Observations on the basic (G'/G)-expansion method for finding solutions to nonlinear evolution equations

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Abstract

The extended tanh-function expansion method for finding solutions to nonlinear evolution equations delivers solutions in a straightforward manner and in a neat and helpful form. On the other hand, the more recent but less efficient (G'/G)-expansion method delivers solutions in a rather cumbersome form. It is shown that these solutions are merely disguised forms of the solutions given by the earlier method so that the two methods are entirely equivalent. An unfortunate consequence of this observation is that, in many papers in which the (G'/G)-expansion method has been used, claims that 'new' solutions have been derived are often erroneous; the so-called 'new' solutions are merely disguised versions of previously known solutions.

Key words: Nonlinear evolution equations; the tanh-function expansion method; the (G'/G)-expansion method; solitary travelling-waves.
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1 Introduction

Over the past two decades several expansion methods for finding travelling-wave solutions to nonlinear evolution equations have been proposed, developed and extended. The solutions to dozens of equations have been found by one or other of these methods. Ref. [1] and references therein mention some of this activity. One of the more recent methods is the (G'/G)-expansion method. Refs. [1–14] are a representative selection of papers in which the basic (G'/G)-expansion method is used. Not surprisingly, the basic method has been extended already (see [15,16], for example).

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The aim of the present paper is to show that the extended tanh-function expansion method proposed by Fan in 2000 [17] and the basic (G'/G)-expansion method proposed eight years later by Wang et al [1] are entirely equivalent in as much as they deliver exactly the same set of solutions to a given evolution equation. However, the former method delivers solutions in a straightforward way and in a neat and helpful form. More effort is required when using the latter method, and the solutions are delivered in a rather cumbersome form. An unfortunate consequence of this observation is that, in many papers in which the latter more recent method has been used, claims that 'new' solutions have been derived are often erroneous; the so-called 'new' solutions are merely disguised versions of previously known solutions.

Recently, in a series of enlightening papers [18–20], Kudryashov has pointed out the danger of not recognizing that apparently different solutions may simply be different forms of the same solution. He has provided numerous examples to illustrate this phenomenon. Recently, we have made some complementary observations [21–23].

In Section 2 we outline the basic tanh-function method and the modification to it that results in the so-called 'extended tanh-function method'. (It should be noted that there is not a consistent usage of this terminology in the literature.) In Section 3 we outline the basic (G'/G)-expansion method. In Section 4 we show that the two methods are entirely equivalent. In Section 5 we give an illustrative example. A brief conclusion is given in Section 6.

2 The extended tanh-function expansion method

Before establishing a convenient formulation of the extended tanh-function method, we outline the basic tanh-function method as given in [24].

Suppose we are given a nonlinear evolution equation in the form of a PDE for a function $u(x,t)$. The tanh-function method for solving this equation proceeds in the following steps:

(1) Seek travelling wave solutions by taking $u(x,t) = U(\eta)$, where

$$\eta = x - ct - x_0$$

with c a real constant and x_0 an arbitrary real constant. Substitution into the evolution equation yields an ODE for $U(\eta)$.

(2) If possible, integrate the ODE from step (1) term by term one or more times. This introduces one or more constants of integration.

(3) Introduce the ansatz

$$U(\eta) = \sum_{i=0}^{M} a_i Y^i, \qquad \text{where} \qquad Y := \tanh(k\eta). \tag{2.1}$$

Y satisfies the differential equation

$$\frac{dY}{d\eta} = k(1 - Y^2) \tag{2.2}$$

so that

$$\frac{d}{d\eta} \equiv k(1 - Y^2)\frac{d}{dY}\,. \tag{2.3}$$

Here M is a positive integer (to be determined). The a_i $(i = 0, \ldots, M)$ are real constants with $a_M \neq 0$, and k is a real non-zero constant. Substitution of (2.1) and (2.3) into the ODE from step (1) or step (2) yields an algebraic equation in powers of Y.

(4) Determine M (if possible); usually this involves balancing the linear term(s) of highest order in the algebraic equation from step (3) with the highest-order nonlinear term(s).

(5) With M as determined in step (4), equate the coefficients of each power of Y to zero in the algebraic equation from step (3). This yields a system of algebraic equations involving the a_i $(i = 0, \ldots, M)$, k, c and, if the integrations in step (2) are performed, the integration constants. If the original evolution equation contains some arbitrary constant coefficients, these will, of course, also appear in the system of algebraic equations. If it is possible to find a real non-trivial solution to these equations, the method has worked successfully.

