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A note on loop-soliton solutions of the short-pulse equation

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Abstract

It is shown that the N-loop soliton solution to the short-pulse equation may be decomposed exactly into N separate soliton elements by using a Moloney–Hodnett type decomposition. For the case $N = 2$, the decomposition is used to calculate the phase shift of each soliton caused by its interaction with the other one. Corrections are made to some previous results in the literature.

Key words: short-pulse equation; sine-Gordon equation; N-loop soliton; Moloney–Hodnett decomposition; phase shift.
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1 Introduction

The short-pulse equation (SPE), namely

$$u_{xt} = u + \tfrac{1}{6}\,(u^3)_{xx}\,, \tag{1.1}$$

models the propagation of ultra-short light pulses in silica optical fibres [1]. (The SPE is also known as the cubic Rabelo equation [2].) In recent years, various aspects of the SPE have received attention in the literature. A useful summary of some of this work is given in [3]. Here we focus on the N-loop soliton solution to the SPE. Such solutions may be found by making use of N-soliton solutions of equations related to the SPE. Our aim is to complement results on this aspect of the SPE as given in [4–7].

In [4,5] it was shown that the SPE is related to a system of coupled nonlinear dispersionless equations (CNDE) that are a special case of the system studied in [8].

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Also, as mentioned in [2,6,7,9–11], the SPE is related to the sine-Gordon equation (SGE)

$$z_{y\tau} = \sin z \tag{1.2}$$

by the transformation

$$u(x,t) = z_\tau(y,\tau), \quad x = w(y,\tau), \quad t = \tau, \tag{1.3}$$

where

$$w_y = \cos z, \quad w_\tau = -\frac{1}{2}\, z_\tau^2. \tag{1.4}$$

In [12], we found some periodic and solitary travelling-waves solutions of the SPE (1.1) by direct integration. In [4], the two-loop soliton solution to the SPE was found via the two-loop soliton solution to the CNDE given in [8]. In the following papers, solutions to the SPE were found via solutions to the SGE: [6] (one- and two-loop solitons, single breather), [5] (N-loop solitons), [9] (travelling waves), [7] (N-loop solitons, multi-breathers) and [11] (one- and two-phase periodic). Hirota's D-operator method was used in [4,5,7,8].

We complement the work in [4–7] as follows. In Section 2 we show that the N-loop soliton solution to the SPE can be decomposed exactly into N separate soliton elements by using a Moloney–Hodnett type decomposition; this is in contrast to the approximate decomposition in [4]. In Section 3 we use our decomposition to calculate the phase shifts in the case $N = 2$. We also point out errors in the corresponding phase-shift calculation given in [4].

2 The decomposition of the N-soliton solution

As noted in [13,14], the original route to the N-soliton solution of the SGE by use of Hirota's method was via the bilinear transformation

$$z = 4\tan^{-1}(G/F), \tag{2.1}$$

where F and G are real functions of ξ_j $(j = 1, 2, \ldots, N)$,

$$\xi_j = \hat{\xi}_j + \xi_{0j}, \quad \text{where} \quad \hat{\xi}_j = p_j\, y + \frac{\tau}{p_j}\,, \tag{2.2}$$

the p_j are arbitrary non-zero constants, and the ξ_{0j} are arbitrary constants. For our purposes it is more convenient to use the equivalent bi-logarithmic bilinear transformation as given in [15,16,7], namely

$$z = 2i\ln(f^*/f), \tag{2.3}$$

where $f := F + iG$ and * denotes the complex conjugate. In view of (1.3) and (2.3), Matsuno [7] deduced that the N-soliton solution of the SPE may be found via the bi-logarithmic bilinear transformation

$$u(x,t) := U(y,\tau) = 2i[\ln(f^*/f)]_\tau. \tag{2.4}$$

He also made the perceptive and important observation that the relation $w_y = \cos z$ in (1.4) can be integrated to obtain

$$x := w(y,\tau) = y - 2[\ln(f^* f)]_\tau + x_0, \tag{2.5}$$

where x_0 is an arbitrary constant.

Hirota [14] showed that the logarithmic bilinear transformation appropriate for the Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2.6}$$

is

$$u = 2(\ln F)_{xx}. \tag{2.7}$$

In [17], Moloney and Hodnett showed how (2.7) may be used to decompose the N-soliton solution of the KdV equation into N separate soliton elements. This procedure has also been used to decompose the N-loop soliton solution to the Vakhnenko equation [18,19]. Here we show how Moloney and Hodnett's procedure may be adapted and applied to (2.4) in order to decompose the N-soliton solution of the SPE into N separate soliton elements.

