for example, [10]). However, the asymptotic behaviour of solutions, including positivity, in the parameter region* $\nu_1 \in (1/2, 1)$ *is unknown: we can make no comparisons there.*

4.2 Proof of Theorems 4.3 and 4.4

Proof of Theorem 4.3 For a fixed $\gamma > 0$, let Ω_γ and $j(\gamma)$ be defined as in Lemma 2.2 and $\varepsilon(h)$ be defined as in (26) with $H = j(\gamma)$. Set

$$\bar{\varepsilon}(h) = \varepsilon(h)\sqrt[\nu]{\frac{2}{h\beta}}$$

and note that, since $1/2 - \nu_1/\nu > 0$, for fixed γ, ν, ν_1 and β

$$\lim_{h\to 0}\bar{\varepsilon}(h) = \lim_{h\to 0}\left\{h^{\frac{1}{2}-\frac{\nu_1}{\nu}} j(\gamma)\beta^{-\frac{\nu_1}{\nu}} 2^{\frac{\nu_1}{\nu}} K_1^{-1}\nu^{-1}\right\} = 0.$$

Choose $\bar{h} = \bar{h}(j(\gamma), \nu, \nu_1, \beta)$ as described in Lemma 2.11 and fix $h \le \bar{h}$. As before, set $q = 1 - \varepsilon(h)$. By (27) in Remark 1 we have $q \in (q^*, 1)$.

From definitions of $\bar{\varepsilon}$ and q, we conclude that $q\sqrt[\nu]{\frac{2}{h\beta}} = \sqrt[\nu]{\frac{2}{h\beta}} - \bar{\varepsilon}(h)$, and thus

$$I_q^M = \left(-\sqrt[\nu]{\frac{2}{h\beta}} + \bar{\varepsilon}(h), \quad \sqrt[\nu]{\frac{2}{h\beta}} - \bar{\varepsilon}(h) \right).$$

The function $f(u) = h\beta u|u|^\nu$ satisfies (8) and (9). Also, when $u \in I_q^M$, f satisfies condition (11) with $\lambda = 2(1 - q^\nu) \in (0, 2)$.

Let $X_0 \in I_q^M$. Fix $\omega \in \Omega_\gamma$, set $x_n = X_n(\omega)$ and set

$$S_n = \sqrt{h}|X_n(\omega)|^{\nu_1}\sigma_n\xi_{n+1}(\omega).$$

Since $|X_n(\omega)|^{\nu_1} \le \left(q\sqrt[\nu]{\frac{2}{h\beta}}\right)^{\nu_1}$ when $\omega \in \Omega_\gamma$, $\lim_{n\to\infty} S_n = 0$ by (5) in Assumption 2.1. Also, by Corollary (2.2), we obtain that $G_B : I_q^M \to I_q^M$, where G_B is defined as in (22). Thus we can apply Lemma 2.3 and conclude that $\lim_{n\to\infty} x_n = 0$.

Proof of Theorem 4.4 For a fixed $\gamma > 0$, let Ω_γ and $j(\gamma)$ be defined as in Lemma 2.2. Fix some $\varepsilon > 0$. Let

$$\delta = \delta(\varepsilon) \in \left(0, 1 - \frac{1}{(1+\varepsilon)^\nu}\right).$$

For arbitrary $h > 0$ and for $|u| \ge \sqrt[\nu]{\frac{2}{h\beta}}(1+\varepsilon)$ we have

$$\begin{aligned} |f(u)| = h\beta|u||u|^\nu &= h\beta(1-\delta)|u|^{1+\nu} + h\beta\delta|u|^{1+\nu} \\ &\ge h\beta(1-\delta)|u|\left|\sqrt[\nu]{\frac{2}{h\beta}}(1+\varepsilon)\right|^\nu + h\beta\delta|u|^{1+\nu} \\ &= 2(1+\varepsilon)^\nu(1-\delta)|u| + h\beta\delta|u|^{1+\nu}. \end{aligned}$$

Set

$$a = a(\gamma, \varepsilon, h, \nu, \nu_1, \beta) = \max\left\{\sqrt[\nu]{\frac{2}{h\beta}}(1+\varepsilon), \left(\frac{j(\gamma)}{\beta\delta\sqrt{h}}\right)^{\frac{1}{1+\nu-\nu_1}}\right\}, \tag{37}$$

and

$$S_n(\omega) = \sqrt{h}\sigma_n\xi_{n+1}, \quad g(x_n) = |x_n|^{\nu_1}, \quad H = \sqrt{h}j(\gamma).$$

Then $|S_n(\omega)| \le H$ on Ω_γ. When $|u| \ge \left(\frac{j(\gamma)}{\beta\delta\sqrt{h}}\right)^{\frac{1}{1+\nu-\nu_1}}$ we have

$$h\beta\delta|u|^{1+\nu} > \sqrt{h}j(\gamma)|u|^{\nu_1}.$$

Thus, on the set $(-\infty, -a) \cap (a, \infty)$, the function f satisfies condition (14) in Lemma 2.5, with $\lambda = 2(1+\varepsilon(h))^\nu(1-\delta) - 2$ and $H = \sqrt{h}j(\gamma)$. The statement of the theorem follows.

Remark 4 *In order to understand which term on the right-hand-side of (37) dominates, we must calculate $j(\gamma)$ for specific distributions and for appropriate values of γ; see Section 5.*

5 Estimating the bound $j(\gamma)$ on the diffusion.

In this section we compute values of $j(\gamma)$ such that

$$\mathbb{P}\{|\sigma_n \xi_{n+1}| < j(\gamma),\, n \in \mathbf{N}\} > 1 - \gamma, \tag{38}$$

holds for various perturbation classes, setting $\gamma = 0.1,\, 0.05,\, 0.01$. For each perturbation class, we must choose the form of the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ so that it satisfies (5) in Assumption 2.1.

