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Observations on the tanh-coth expansion method for finding solutions to nonlinear evolution equations

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Abstract

The ‘tanh-coth expansion method’ for finding solitary travelling-wave solutions to nonlinear evolution equations has been used extensively in the literature. It is a natural extension to the basic tanh-function expansion method which was developed in the 1990s. It usually delivers three types of solution, namely a tanh-function expansion, a coth-function expansion, and a tanh-coth expansion. It is known that, for every tanh-function expansion solution, there is a corresponding coth-function expansion solution. It is shown that there is a tanh-coth expansion solution that is merely a disguised version of the coth solution. In many papers, such tanh-coth solutions are erroneously claimed to be ‘new’. However, other tanh-coth solutions may be delivered that are genuinely new in the sense that they would not be delivered via the basic tanh-function method. Similar remarks apply to tan, cot and tan-cot expansion solutions.

Key words: Nonlinear evolution equations; the tanh-function expansion method; the tanh-coth function expansion method; solitary travelling-waves.

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1 Introduction

‘Expansion’ methods for finding solitary travelling-wave solutions to nonlinear evolution equations have received considerable attention over the past dozen years or so. The basic tanh-function method was introduced in the early 1990s (see [1] and

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references therein, for example). Since then the method has been extended, generalized and adapted, the motivation being to obtain more solutions than are delivered by the basic tanh-function method.

The aim of this paper is to discuss one such extension of the basic tanh-function method. In this paper we refer to this extension as the ‘tanh-coth method’. It was proposed in its simplest form in [2], for example. The reader should be warned that there is no uniform terminology in the literature: the method, and slight variants upon it, have been referred to as the ‘tanh-coth method’ (see [3], for example), the ‘extended tanh-function method’ (see [2], for example) or the ‘modified extended tanh-function method’ (see [4], for example).

We will show that if the tanh-coth method is applied to a nonlinear evolution equation that has a tanh-function expansion solution, then the method will also deliver a coth-function expansion solution and a tanh-coth expansion solution that is merely a disguised version of the coth solution. One unfortunate consequence of this observation is that many papers have been published in which it is claimed that such tanh-coth solutions are ‘new’, whereas the truth is that these ‘new’ solutions are merely disguised versions of previously known solutions. We shall give some examples of tanh-coth solutions that are not simply disguised versions of a coth solution and so may indeed be regarded as new in the sense that they would not be delivered by the basic tanh-function method.

In a series of enlightening papers [5–7], Kudryashov has pointed out the danger of not recognizing that apparently different solutions may simply be different forms of the same solution. He has provided numerous examples to illustrate this phenomenon. Recently, we have given two more examples [8,9].

In Section 2 we summarize the basic tanh-function method and explain how the tanh-coth method is an obvious extension. In Section 3 we make various observations on the consequences of the application of the method. An illustrative example is given in Section 4 and concluding comments are made in Section 5. Some useful identities are given in the Appendix.

2 The tanh-coth method

Suppose we are given a nonlinear evolution equation in the form of a PDE for a function $u(x, t)$. Briefly, the tanh-function method [1] for solving this equation proceeds as follows. First, we seek travelling wave solutions by taking $u(x, t) = U(\eta)$, where $\eta := x - ct - x_0$ with c a real constant and x_0 an arbitrary real constant. Substitution into the PDE yields an ODE for $U(\eta)$. If possible, the ODE is integrated term by term one or more times and this introduces one or more constants of integration. (This step is optional.) The key step is to introduce the

ansatz

$$U(\eta) = \sum_{j=0}^M A_j Y^j, \quad \text{where } Y := \tanh(k\eta). \quad (2.1)$$