Notice that (2.2) is also satisfied by $Y := \coth(k\eta)$. This was pointed out in [25]. It follows that a coth-function expansion solution may be obtained from a tanh-function expansion solution simply by replacing 'tanh' by 'coth'. This result can also be deduced by applying the transformation $kx_0 \to kx_0 + i\pi/2$ to $\tanh(k\eta)$ to obtain $\coth(k\eta)$. With $Y := \tanh(k\eta)$, a bounded solitary-wave solution is obtained, whereas with $Y := \coth(k\eta)$, an unbounded solitary-wave solution is obtained.

Fan [17] extended the basic method as follows. In step (3), (2.1)–(2.3) are replaced by (2.4)–(2.6) respectively. The ansatz is

$$U(\eta) = \sum_{i=0}^{M} a_i Y^i, \tag{2.4}$$

where Y now satisfies the differential equation

$$\frac{dY}{d\eta} = k^2 - Y^2 \tag{2.5}$$

so that

$$\frac{d}{d\eta} \equiv (k^2 - Y^2)\frac{d}{dY}\,. \tag{2.6}$$

If $k^2 > 0$ (so that k is real),

$$Y := k\tanh(k\eta) \quad \text{or} \quad Y := k\coth(k\eta). \tag{2.7}$$

If $k^2 < 0$ (so that k is imaginary), write $k = iK$, where K is real, and then

$$Y := -K\tan(K\eta) \quad \text{or} \quad Y := K\cot(K\eta). \tag{2.8}$$

If $k = 0$,

$$Y := \frac{1}{\eta}\,. \tag{2.9}$$

Step (4) is unchanged. However, at the end of step (5), there are now five possible expressions for Y, as given by (2.7)–(2.9), that can be substituted back into (2.4) as appropriate.

3 The basic (G'/G)-expansion method

In some respects, the formulation of the basic (G'/G)-expansion method, as proposed by Wang et al [1], is similar to that of the tanh-function expansion method. The procedure is as follows.

(1) This step is similar to step (1) in Section 2 but with $u(x,t) = U(\xi)$, where

$$\xi = x - ct.$$

(2) This is the same as step (2) in Section 2.

(3) Introduce the ansatz

$$U(\xi) = \sum_{i=0}^{M} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{3.1}$$

where $G = G(\xi)$ satisfies the differential equation

$$G'' + \lambda G' + \mu G = 0, \tag{3.2}$$

λ, μ and the α_i $(i = 0, \ldots, M)$ are real constants with $\alpha_M \neq 0$, and the prime denotes differentiation with respect to ξ. Substitution of (3.1) into the ODE from step (1) or step (2), and use of (3.2), yields an algebraic equation in powers of G'/G.

(4) Determine M by the balancing act described in step (4) in Section 2.

(5) With M as determined in step (4), equate the coefficients of each power of G'/G to zero in the algebraic equation from step (3). This yields a system of algebraic equations involving the α_i $(i = 0, \ldots, M)$, λ, μ, c and, if appropriate, constant coefficients from the original evolution equation and integration constants. If it is possible to find a real non-trivial solution to these equations, the method has worked successfully. Finally, the general solution of (3.2) can be substituted into (3.1).

4 Equivalence of the extended tanh and (G'/G)-expansion methods

First let us suppose that $\lambda^2 - 4\mu \neq 0$. Then the solution to (3.2) is

$$G = Ae^{m_1\xi} + Be^{m_2\xi}, \tag{4.1}$$

where A and B are constants and

$$m_1 = -\frac{\lambda}{2} + k, \quad m_2 = -\frac{\lambda}{2} - k, \quad k = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,. \tag{4.2}$$

After some manipulation, we find that

$$\frac{G'}{G} = -\frac{\lambda}{2} + k\tanh[k(\xi - \xi_0)], \tag{4.3}$$

where

$$e^{2k\xi_0} = \frac{B}{A}, \quad \text{i. e.} \quad \tanh(k\xi_0) = \frac{B-A}{B+A}\,. \tag{4.4}$$

If $\lambda^2 - 4\mu > 0$, k, A and B are real. With $B/A > 0$, (4.3) becomes

$$\frac{G'}{G} = -\frac{\lambda}{2} + k\tanh(k\eta) \tag{4.5}$$

and with $B/A < 0$, (4.3) becomes

$$\frac{G'}{G} = -\frac{\lambda}{2} + k\coth(k\eta), \tag{4.6}$$

where, in (4.5) and (4.6), $2kx_0 = \ln|B/A|$.