5.1 First example: Normal perturbations

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables, each with an $\mathcal{N}(0,1)$ distribution function. In order to compute values of $j(\gamma)$ such that

$$\mathbb{P}\left[|\sigma_n \xi_{n+1}| < j(\gamma),\, n \in \mathbb{N}\right] > 1 - \gamma,$$

we require the following set of tight bounds on the Normal distribution function, (found in Sasvari & Chen [16]): for all $u \in \mathbb{R}$,

$$\sqrt{1 - e^{-\frac{u^2}{2}}} < \frac{1}{\sqrt{2\pi}} \int_{-u}^{u} e^{-\frac{x^2}{2}}\, dx < \sqrt{1 - e^{-\frac{2u^2}{\pi}}}. \tag{39}$$

Remark 5 *It was proved in [1] that when the distribution of each ξ_n is standard Normal,* (5) *holds if and only if*

$$\lim_{n\to\infty} \sigma_n^2 \ln n = 0. \tag{40}$$

In fact the result in [1] was more general than this: it stated that (40) is necessary and sufficient to guarantee $\lim_{n\to\infty} \sigma_n \xi_{n+1} = 0$ *a.s. when the distribution $\Psi_n \equiv \Psi$ of each ξ_n behaves asymptotically as follows:*

$$1 - \Psi(a) + \Psi(-a) \sim \frac{b}{a} e^{-\frac{1}{k}a^2}, \quad k > 0, \quad a, b \in \mathbb{R}$$

The distribution of a standard Normal random variable is included in this case, as may be seen from Mills' estimate (see, for example [10]):

$$\frac{a}{1+a^2} e^{-a^2/2} \le \int_a^\infty e^{-u^2/2} du \le \frac{1}{a} e^{-a^2/2}, \quad a \in \mathbb{R}.$$

Example 5.1 *We define* $\{\sigma_n\}_{n\in\mathbb{N}}$ *as the monotone decreasing sequence*

$$\sigma_n = \frac{1}{[\ln(n+1)]^{1.1}}, \quad n \in \mathbb{N},$$

ensuring that (40) *is satisfied. Thus, for* $n \in \mathbb{N}$,

$$\sigma_n \le \sigma_1 = \frac{1}{[\ln 2]^{1.1}} = 1.4966 < 1.5. \tag{41}$$

Since $\{\xi_n\}_{n\in\mathbb{N}}$ *is a sequence of independent random variables, and applying the lower bound in* (39)*, we see that*

$$\begin{aligned}
\mathbb{P}[|\sigma_n\xi_{n+1}| > j(\gamma),\, n\in\mathbb{N}] &= \mathbb{P}\left[|\xi_{n+1}| > \frac{j(\gamma)}{\sigma_n},\, n\in\mathbb{N}\right] \\
&= \mathbb{P}\left[\bigcap_{n=0}^{\infty}\left\{|\xi_{n+1}| > \frac{j(\gamma)}{\sigma_n}\right\}\right] \\
&\le \mathbb{P}\left[|\xi_N| > \frac{j(\gamma)}{\sigma_1}\right] \\
&\le 1 - \sqrt{1 - e^{-\frac{1}{2}\left(\frac{j(\gamma)}{\sigma_1}\right)^2}},
\end{aligned}$$

for any particular $N \in \mathbb{N}$*. Applying the upper bound on* σ_1 *from* (41)*, and rearranging, we find that we must choose* $j(\gamma)$ *so that*

$$j(\gamma) > \sqrt{-3\ln\left[1-(1-\gamma)^2\right]}. \tag{42}$$

Note that $1-(1-\gamma)^2 \in (0,1)$*, and therefore the inequality* (42) *is well defined.*

- *If* $\gamma = 0.1$*, then* $j(0.1) > 4.99$ *suffices for* (42) *to hold, and*

  $$\mathbb{P}\left[|\sigma_n\xi_{n+1}| < 5,\, n\in\mathbb{N}\right] > 0.9.$$

- *If* $\gamma = 0.05$*, then* $j(0.05) > 6.99$ *suffices for* (42) *to hold, and*

  $$\mathbb{P}\left[|\sigma_n\xi_{n+1}| < 7,\, n\in\mathbb{N}\right] > 0.95.$$

- *If* $\gamma = 0.01$*, then* $j(0.01) > 11.76$ *suffices for* (42) *to hold, and*

  $$\mathbb{P}\left[|\sigma_n\xi_{n+1}| < 11.77,\, n\in\mathbb{N}\right] > 0.99.$$

5.2 Second example: a distribution with polynomial tails

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed random variables. Suppose there exists $a^* > 0$ such that the distribution Ψ of each ξ_n satisfies, for some $m > 2$,

$$\mathbb{P}\{|\xi_n| > a\} = 1 - \Psi(a) + \Psi(-a) = a^{-m}, \quad a > a^*. \tag{43}$$

Suppose also that there exists $c < \infty$ such that

$$\lim_{a\to\infty}(1-\Psi(a))a^m = c. \tag{44}$$

Remark 6 *$m > 2$ ensures that each ξ_n has a finite second moment. Note also that in this demonstration we require that (43) hold precisely for $a > a^*$, and that a^* be sufficiently small not to affect our computations. In practice, the relationship would generally be asymptotic, and the computations described in this example are thus approximate.*

Remark 7 *It was proved in [1] that, under constraint (44), condition (5) holds if and only if*

$$\|\sigma\|_m = \sum_{i=1}^{\infty}|\sigma_i|^m < \infty. \tag{45}$$

Example 5.2 *Set $m = 3$, $a^* = 2$, and define $\{\sigma_n\}_{n\in\mathbb{N}}$ to be the monotone decreasing sequence*

$$\sigma_n = n^{-\frac{1}{2}}, \quad n \in \mathbb{N},$$

which satisfies (45). Now we fix $j(\gamma) > a^* \max_{i\in\mathbb{N}}\sigma_i = 2$, and estimate

$$\begin{aligned}\mathbb{P}\{|\sigma_n\xi_{n+1}| \ge j(\gamma), \quad n\in\mathbf{N}\} &= \mathbb{P}\left\{|\xi_{n+1}| \ge \frac{j(\gamma)}{\sigma_n}, \quad n\in\mathbf{N}\right\}\\ &\le \mathbb{P}\left[\bigcup_{i=1}^{\infty}\left\{|\xi_{i+1}| \ge \frac{j(\gamma)}{\sigma_i}\right\}\right]\\ &\le \sum_{i=1}^{\infty}\mathbb{P}\left\{|\xi_{i+1}| \ge \frac{j(\gamma)}{\sigma_i}\right\}\\ &= \sum_{i=1}^{\infty}\left(\frac{j(\gamma)}{\sigma_i}\right)^{-3}.\end{aligned}$$