Y satisfies the differential equation

$$\frac{dY}{d\eta} = k(1 - Y^2) \quad (2.2)$$

so that

$$\frac{d}{d\eta} \equiv k(1 - Y^2) \frac{d}{dY}. \quad (2.3)$$

In (2.1), M is a positive integer (to be determined). The A_j ($j = 0, \dots, M$) are real constants with $A_M \neq 0$, and k is a real non-zero constant. Substitution of (2.1) and (2.3) into the ODE, integrated if possible, yields an algebraic equation in powers of Y . If possible, M is determined; usually this involves balancing the linear term(s) of highest order in the algebraic equation with the highest-order nonlinear term(s). With M determined, the coefficients of each power of Y are equated to zero in the algebraic equation. This yields a system of algebraic equations involving the A_j ($j = 0, \dots, M$), k , c and, if applicable, the integration constants. If the original evolution equation contains some arbitrary constant coefficients, these will, of course, also appear in the system of algebraic equations. If it is possible to find a real non-trivial solution to these equations, then the method has worked successfully.

Notice that (2.2) is also satisfied by $\coth(k\eta)$. It follows that a \coth -function expansion solution may be obtained from a \tanh -function expansion solution simply by replacing ‘ \tanh ’ by ‘ \coth ’. In other words, if (2.1) is a solution of the evolution equation then

$$U(\eta) = \sum_{j=0}^M A_j Y^{-j} \quad (2.4)$$

is also a solution, and vice versa. This result is well known; it was pointed out in [10], for example. Note that (2.4) may be obtained from (2.1) by replacing kx_0 in (2.1) by $kx_0 + i\pi/2$ and then using (A.9).

Based on the above observations, it is tempting to propose an extension to the basic \tanh method by replacing the ansatz (2.1) by a more general ansatz, namely

$$U(\eta) = \sum_{j=-M}^M B_j Y^j, \quad \text{where } Y := \tanh(k\eta). \quad (2.5)$$

This is the ‘ \tanh - \coth ’ expansion method. It was proposed in this form in [2], for example. The motivation is to try to find more solutions than can be found by using the basic \tanh -function method.

3 Observations on the tanh-coth method

Use of (A.4) in (2.4) gives a tanh-coth solution of the form

$$U(\eta) = a_0 + \sum_{j=1}^M a_j (y^j + y^{-j}), \quad \text{where } y := \tanh(k\eta/2) \quad (3.1)$$

and the a_j ($j = 0, \dots, M$) are linear combinations of the A_j ($j = 0, \dots, M$). Our crucial observation is that (3.1) is just a special case of (2.5) with $B_{-j} = B_j (= a_j)$ and $k \rightarrow k/2$. Refs. [2–4,11–21] are examples of papers in which all the derived tanh-coth solutions are just examples of (3.1) and so are disguised versions of the corresponding coth expansion solutions.

Soliman [11] used the tanh-coth method to solve the KdV–Burgers equation. In [11], (19) is a tanh solution and (18) is the corresponding coth solution. Two tanh-coth solutions are also given. The one given by (20) is a disguised version of the coth solution (18); this was noted by Kudryashov [7]. The second one, given by (21), is actually a disguised version of a coth solution which, in turn, corresponds to a second tanh solution that is not given in [11]. The two tanh solutions agree with the two solutions stated in (5) in [22], for example.

Note that in Section 7 of [19], the basic tanh-function method is used to derive a tanh solution and a coth solution. The author claims that the tanh-coth method “does not work”. However, we have verified that it does work and, not surprisingly, that it delivers the tanh and coth solutions and also a tanh-coth solution that is the coth solution in disguise.

Our observations involving hyperbolic functions can be extended to trigonometric functions as follows. By writing $k = iK$, we have

$$\tanh(k\eta) = i \tan(K\eta), \quad \coth(k\eta) = -i \cot(K\eta). \quad (3.2)$$

Consequently, we can generate a tan-function expansion solution from a tanh-function expansion solution, and a cot-function expansion solution from a coth-function expansion solution. (These results are also delivered by the extended tanh-function method as proposed in [23].) These generated solutions are real provided the resulting coefficients are real. Here we make the further observation that, in view of (3.2), the tanh-coth expansion solution (3.1) generates a tan-cot expansion solution of the form

$$U(\eta) = b_0 + \sum_{j=1}^M b_j [(-1)^j w^j + w^{-j}], \quad \text{where } w := \tan(K\eta/2) \quad (3.3)$$

and $b_j = a_j i^{-j}$. This solution is, in fact, a disguised version of the cot-function expansion solution. Indeed, (3.3) can be obtained directly from the cot-function

expansion solution by use of (A.6). Refs. [2–4,15–21] are examples of papers in which all the derived tan-cot solutions are just examples of (3.3) and so are disguised versions of the corresponding cot expansion solutions.