If $\lambda^2 - 4\mu < 0$, $K := \frac{1}{2}\sqrt{4\mu - \lambda^2}$ and $A+B$ are real and $A-B$ is imaginary. Then with $k = iK$, (4.5) and (4.6) become

$$\frac{G'}{G} = -\frac{\lambda}{2} - K\tan(K\eta) \tag{4.7}$$

and

$$\frac{G'}{G} = -\frac{\lambda}{2} + K\cot(K\eta) \tag{4.8}$$

respectively.

In passing we note that the solution (4.1) may be written in the alternative form

$$G = e^{-\lambda/2}(C_1\cosh k\xi + C_2\sinh k\xi), \tag{4.9}$$

where $C_1 \equiv A + B$ and $C_2 \equiv A - B$. After some manipulation we find that

$$\frac{G'}{G} = -\frac{\lambda}{2} + k\left(\frac{C_1\sinh k\xi + C_2\cosh k\xi}{C_1\cosh k\xi + C_2\sinh k\xi}\right). \tag{4.10}$$

With $\tanh k\xi_0 := -C_2/C_1$, (4.10) may be written exactly as (4.3) by use of the identities

$$\cosh(P-Q) = \cosh P \cosh Q - \sinh P \sinh Q,$$
$$\sinh(P-Q) = \sinh P \cosh Q - \cosh P \sinh Q.$$

If $\lambda^2 - 4\mu = 0$ then $m_1 = m_2 = -\lambda/2$ and

$$G = (D\xi + E)e^{-\lambda\xi/2}, \tag{4.11}$$

where D and E are real constants. After some manipulation, we find that

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{1}{\eta}, \tag{4.12}$$

where $x_0 = -E/D$.

When G'/G, as given by one of (4.5)–(4.8) or (4.12), is substituted into (3.1) we obtain exactly the same results as by substituting Y, as given by one of the expressions in (2.6)–(2.9), into (2.4).

5 Illustrative example

As a simple illustrative example, consider the KdV equation in the form

$$u_t + uu_x + u_{xxx} = 0. \tag{5.1}$$

For the extended tanh-function method, (2.4) is

$$U(\eta) = c + 8k^2 - 12Y^2. \tag{5.2}$$

By using (2.7)–(2.9) in (5.2), it is straightforward to obtain the following five solutions:

$$u = c + 8k^2 - 12k^2 \tanh^2(k\eta), \tag{5.3}$$
$$u = c + 8k^2 - 12k^2 \coth^2(k\eta), \tag{5.4}$$
$$u = c - 8K^2 - 12K^2 \tan^2(K\eta), \tag{5.5}$$
$$u = c - 8K^2 - 12K^2 \cot^2(K\eta), \tag{5.6}$$
$$u = c - \frac{12}{\eta^2}. \tag{5.7}$$

For the (G'/G)-expansion method, (3.1) is

$$U(\eta) = c - \lambda^2 - 8\mu - 12\lambda\left(\frac{G'}{G}\right) - 12\left(\frac{G'}{G}\right)^2. \tag{5.8}$$

By using (4.5)–(4.8) and (4.12) in (5.8), and after some manipulation, we obtain the five solutions given by (5.3)–(5.7).

In [1], the solution with $\lambda^2 - 4\mu > 0$ was arrived at by using the expression for G'/G given in (4.10), and presented in the form

$$u = \alpha_0 + 3\lambda^2 - 3(\lambda^2 - 4\mu)\left(\frac{C_1 \sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi) + C_2 \cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi)}{C_1 \cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi) + C_2 \sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi)}\right)^2, \quad (5.9)$$

where

$$\xi = x - (\alpha_0 + 8\mu + \lambda^2)t. \quad (5.10)$$

In our opinion, this form is most unhelpful to the reader; this solution in the form (5.3) or (5.4) is more user-friendly. Similar comments apply to the solution in [1] with $\lambda^2 - 4\mu < 0$.

Note that the solutions (5.4)–(5.7) are unbounded. The bounded solution (5.3) is probably the most useful solution. In Section 3 of [24] it was obtained directly by using an automated version of the basic tanh-function method, namely the Mathematica package known as `ATFM`.

6 Conclusion

We have shown that, for a given nonlinear evolution equation, the extended tanh-function method and the basic (G'/G)-expansion method give exactly the same set of solutions. In our opinion the former method is more efficient.

Many authors claim to have derived new solutions by using the (G'/G)-expansion method when they have merely derived solutions that are disguised versions of previously known solutions.

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