Then

$$\mathbb{P}\left\{|\sigma_n\xi_{n+1}| < j(\gamma),\, n\in\mathbf{N}\right\} \ge 1 - \sum_{i=1}^{\infty}\left(\frac{j(\gamma)}{\sigma_i}\right)^{-3}. \tag{46}$$

Therefore it suffices to choose $j(\gamma) > 2$ to satisfy

$$j(\gamma)^{-3}\sum_{i=1}^{\infty}\sigma_i^m < \gamma, \tag{47}$$

which, since $\|\sigma\|_3 < \infty$, may be rewritten

$$j(\gamma) > \left[\frac{\|\sigma\|_3}{\gamma}\right]^{\frac{1}{3}}. \tag{48}$$

It remains to calculate

$$\begin{aligned} \|\sigma\|_3 &= \sum_{i=1}^{\infty} \sigma_i^3 = \sum_{i=1}^{\infty} i^{-\frac{3}{2}} = 1 + \sum_{i=2}^{\infty} i^{-\frac{3}{2}} \leq 1 + \int_1^{\infty} x^{-\frac{3}{2}} dx \\ &= 1 + \left. \frac{x^{1-3/2}}{1-3/2} \right|_1^{\infty} = 3. \end{aligned}$$

- *If* $\gamma = 0.1$ *then* $j(0.1) > 3.107$ *suffices for* (48) *to hold, and*

$$\mathbb{P}\left[|\sigma_n \xi_{n+1}| < 3.108,\ n \in \mathbb{N}\right] > 0.9.$$

- *If* $\gamma = 0.05$ *then* $j(0.05) > 3.914$ *suffices for* (48) *to hold, and*

$$\mathbb{P}\left[|\sigma_n \xi_{n+1}| < 3.915,\ n \in \mathbb{N}\right] > 0.95.$$

- *If* $\gamma = 0.01$ *then* $j(0.01) > 6.694$ *suffices for* (48) *to hold, and*

$$\mathbb{P}\left[|\sigma_n \xi_{n+1}| < 6.695,\ n \in \mathbb{N}\right] > 0.99.$$

6 Computing sufficient bounds on the stepsize parameter h

Recall the definition of $\bar{h}$ from the statement of Theorem 3.1. In this section we find implicit formulae for $\bar{h}$, and illustrate them with the examples from Section 5. Additionally, for any particular $X_0 \in \mathbf{R}$, we estimate an upper bound on the stepsize h in equation (1) that will place X_0 in the region of convergence $\mathcal{A}$.

6.1 Estimation of $\bar{h}$

Taking into consideration the proof of Lemma 2.10, $\bar{h}$ must be sufficiently small that $\varepsilon(h)$, defined in (23) by

$$\varepsilon(h) = 2^{1-\frac{1}{\nu}} h^{\frac{1}{2}+\frac{1}{\nu}} \nu^{-1} \beta^{\frac{1}{\nu}} j(\gamma),$$

satisfies (27) in Remark 1, and the following inequalities:

$$\frac{\nu\varepsilon(h)}{2} < 1 - (1-\varepsilon(h))^{\nu}, \tag{49}$$

$$\frac{\nu\varepsilon(h)}{2} < (1+\varepsilon(h))^{\nu} - 1. \tag{50}$$

We start by assuming that $\varepsilon(h) < \frac{1}{2}$. To keep the calculations simple we will restrict our examination to the case where $\nu > 1$. The case where $\nu < 1$ is similar.

We begin by determining when (49) holds. By examining a truncated series expansion of $(1-\varepsilon)^\nu$ we see that for $\varepsilon(h)=\varepsilon\in(0,1/2)$ there exists $\theta\in(0,\varepsilon)$ such that

$$(1-\varepsilon)^\nu = 1-\nu\varepsilon+\frac{\nu(\nu-1)(1-\theta)^{\nu-2}}{2}\varepsilon^2, \tag{51}$$

where $1-\theta\in(1/2,1)$ and

$$\begin{cases} 1>(1-\theta)^{2-\nu}>\left(\frac{1}{2}\right)^{2-\nu}, & \text{if} \quad \nu<2; \\ 1<(1-\theta)^{2-\nu}<\left(\frac{1}{2}\right)^{2-\nu}, & \text{otherwise.} \end{cases} \tag{52}$$

From (51) we see that (49) holds if

$$\frac{\nu(\nu-1)(1-\theta)^{\nu-2}}{2}\varepsilon^2<\frac{\nu\varepsilon}{2},$$

which will be the case if

$$\varepsilon=\varepsilon(h)<\frac{(1-\theta)^{2-\nu}}{\nu-1}. \tag{53}$$

Similarly, to determine when (50) holds, we see that for $\varepsilon\in\left(0,\frac{1}{2}\right)$, there exists $\theta\in(0,\varepsilon)$ such that

$$(1+\varepsilon)^\nu = 1+\nu\varepsilon+\frac{\nu(\nu-1)(1+\theta)^{\nu-2}}{2}\varepsilon^2, \tag{54}$$

where $1<1+\theta<3/2$ and

$$\begin{cases} 1<(1+\theta)^{2-\nu}<\left(\frac{3}{2}\right)^{2-\nu}, & \text{if} \quad \nu<2; \\ 1>(1+\theta)^{2-\nu}>\left(\frac{3}{2}\right)^{2-\nu}, & \text{otherwise.} \end{cases} \tag{55}$$

From (54) we see that (50) holds if

$$\frac{\nu(\nu-1)(1+\theta)^{\nu-2}}{2}\varepsilon^2<\frac{\nu\varepsilon}{2},$$

which will be the case if

$$\varepsilon=\varepsilon(h)<\frac{(1+\theta)^{2-\nu}}{\nu-1}. \tag{56}$$

Estimates (53) and (56) imply that

$$\varepsilon(h)<\frac{\min\{(1+\theta)^{2-\nu},\,(1-\theta)^{2-\nu}\}}{\nu-1}. \tag{57}$$

Let

$$G(\nu):=\begin{cases} \frac{1}{\nu-1}\left(\frac{1}{2}\right)^{2-\nu}, & \text{if } \nu<2; \\ \frac{1}{\nu-1}\left(\frac{3}{2}\right)^{2-\nu}, & \text{otherwise.} \end{cases} \tag{58}$$

In light of (52) and (55), the estimates (49) and (50) hold when $\varepsilon(h) < G(\nu)$, defined by (58). We note that $G(2) = 1$.