Application of the tanh-coth method may also lead to tanh-coth (or tan-cot) solutions that are not disguised versions of coth (or cot) solutions. For example, with $M = 1$, suppose there is a tanh-coth solution of the form

$$U(\eta) = a_0 + a_1(-y + y^{-1}). \quad (3.4)$$

Then, by use of (A.7), (3.4) becomes the cosech solution

$$U(\eta) = a_0 + 2a_1 \operatorname{cosech}(k\eta). \quad (3.5)$$

Furthermore, by use of (A.10), it may be possible to obtain a real sech solution from (3.5). (This procedure is illustrated in Section 4.)

Examples are to be found in [24–27]. In [24], (29) and (32) are tanh-coth solutions equivalent to the coth solutions (28) and (31) respectively. However, (33) is a tanh-coth solution that can be written as the cosech solution (38) by use of (A.7). In (41) and (42) in [25], the tanh-coth and tan-cot solutions are disguised versions of the coth and cot solutions respectively. However, the tanh-coth solution (50) may be written as the cosech solution (57) by use of (A.7) and then the sech solution (56) may be deduced by use of (A.10). In [26] the tanh-coth solution (5.15) and the tan-cot solution (5.25) are equivalent to the coth solution (5.12) and cot solution (5.22), respectively. However, the tanh-coth solution (5.14) can be written as a cosech solution by use of (A.7) and then the sech solution (5.37b) may be derived by use of (A.10). The tan-cot solution (5.24) may be written as a cosec solution by using (A.8). In [27] the tan-cot solutions (34) and (35) are equivalent to the cot solutions (30) and (31), respectively. However, the tan-cot solution (38) can be written as a cosec solution. Then, with $\alpha \rightarrow i\alpha/2$, a cosech solution is obtained from which the sech solution (12) (with $\beta = -\alpha/\sqrt{2}$) may be derived by use of (A.10).

4 Illustrative example

As a simple example to illustrate our observations, consider the mKdV equation in the form

$$u_t + \alpha u^2 u_x + u_{xxx} = 0, \quad (4.1)$$

where α is a non-zero constant. After one integration, the ODE for $U(\eta)$ is

$$-cU + \alpha U^3/3 + U'' = A, \quad (4.2)$$

where A is a constant of integration and the prime denotes differentiation with respect to η . On balancing the second and third terms in (4.2), we obtain $3M =$

$M + 2$ so that $M = 1$ in (2.5). U given by (2.5) is substituted into (4.2) and the coefficients of Y^i ($i = -3, \dots, 3$) are then set to zero. The resulting seven algebraic equations involving $B_{-1}, B_0, B_1, k, c, A$ and α have the following four solution-sets in each of which $A = 0, B_0 = 0$ and k is arbitrary:

$$B_{-1} = 0, \quad B_1 = \pm k\sqrt{-6/\alpha}, \quad c = -2k^2, \quad (4.3)$$

$$B_1 = 0, \quad B_{-1} = \pm k\sqrt{-6/\alpha}, \quad c = -2k^2, \quad (4.4)$$

$$B_{-1} = B_1, \quad B_1 = \pm k\sqrt{-6/\alpha}, \quad c = -8k^2, \quad (4.5)$$

$$B_{-1} = -B_1, \quad B_1 = \pm k\sqrt{-6/\alpha}, \quad c = 4k^2. \quad (4.6)$$

The set (4.3) gives the solution

$$u(x, t) = \pm k\sqrt{-6/\alpha} \tanh(k\eta), \quad \eta = x + 2k^2t - x_0, \quad \alpha < 0. \quad (4.7)$$