Taking (27) into consideration, we now see that h must be sufficiently small that

$$2^{1-\frac{1}{\nu}} h^{\frac{1}{2}+\frac{1}{\nu}} \nu^{-1} \beta^{\frac{1}{\nu}} j(\gamma) < \min\left\{\frac{1}{2},\, 1-q^*,\, G(\nu)\right\},$$

or, alternatively,

$$h < \bar{h} = \left(\frac{\nu \min\left\{\frac{1}{2},\, 1-q^*,\, G(\nu)\right\}}{2^{1-\frac{1}{\nu}} \beta^{\frac{1}{\nu}} j(\gamma)}\right)^{\frac{2\nu}{2+\nu}}. \tag{59}$$

The following example illustrates the use of this formula.

Example 6.1 *Consider equation*

$$X_{n+1} = X_n - hX_n^3 + \sqrt{h}\sigma_n \xi_{n+1}, \quad n = 1, 2 \dots. \tag{60}$$

Here, $\beta = 1$, $\nu = 2$. First we find q^* by following the argument in the proof of Lemma 2.7, which is contained in Section 8. Thus we must solve equation (69), which in this example takes the form

$$c^3 - 3c - 2 = 0.$$

This equation has 3 solutions, $c_{1,2} = -1$ and $c_3 = 2$. Since we are interested only in the positive root, we use $c_3 = 2$ in our calculations:

$$q^* = \frac{c(\nu)}{\sqrt[\nu]{2(1+\nu)}} = \frac{c(\nu)}{\sqrt[\nu]{2\times 3}} = \frac{2}{\sqrt{6}} = 0.81. \tag{61}$$

Since $G(2) = 1$ we have

$$\min\left\{\frac{1}{2},\, 1-q^*,\, G(\nu)\right\} = \min\{0.5, 0.19, 1\} = 0.19.$$

Thus, from (59) we obtain

$$\bar{h} = \frac{2\times 0.19}{2^{\frac{1}{2}} j(\gamma)} = \frac{0.269}{j(\gamma)}.$$

Now we use results from the examples given in Section 5 to fill in the value of $j(\gamma)$. Fix $\gamma = 0.05$ and consider the two classes of perturbation given in Examples 5.1 and 5.2. Example 5.1 indicates that for Normal perturbations we can take $j(0.05) = 7$ while Example 5.2 indicates that for perturbations with polynomial tails we can take $j(0.05) = 3.915$. Thus

$$\bar{h}_{(Normal)} = \frac{0.269}{7} \approx 0.0384, \quad \bar{h}_{(Polynomial)} = \frac{0.269}{3.915} \approx 0.0687.$$

6.2 Computing h sufficient for stability with confidence $1-\gamma$ when the initial value X_0 is given

Fix $X_0 \in \mathbb{R}$. We want to find h such that $X_0 \in \mathcal{A}$, where $\mathcal{A}$ is the region of convergence for solutions of (1). For simplicity we examine only initial values satisfying

$$|X_0| \le \sqrt[\nu]{\frac{2}{h\beta}},$$

where $h < \bar{h}$ as defined in (59), to guarantee the conditions of Theorem 3.1. Thus we obtain the following estimation for h:

$$h < \min\left\{\frac{2}{\beta X_0^{\nu}},\quad \left(\frac{\nu \min\left\{\frac{1}{2},\, 1-q^*,\, G(\nu)\right\}}{2^{1-\frac{1}{\nu}}\beta^{\frac{1}{\nu}} j(\gamma)}\right)^{\frac{2\nu}{2+\nu}}\right\}. \tag{62}$$

Since we have already computed values for the second term on the right hand side of (62) for two perturbation classes, it remains to compute values for the first term, and compare.

Example 6.2 *Consider again equation (60) from Example 6.1. For $X_0 =$ 1.9 we have*

$$\frac{2}{\beta X_0^{\nu}} = \frac{2}{1.9^2} = 0.055,$$

for $X_0 = 20$ we have

$$\frac{2}{\beta X_0^{\nu}} = \frac{2}{20^2} = 0.005,$$

while for $X_0 = 100$ we have

$$\frac{2}{\beta X_0^{\nu}} = \frac{2}{100^2} = 0.0002.$$

In the case where the perturbations are Normal, $\bar{h}_{(Normal)} \approx 0.0384$ will adequately simulate the $X_0 = 1.9$ solution at a 95% confidence level, but for $X_0 = 20$ or $X_0 = 100$, the stepsize h must be decreased accordingly. In the case where the perturbation has polynomial tails, $\bar{h}_{(Polynomial)} \approx 0.0684$ will not adequately simulate the solution at a 95% confidence level for any of these initial values; the stepsize h must be decreased.

7 Example: the dynamics of an Euler-Maruyama difference equation with slowly vanishing noise

7.1 Theoretical description of dynamics

Consider the Itô equation

$$dX(t) = X^3(t) + \frac{1}{[\log(t+1)]^{1.1}} dW(t), \quad t > 0, \tag{63}$$

where W is a standard Brownian motion. All solutions of (63) are a.s. asymptotically stable, by Chan & Williams [8]. A 95%-level description of the stability dynamics of the Euler-Maruyama discretisation

$$X_{n+1} = X_n - hX_n^3 + \sqrt{h}\frac{1}{[\log(n+1)]^{1.1}}\xi_{n+1}, \quad n \in \mathbb{N}, \tag{64}$$

where $\{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of independent standard Normal random variables, is an easy corollary of Theorem 3.1.

Corollary 7.1 *Let the sequence* $\{X_n\}_{n\in\mathbb{N}}$ *be a solution of* (64). *Then, for all* $h < 0.0384$*,*

1. $\mathbb{P}[\lim_{n\to\infty} X_n = 0] > 0.95$ *when*

$$X_0 \in \left(-\sqrt{\frac{2}{h}} + 7\sqrt{h}, \sqrt{\frac{2}{h}} - 7\sqrt{h}\right),$$

2. $\mathbb{P}[\limsup_{n\to\infty} X_n = \infty \text{ \& } \liminf_{n\to\infty} X_n = -\infty] > 0.95$*, when*

$$X_0 \in \left(-\infty, -\sqrt{\frac{2}{h}} - 7\sqrt{h}\right) \bigcup \left(\sqrt{\frac{2}{h}} + 7\sqrt{h}, \infty\right).$$

Proof The explicit formulae in (23) and (31) give $\bar{\varepsilon}(h) = 7\sqrt{h}$, and recall from Example 6.1 that $\bar{h}_{(\text{Normal})} = 0.0357$ when $\gamma = 0.05$. The statement of the corollary then follows from that of Theorem 3.1.

7.2 Numerical investigation of dynamics

It is reasonable to use numerical simulation to determine how fully Corollary 7.1 represents the stability-instability dynamics of solutions of (64). That is the purpose of this subsection. For all simulations that follow, real numbers are rationally approximated with 64-bit floating-point numbers satisfying the IEEE-754 standard, and the sequence of Gaussian numbers $\{\xi_n\}_{n\in\mathbb{N}}$ has been approximated with a sequence of pseudo-random Gaussian numbers generated with the `nextGaussian()` method of the `java.util.Random()` class. This method implements the polar form of the Box-Muller-Marsaglia transformation to generate independent pairs of Normally distributed pseudo-random numbers from independent pairs of pseudo-random numbers uniformly distributed over the interval $[0, 1]$. A full description can be found, for example, in Section 3.4.1, Subsection C of Knuth [12].

7.2.1 An algorithm for approximating the probability of stability of solutions of (64)

The following procedure will be applied across a range of initial values of (64):

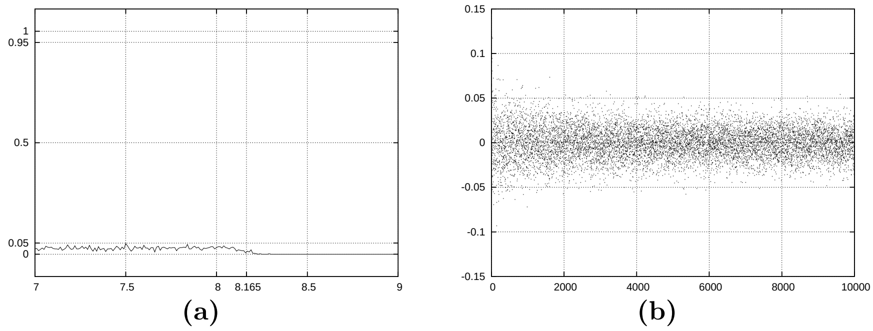


Figure 1: **(a)**: simulated ($\varepsilon = 0.01$) probabilities of stability of solutions of (64), across initial values uniformly spaced 0.01 apart on the interval $[7, 9]$. $h = 0.03$ and $\sqrt{2/h} = 8.165$. **(b)**: a simulation of $\left\{\sqrt{h}\left(1/[\log(n+1)]^{1.1}\right)\xi_{n+1}\right\}_{n=1}^{100{,}000}$ with $h = 0.03$.

1. Choose an initial value X_0;
2. Initialise a counter `numStabPaths` to zero;
3. Simulate a path of length $N = 1000$;
4. If $|X_{999}| < \varepsilon$, for some $\varepsilon > 0$, then increment `numStabPaths`. Otherwise, do nothing;
5. Repeat 500 times with independent sets of Gaussian random numbers;
6. The quantity `numStabPaths`/500 represents the simulated probability of stability of the solution of (64) corresponding to X_0.

The choice of ε is important. In Figure 1, Part (a), we see that, when $\varepsilon = 0.01$, the simulated probabilities of stability are inconsistent with the statement of Corollary 7.1: they are close to zero everywhere. In Figure 1, Part (b), we see why: the magnitude of state-independent stochastic perturbation has frequent deviations above 0.01 over an interval 100 times longer than the interval of simulation. Although $\{\sigma_n\xi_n+1\}_{n\in\mathbb{N}}$ obeys (5) and therefore converges asymptotically to zero with probability one, it does so very slowly when $\sigma_n = 1/[\log(n+1)]^{1.1}$, $n \in \mathbb{N}$. Since a finite state machine must classify the asymptotic stability of paths by observing each one over a finite time-set, setting the value of ε too low will yield false negatives: stable paths will be mistaken for unstable paths.

The solution is to raise the value of ε. In fact we can choose ε to be very large, without incurring false positives, since solutions of the unperturbed difference equation

$$x_{n+1} = x_n - hx_n^3, \quad n \in \mathbb{N}, \quad x_0 > \sqrt{\frac{2}{h}},$$

grow in magnitude so quickly that their values exceed the overflow bounds of 64-bit floating point number within a few timesteps. Therefore, unstable paths are unlikely to be mistaken for stable paths. A more detailed numerical investigation of this phenomenon, for polynomial difference equations with state-dependent perturbations, was presented in Kelly & Morgan [11]. To generate all following simulations (illustrated in Figures 2 and 3), we set $\varepsilon = 1.79769313486231570 \times 10^{308}$. This is the largest rational number that can be represented in a 64-bit floating point format satisfying the IEEE-754 standard.

7.2.2 The admissible range of values of h

The statement of Corollary 7.1 holds for all $h < \bar{h} = 0.0384$, but gives no indication whether or not larger values of h will provide useful numerical results. We can investigate this directly. Consider Figure 2. Parts (a)-(d) show simulated probabilities of stability for solutions of (64) for increasing values of h. In part (a), $h = 0.03 < 0.0384$, and the simulated probabilities of stability are consistent with the statement of Corollary 7.1. However, we see that the simulated probability of stability in the region of stability remains above 0.95 when $h \lesssim 0.375$. This indicates that $\bar{h}$ is conservatively low.

7.2.3 The significance of the 'blind spot'

h	$[\tilde{f}_h, \tilde{l}_h]$	$[f_h, l_h]$
0.02	$[9.93, 10.07]$	$[9.01, 10.99]$
0.01	$[14.1, 14.2]$	$[13.442, 14.842]$

Table 1: Comparative values of $\tilde{f}_h$, $\tilde{l}_h$, f_h, l_h, when $h < 0.0384 = \bar{h}$.