The set (4.4) gives the solution

$$u(x, t) = \pm k\sqrt{-6/\alpha} \coth(k\eta), \quad \eta = x + 2k^2t - x_0, \quad \alpha < 0. \quad (4.8)$$

From our observation in Section 2, we know that once (4.7) is obtained, the corresponding coth solution given by (4.8) follows directly. Furthermore, by using (A.1) in (4.8), and then taking $k/2 \rightarrow k$, we obtain

$$u(x, t) = \pm k\sqrt{-6/\alpha} [\tanh(k\eta) + \coth(k\eta)], \quad \eta = x + 8k^2t - x_0, \quad \alpha < 0, \quad (4.9)$$

which is precisely the tanh-coth solution obtained from the set (4.5). Hence, as predicted in Section 3, (4.9) is just a disguised version of (4.8). The set (4.6) also gives a tanh-coth solution, namely

$$u(x, t) = \pm k\sqrt{-6/\alpha} [\coth(k\eta) - \tanh(k\eta)], \quad \eta = x - 4k^2t - x_0, \quad \alpha < 0. \quad (4.10)$$

In view of the identity (A.7), (4.10) with $k \rightarrow k/2$ may be written

$$u(x, t) = \pm k\sqrt{-6/\alpha} \operatorname{cosech}(k\eta), \quad \eta = x - k^2t - x_0, \quad \alpha < 0. \quad (4.11)$$

In view of the identity (A.10), (4.11) with $kx_0 \rightarrow kx_0 + i\pi/2$ gives the imaginary solution

$$u(x, t) = \pm ki\sqrt{-6/\alpha} \operatorname{sech}(k\eta), \quad \eta = x - k^2t - x_0, \quad \alpha < 0, \quad (4.12)$$

which, in turn, gives the real solution

$$u(x, t) = \pm k\sqrt{6/\alpha} \operatorname{sech}(k\eta), \quad \eta = x - k^2t - x_0, \quad \alpha > 0. \quad (4.13)$$

Note that the solutions given by (4.7)–(4.9) are derivable via the basic tanh-function method whereas, as also noted in [1], for example, this is not true for the solution

given by (4.13). The solutions given by (4.7) and (4.13) are the well-known bounded solitary-wave solutions corresponding to (2.24) and (2.23) respectively in [1], for example. The solutions given by (4.8) and (4.11) are unbounded; as they are less useful, they are mentioned less frequently in the literature.

For completeness, we give the corresponding trigonometric solutions in each of which $\alpha < 0$:

$$u(x, t) = \pm K \sqrt{-6/\alpha} \tan(K\eta), \quad \eta = x - 2K^2t - x_0, \quad (4.14)$$

$$u(x, t) = \pm K \sqrt{-6/\alpha} \cot(K\eta), \quad \eta = x - 2K^2t - x_0, \quad (4.15)$$

$$u(x, t) = \pm K \sqrt{-6/\alpha} [-\tan(K\eta) + \cot(K\eta)], \quad \eta = x - 8K^2t - x_0, \quad (4.16)$$

$$u(x, t) = \pm K \sqrt{-6/\alpha} [\tan(K\eta) + \cot(K\eta)], \quad \eta = x + 4K^2t - x_0, \quad (4.17)$$

$$u(x, t) = \pm K \sqrt{-6/\alpha} \operatorname{cosec}(K\eta), \quad \eta = x + K^2t - x_0, \quad (4.18)$$

$$u(x, t) = \pm K \sqrt{-6/\alpha} \sec(K\eta), \quad \eta = x + K^2t - x_0. \quad (4.19)$$

Solution (4.16) is a disguised version of (4.15), and solution (4.17) is a disguised version of (4.18). With $Kx_0 \rightarrow Kx_0 + \pi/2$, (4.15) and (4.18) may be obtained from (4.14) and (4.19) respectively.