The four cases illustrated in Figure 2 indicate that, in our simulations, the transition from stability with high probability to stability with low probability occurs smoothly over an interval roughly centred on $\sqrt{2/h}$. For fixed h, we denote the simulated interval of transition $[\tilde{f}_h, \tilde{l}_h]$, where

$$\begin{aligned} \tilde{f}_h &:= \min\{X_0 : \text{the simulated probability of stability is less than } 0.95\}, \\ \tilde{l}_h &:= \min\{X_0 : \text{the simulated probability of stability is less than } 0.05\}. \end{aligned}$$

Corollary 7.1 has nothing to say about the asymptotic stability of solu-

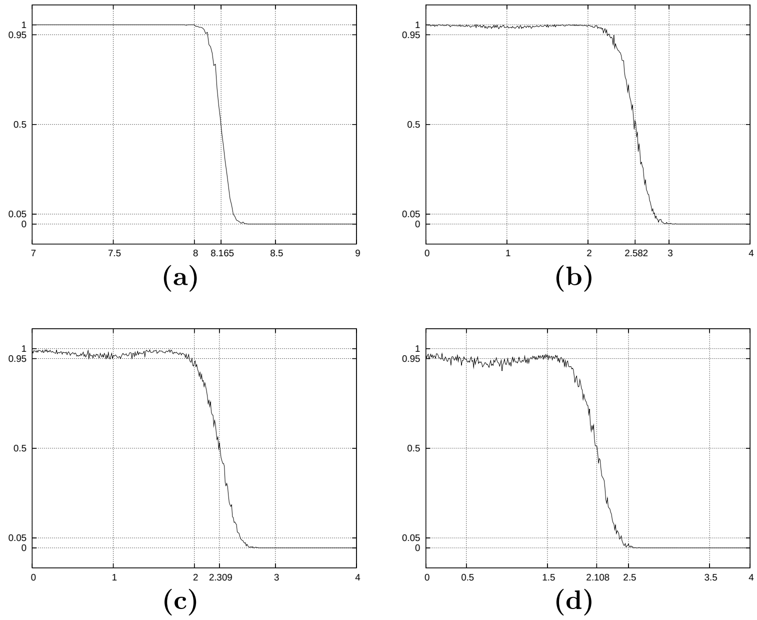


Figure 2: Simulated ($\varepsilon = 1.79769313486231570 \times 10^{308}$) probabilities of stability of solutions of (64) across initial values uniformly spaced 0.01 apart on an interval $[a, b]$. In **(a)**, $[a, b] = [7, 9]$, $h = 0.03$, $\sqrt{2/h} = 8.165$. In **(b)**, $[a, b] = [0, 4]$, $h = 0.3$, $\sqrt{2/h} = 2.582$. In **(c)**, $[a, b] = [0, 4]$, $h = 0.375$, $\sqrt{2/h} = 2.309$. In **(d)**, $[a, b] = [0, 4]$, $h = 0.45$, $\sqrt{2/h} = 2.108$.

tions of (64) when

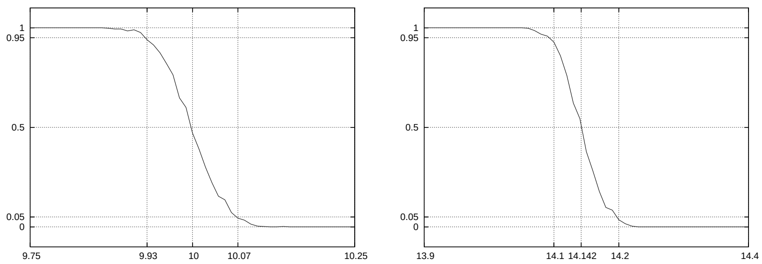
$$X_0 \in [f_h, l_h] := \left[\sqrt{\frac{2}{h}} \pm 7\sqrt{h} \right], \tag{65}$$

and it is reasonable to investigate the relationship between this 'blind spot' and the simulated intervals of transition.

In Table 1, we compare $[\tilde{f}_h, \tilde{l}_h]$ with the corresponding $[f_h, l_h]$ for two values of $h < \bar{h}$. We see that, in each case, the 'blind spot' $[f_h, l_h]$ is larger than the simulated interval of transition $[\tilde{f}_h, \tilde{l}_h]$. The corresponding simulations are illustrated in Figure 3

7.2.4 Lessons drawn from numerical simulation

With a carefully designed algorithm for simulating the probabilities of stability of solutions of (64), we see that, although our simulations are consistent with the predictions of Corollary 7.1, the statement of the corollary appears



(a) (b)

Figure 3: Simulated ($\varepsilon = 1.79769313486231570 \times 10^{308}$) probabilities of stability of solutions of (64) across initial values uniformly spaced 0.01 apart on an interval $[a, b]$. In **(a)**, $[a, b] = [9.75, 10.25]$, $h = 0.02$, $\sqrt{2/h} = 10$, $[\tilde{f}_h, \tilde{l}_h] = [9.93, 10.07]$. In **(b)**, $[a, b] = [13.9, 14.4]$, $h = 0.01$, $\sqrt{2/h} = 14.142$, $[\tilde{f}_h, \tilde{l}_h] = [14.1, 14.2]$

to be incomplete in two ways. First, the upper limit $\bar{h}$ on the stepsize h is too small. Second, the corollary ignores regions of the initial value set in which solutions display stability or instability with simulated probabilities greater than 0.95.

8 Proofs of Lemmas 2.7, 2.10 and 2.11

Proof of Lemma 2.7 We address each part in turn.

(a) Since $[F_h]'(x) = 1 - h(1+\nu)\beta x^{\nu}$, we have

$$x_{max} = \frac{1}{\sqrt[\nu]{h(1+\nu)\beta}}, \quad [F_h]_{max} = \frac{1}{\sqrt[\nu]{h(1+\nu)\beta}} \frac{\nu}{1+\nu}.$$

Note that the function F_h is strictly decreasing on $\left[1/\sqrt[\nu]{h(1+\nu)\beta}, \infty\right)$.
(b) Let x_1 be a point of the intersection of $y = F_h(x)$ with OX:

$$F_h(x) = x - h\beta x^{1+\nu} = 0, \quad x_1 = \frac{1}{\sqrt[\nu]{h\beta}}.$$

(c) Let x_3 be a point of the intersection of $y = F_h(x)$ with $y = -x$:

$$F_h(x) = x - h\beta x^{1+\nu} = -x, \quad x_3 = \sqrt[\nu]{\frac{2}{h\beta}}.$$

(d) To find a point $x_2 \in \left(\frac{1}{\sqrt[\nu]{h(1+\nu)}}, \infty\right)$ such that $F_h(x_2) = -[F_h]_{max}$, we need to solve the equation

$$F_h(x_2) = -\frac{1}{\sqrt[\nu]{h(1+\nu)\beta}}\frac{\nu}{1+\nu}. \tag{66}$$

Since

$$[F_h]_{max} = \frac{1}{\sqrt[\nu]{h(1+\nu)\beta}}\frac{\nu}{1+\nu} < x_{max} = \frac{1}{\sqrt[\nu]{h(1+\nu)\beta}},$$

point x_2 lies between x_1 and x_3, i.e.

$$x_1 < x_2 < x_3. \tag{67}$$

In order to solve equation (66), we represent point x_2 in the form

$$x_2 = \frac{c(\nu)}{\sqrt[\nu]{h(1+\nu)\beta}}$$

and show that it is possible to find $c = c(\nu)$, which does not depend on h, such that

$$F_h\left(\frac{c(\nu)}{\sqrt[\nu]{h(1+\nu)\beta}}\right) = -\frac{1}{\sqrt[\nu]{h(1+\nu)\beta}}\frac{\nu}{1+\nu}.$$

In other words we need to solve the equation

$$\frac{c}{\sqrt[\nu]{h(1+\nu)\beta}}\left(1 - \frac{c^\nu}{1+\nu}\right) = -\frac{1}{\sqrt[\nu]{h(1+\nu)\beta}}\frac{\nu}{1+\nu}. \tag{68}$$

After multiplying (68) by $\sqrt[\nu]{h(1+\nu)\beta}$ we obtain for $c = c(\nu)$,

$$c^{\nu+1} - (1+\nu)c - \nu = 0. \tag{69}$$

Note that the solution, $c = c(\nu)$, does not depend on h, but only on ν.

Substituting the values of x_1, x_2 and x_3 into (67), we arrive at

$$\frac{1}{\sqrt[\nu]{h\beta}} < \frac{c(\nu)}{\sqrt[\nu]{h(1+\nu)\beta}} < \frac{\sqrt[\nu]{2}}{\sqrt[\nu]{h\beta}},$$

which is equivalent to

$$\sqrt[\nu]{1+\nu} < c(\nu) < \sqrt[\nu]{2(1+\nu)}. \tag{70}$$

To show that equation (66) has the root $c(\nu)$ which satisfies inequality (70), we consider the function

$$\chi(c) = c^{\nu+1} - (1+\nu)c - \nu,$$

and note that $\chi(\sqrt[\nu]{1+\nu}) < 0$ and $\chi(\sqrt[\nu]{2(1+\nu)}) > 0$.

Since the function $y = \chi(x)$ increases for $x > 1$, the equation $\chi(c) = 0$ has only one positive root

$$c(\nu) \in \left(\sqrt[\nu]{1+\nu},\ \sqrt[\nu]{2(1+\nu)} \right).$$

Then

$$\frac{x_2}{x_3} = \frac{\frac{c(\nu)}{\sqrt[\nu]{h(1+\nu)\beta}}}{\sqrt[\nu]{\frac{2}{h\beta}}} = \frac{c(\nu)}{\sqrt[\nu]{2(1+\nu)}} < 1.$$

Proof of Lemma 2.10 If $q \in (q^*, 1)$, then

$$q\sqrt[\nu]{\frac{2}{h\beta}} \in \left(\frac{c(\nu)}{\sqrt[\nu]{h(1+\nu)\beta}}, \sqrt[\nu]{\frac{2}{h\beta}} \right) = (x_2, x_3).$$

Since F decreases on (x_2, ∞) and is symmetric with respect to OY, we have:

$$\begin{aligned} \max_{x \in I_q^M} F(x) = -F\left(q\sqrt[\nu]{\frac{2}{h\beta}}\right) &= -q\sqrt[\nu]{\frac{2}{h\beta}} + h\beta\left(q\sqrt[\nu]{\frac{2}{h\beta}}\right)^{1+\nu} \\ &= q(2q^\nu - 1)\sqrt[\nu]{\frac{2}{h\beta}}. \end{aligned}$$

Also,

$$\begin{aligned} \min_{x \in I_q^M} F(x) = F\left(q\sqrt[\nu]{\frac{2}{h\beta}}\right) &= q\sqrt[\nu]{\frac{2}{h\beta}} - h\beta\left(q\sqrt[\nu]{\frac{2}{h\beta}}\right)^{1+\nu} \\ &= -q(2q^\nu - 1)\sqrt[\nu]{\frac{2}{h\beta}}. \end{aligned}$$

Now we find the bounds for C which ensure that (21) holds true. In order for (21) to be fulfilled for all $x \in I_q^M$, we need to have

$$x - h\beta x^{1+\nu} + \sqrt{h}C \le q\sqrt[\nu]{\frac{2}{h\beta}} \qquad \text{and} \qquad x - h\beta x^{1+\nu} - \sqrt{h}C \ge -q\sqrt[\nu]{\frac{2}{h\beta}}.$$

We estimate

$$x - h\beta x^{1+\nu} + \sqrt{h}C \le q(2q^\nu - 1)\sqrt[\nu]{\frac{2}{h\beta}} + \sqrt{h}|C| \le q\sqrt[\nu]{\frac{2}{h\beta}}, \tag{71}$$

and

$$x - h\beta x^{1+\nu} - \sqrt{h}C \ge -q(2q^\nu - 1)\sqrt[\nu]{\frac{2}{h\beta}} - \sqrt{h}|C| \ge -q\sqrt[\nu]{\frac{2}{h\beta}}. \tag{72}$$