5 Concluding comments

The tanh-coth expansion method is a natural extension to the basic tanh-function expansion method which was developed in the 1990s. It usually delivers three types of solution, namely a tanh-function expansion, a coth-function expansion, and a tanh-coth expansion. It is known that, for every tanh-function expansion solution (2.1), there is a corresponding coth-function expansion solution (2.4); we have shown that in addition there is a tanh-coth expansion solution (3.1) that is merely a disguised version of the coth solution. One consequence of this observation is that, in many papers, such tanh-coth solutions are erroneously claimed to be new. Nevertheless, the tanh-coth method is worthwhile because it may deliver other types of tanh-coth solutions which are genuinely new. Such solutions would not be delivered by using the basic tanh-function method. We have given examples where these solutions may be more conveniently expressed in terms of the cosech function, and sech-function solutions can be derived from them. Similar remarks apply to tan, cot and tan-cot expansion solutions.

Appendix: Useful identities

It is straightforward to show that

$$\coth 2\theta = 2^{-1} (y + y^{-1}), \quad \text{where } y := \tanh \theta. \quad (\text{A.1})$$

It follows that

$$\coth^2 2\theta = 2^{-2} [(y^2 + y^{-2}) + 2], \quad (\text{A.2})$$

$$\coth^3 2\theta = 2^{-3} [(y^3 + y^{-3}) + 3(y + y^{-1})], \quad (\text{A.3})$$

and, more generally,

$$\coth^n 2\theta = 2^{-n} \sum_{r=0}^N \binom{n}{r} [y^{(n-2r)} + y^{-(n-2r)}], \quad (\text{A.4})$$

where

$$N = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Similarly,

$$\cot 2\theta = 2^{-1} (-w + w^{-1}), \quad \text{where } w := \tan \theta, \quad (\text{A.5})$$

leads to

$$\cot^n 2\theta = 2^{-n} \sum_{r=0}^N \binom{n}{r} (-1)^r [(-1)^n w^{(n-2r)} + w^{-(n-2r)}]. \quad (\text{A.6})$$

Note also the identities

$$\operatorname{cosech} 2\theta = 2^{-1} (-y + y^{-1}), \quad (\text{A.7})$$

$$\operatorname{cosec} 2\theta = 2^{-1} (w + w^{-1}), \quad (\text{A.8})$$

$$\tanh(\theta - i\pi/2) = \coth \theta, \quad (\text{A.9})$$

$$\operatorname{cosech}(\theta - i\pi/2) = i \operatorname{sech} \theta. \quad (\text{A.10})$$

References

- [1] E.J. Parkes, B.R. Duffy, An automated tanh-function method for finding solitary wave solutions to nonlinear evolution equations, *Comput. Phys. Commun.* 98 (1996) 288–300.
- [2] A.M. Wazwaz, New solitary wave solutions to the Kuramoto-Sivashinsky and the Kawahara equations, *Appl. Math. Comput.* 182 (2006) 1642–1650.
- [3] A.M. Wazwaz, The tanh-coth method for solitons and kink solutions for nonlinear parabolic equations, *Appl. Math. Comput.* 188 (2007) 1467–1475.

- [4] S.A. Elwakil, S.K. El-labany, M.A. Zahran, R. Sabry, Modified extended tanh-function method for solving nonlinear partial differential equations, *Phys. Lett. A* 299 (2002) 179–188.
- [5] N.A. Kudryashov, N.B. Loguinova, Be careful with the Exp-function method, *Commun. Nonlinear Sci. Numer. Simulat.* 14 (2009) 1881–1890.
- [6] N.A. Kudryashov, On “new travelling wave solutions” of the KdV and KdV-Burgers equations, *Commun. Nonlinear Sci. Numer. Simulat.* 14 (2009) 1891–1900.
- [7] N.A. Kudryashov, Seven common errors in finding exact solutions of nonlinear differential equations, *Commun. Nonlinear Sci. Numer. Simulat.* 14 (2009) 3507–3529.
- [8] E.J. Parkes, A note on travelling-wave solutions to Lax’s seventh-order KdV equation, *Appl. Math. Comput.* 215 (2009) 864–865.
- [9] E.J. Parkes, A note on solitary travelling-wave solutions to the transformed reduced Ostrovsky equation, *Commun. Nonlinear Sci. Numer. Simulat.* (2009), doi:10.1016/j.cnsns.2009.11.016.
- [10] E. Fan, H. Zhang, A note on the homogeneous balance method, *Phys. Lett. A* 246 (1998) 403–406.
- [11] A.A. Soliman, The modified extended tanh-function method for solving Burgers-type equations, *Physica A* 361 (2006) 394–404.
- [12] A.A. Soliman, Exact traveling wave solution of nonlinear variants of the RLW and PHI-four equations, *Phys. Lett. A* 368 (2007) 383–390.
- [13] A.M. Wazwaz, The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations, *Appl. Math. Comput.* 184 (2007) 1002–1014.
- [14] A. Bekir, A.C. Cevikel, Solitary wave solutions of two nonlinear physical models by tanh-coth method, *Commun. Nonlinear Sci. Numer. Simulat.* 14 (2009) 1804–1809.
- [15] S.A. Elwakil, S.K. El-labany, M.A. Zahran, R. Sabry, Modified extended tanh-function method and its applications to nonlinear equations, *Appl. Math. Comput.* 161 (2005) 403–412.
- [16] A.M. Wazwaz, New solitary wave solutions to the modified forms of Degasperis–Procesi and Camassa–Holm equations, *Appl. Math. Comput.* 186 (2007) 130–141.
- [17] A.M. Wazwaz, The extended tanh method for abundant solitary wave solutions of nonlinear wave equations *Appl. Math. Comput.* 187 (2007) 1131–1142.
- [18] A.M. Wazwaz, The tanh-coth method for new compactons and solitons solutions for the $K(n, n)$ and the $K(n + 1, n + 1)$ equations, *Appl. Math. Comput.* 188 (2007) 1930–1940.
- [19] A.M. Wazwaz, The tanh method for travelling wave solutions to the Zhiber–Shabat equation and other related equations, *Commun. Nonlinear Sci. Numer. Simulat.* 13 (2008) 584–592.

- [20] A.M. Wazwaz, New travelling wave solutions to the Boussinesq and the Klein-Gordon equations, *Commun. Nonlinear Sci. Numer. Simulat.* 13 (2008) 889–901.
- [21] A.M. Wazwaz, The extended tanh method for the Zakharov–Kuznetsov (ZK) equation, the modified ZK equation, and its generalized forms, *Commun. Nonlinear Sci. Numer. Simulat.* 13 (2008) 1039–1047.
- [22] E.J. Parkes, Exact solutions to the two-dimensional Korteweg–de Vries–Burgers equation, *J. Phys. A: Math. Gen.* 27 (1994) L497–L501.
- [23] E. Fan, Extended tanh-function method and its applications to nonlinear equations, *Phys. Lett. A* 277 (2000) 212–218.
- [24] Y. Wang, L. Wei, New exact solutions to the $(2 + 1)$ -dimensional Konopelchenko–Dubrovsky equation, *Commun. Nonlinear Sci. Numer. Simulat.* 15 (2010) 216–224.
- [25] A.M. Wazwaz, The tanh-coth and the sine-cosine methods for kinks, solitons, and periodic solutions for the Pochhammer–Chree equations, *Appl. Math. Comput.* 195 (2008) 24–33.
- [26] M. Yaghobi Moghaddam, A. Asgari, H. Yazdani, Exact travelling wave solutions for the generalized nonlinear Schrödinger (GNLS) equation with a source by Extended tanh-coth, sine-cosine and Exp-Function methods, *Appl. Math. Comput.* 210 (2009) 422–435.
- [27] A.M. Wazwaz, The tanh-coth and the sech methods for exact solutions of the Jaulent–Miodeck equation, *Phys. Lett. A* 366 (2007) 85–90.