Both inequalities (71) and (72) hold if $|C|$ satisfies

$$\sqrt{h}|C| \le 2q\sqrt[\nu]{\frac{2}{h\beta}}(1 - q^\nu). \tag{73}$$

Fix $H > 0$ and let $\varepsilon(h)$ be as defined in (23). We find $\bar{h}(\gamma,\nu,\beta)$ such that (27) in Remark 1 holds, and so that for all $h \le \bar{h}(\gamma,\nu,\beta)$

$$1-(1-\varepsilon(h))^\nu > \frac{\nu\varepsilon(h)}{2} \quad \text{and} \quad (1+\varepsilon(h))^\nu - 1 > \frac{\nu\varepsilon(h)}{2}. \tag{74}$$

We note that (27) implies that, for all $h \le \bar{h}(\gamma,\nu,\beta)$,

$$\varepsilon(h)-\varepsilon^2(h) \ge \frac{\varepsilon(h)}{2}.$$

Then for all $h \le \bar{h}(\gamma,\nu,\beta)$ we have

$$\begin{aligned}
2q(1-q^\nu)\sqrt[\nu]{\frac{2}{h\beta}} &= 2(1-\varepsilon(h))\left(1-(1-\varepsilon(h))^\nu\right)\sqrt[\nu]{\frac{2}{h\beta}} \\
&\ge 2(1-\varepsilon(h))\frac{\nu\varepsilon(h)}{2}\sqrt[\nu]{\frac{2}{h\beta}} \\
&\ge \frac{\nu\varepsilon(h)}{2}\sqrt[\nu]{\frac{2}{h\beta}} \\
&= \nu 2^{-1+1/\nu}\beta^{-1/\nu}h^{-1/\nu}2^{1-\frac{1}{\nu}}h^{\frac{1}{2}+\frac{1}{\nu}}\nu^{-1}\beta^{\frac{1}{\nu}}H \\
&= \sqrt{h}H,
\end{aligned} \tag{75}$$

which implies (73), and the statement of the lemma.

Proof of Lemma 2.11 Let F be defined in (15) and q is some number from $(q^*,1)$. From Lemma 2.10 we have:

$$\max_{x\in I_q^M} F(x) = q(2q^\nu-1)\sqrt[\nu]{\frac{2}{h\beta}}, \qquad \min_{x\in I_q^M} F(x) = -q(2q^\nu-1)\sqrt[\nu]{\frac{2}{h\beta}}.$$

Condition (25) is fulfilled if for all $x \in I_q^M$,

$$\begin{aligned}
&x - h\beta x^{1+\nu} + \sqrt{h}x^{\nu_1}|C| \\
&\qquad \le q(2q^\nu-1)\sqrt[\nu]{\frac{2}{h\beta}} + \sqrt{h}q^{\nu_1}\left(\sqrt[\nu]{\frac{2}{h\beta}}\right)^{\nu_1}|C| \le q\sqrt[\nu]{\frac{2}{h\beta}},
\end{aligned}$$

and

$$\begin{aligned}
&x - h\beta x^{1+\nu} - \sqrt{h}x^{\nu_1}|C| \\
&\qquad \ge -q(2q^\nu-1)\sqrt[\nu]{\frac{2}{h\beta}} - \sqrt{h}q^{\nu_1}\left(\sqrt[\nu]{\frac{2}{h\beta}}\right)^{\nu_1}|C| \ge -q\sqrt[\nu]{\frac{2}{h\beta}}.
\end{aligned}$$

Both inequalities hold when

$$\sqrt{h}q^{\nu_1}\left(\sqrt[\nu]{\frac{2}{h\beta}}\right)^{\nu_1}|C| \le q\sqrt[\nu]{\frac{2}{h\beta}} - q(2q^\nu-1)\sqrt[\nu]{\frac{2}{h\beta}} = 2q\sqrt[\nu]{\frac{2}{\beta h}}(1-q^\nu).$$

Thus (25) is fulfilled when

$$\sqrt{h}|C| \quad \leq \quad \left(\sqrt[\nu]{\frac{2}{h\beta}}\right)^{1-\nu_1} 2q^{1-\nu_1}(1-q^{\nu}). \tag{76}$$

Now we define $K_1 = \min\{1, 2^{1-\nu_1}\}$ and define $\varepsilon(h)$ as in (26), for fixed H. Since $\frac{1}{2} + \frac{1}{\nu} - \frac{\nu_1}{\nu} > 0$, for any fixed ν, ν_1, H and β,

$$\lim_{h\to 0} \varepsilon(h) = 0.$$

Therefore we can find $\bar{h}(\gamma, \nu, \nu_1, \beta)$ such that for all $h \leq \bar{h}(\gamma, \nu, \nu_1, \beta)$

$$1 - (1 - \varepsilon(h))^{\nu} > \frac{\nu\varepsilon(h)}{2},$$

and such that (27) in Remark 1 holds. (27) implies that $\frac{1}{2} < 1 - \varepsilon(h) < 1$ for all $h \leq \bar{h}(\gamma, \nu, \nu_1, \beta)$, which in turn implies that

$$(1 - \varepsilon(h))^{\nu_1 - 1} > K_1 = \min\{1, 2^{1-\nu_1}\}.$$

We define $q = q(h) = 1 - \varepsilon(h)$. Then for all $h \leq \bar{h}(\gamma, \nu, \nu_1, \beta)$ we have

$$\begin{aligned}
&2q^{1-\nu_1}(1-q^{\nu})\left(\frac{2}{h\beta}\right)^{\frac{1-\nu_1}{\nu}} \\
&= \quad 2(1-\varepsilon(h))^{1-\nu_1}(1-(1-\varepsilon(h))^{\nu})\left(\frac{2}{h\beta}\right)^{\frac{1-\nu_1}{\nu}} \\
&\geq \quad 2K_1\frac{\nu\varepsilon(h)}{2}\left(\frac{2}{h\beta}\right)^{\frac{1-\nu_1}{\nu}} \\
&= \quad K_1\nu 2^{\frac{1}{\nu}-\frac{\nu_1}{\nu}}\beta^{-\frac{1}{\nu}+\frac{\nu_1}{\nu}}h^{-\frac{1}{\nu}+\frac{\nu_1}{\nu}}\varepsilon(h) \\
&= \quad H\sqrt{h},
\end{aligned}$$

which implies (76) and completes the proof.

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