Firstly, we note that from (2.2) – (2.4)

$$U = \sum_{j=1}^{N} \frac{\partial z}{\partial \xi_j}\frac{\partial \xi_j}{\partial \tau} = \sum_{j=1}^{N} U_j, \tag{2.8}$$

where

$$U_j = \frac{2i}{p_j}\frac{\partial}{\partial \xi_j}\left[(\ln f)^* - (\ln f)\right]. \tag{2.9}$$

Secondly, we note that when Hirota's method is used [7], it turns out that f depends on the ξ_j $(j = 1, 2, \ldots, N)$ via $\exp(\xi_j)$, and for each value of j it is possible to write f in the form

$$f = f_j[1 + q_j \exp(\xi_j)], \tag{2.10}$$

where f_j and q_j do not involve $\exp(\xi_j)$, i.e. they are independent of ξ_j. On combining (2.9) and (2.10), and using the identity

$$\frac{2e^{2\theta}}{1 + e^{2\theta}} = 1 + \tanh(\theta), \tag{2.11}$$

we obtain

$$U_j = \frac{2}{p_j}\,\mathfrak{Im}\left[\tanh\left(\frac{g_j}{2}\right)\right], \tag{2.12}$$

where

$$\exp(g_j) = q_j \exp(\xi_j). \tag{2.13}$$

Similarly, from (2.5), we obtain

$$x = y - \sum_{j=1}^{N} \frac{2}{p_j}\left\{1 + \mathfrak{Re}\left[\tanh\left(\frac{g_j}{2}\right)\right]\right\} + x_0. \tag{2.14}$$

The $u_j(x,t) = U_j(y,\tau)$ given by (2.12), (2.14) and $t = \tau$ are the required separate soliton elements.

For reference purposes, we give the one-soliton solution. When $N = 1$, $q_1 = i$ and then from (2.12) and (2.14) we obtain

$$u := U_1 = \frac{2}{p_1}\,\mathrm{sech}(\xi_1) \tag{2.15}$$

and

$$x = y - \frac{2}{p_1}\left\{1 + \tanh(\xi_1)\right\} + x_0, \tag{2.16}$$

respectively. Equation (2.16) may be written in the form

$$\eta := x + \frac{1}{p_1^2}\,t = \frac{\xi_1}{p_1} - \frac{2}{p_1}\tanh(\xi_1) + \eta_0, \tag{2.17}$$

where η_0 is an arbitrary constant. (2.15) and (2.17) are a solution in parametric form for u as a function of η via the parameter ξ_1. This solution represents a loop soliton moving in the negative x-direction with speed $1/p_1^2$. The solution corresponds to (13) in [6], (3.5) in [12], and (3.2a) and (3.3) in [7]. (There are misprints in (3.2a) and in the text after (3.3) in [7].)

Now we illustrate the decomposition that we have presented by considering the two-loop soliton solution. When $N = 2$,

$$f = 1 + i(e^{\xi_1} + e^{\xi_2}) - e^{\xi_1 + \xi_2 - 2\delta}, \tag{2.18}$$

where

$$e^{-2\delta} = \left(\frac{p_1 - p_2}{p_1 + p_2}\right)^2, \tag{2.19}$$

so that the q_j in (2.13) are given by

$$q_1 = \frac{i - e^{\xi_2 - 2\delta}}{1 + ie^{\xi_2}}, \quad q_2 = \frac{i - e^{\xi_1 - 2\delta}}{1 + ie^{\xi_1}}. \tag{2.20}$$

Now $U = U_1 + U_2$, where U_1 and U_2 are given by (2.12) and (2.13).

Equivalent solutions, but in undecomposed form, were derived in [6] and [7]. Hirota's method was used in [7] but not in [6]; in the latter, solutions for the SPE were derived from the kink–kink and kink–antikink solutions of the SGE as given in [20].

In [4], the SPE was transformed into the CNDE by using a transformation similar to the one used in [18,19] to transform the Vakhnenko equation into a more convenient form. Then the CNDE was put into Hirota form and the cases $N = 1$ and $N = 2$ considered in detail. For $N = 2$, the solution was expressed as an approximate Moloney–Hodnett decomposition. This is in contrast to our decomposition which is exact.

3 The phase-shift calculation for the case $N = 2$

We now consider the phase-shift calculation for the case $N = 2$. First we give a calculation that makes use of our decomposition. Then we indicate why the calculation in [4] is incorrect.

In order to discuss the phase shift of each soliton due to its interaction with the other one, it is convenient to introduce the variables η_j $(j = 1, 2)$ defined by

$$\eta_j := x + \frac{1}{p_j^2}\,t = \frac{\hat{\xi}_j}{p_j} - \sum_{j=1}^{2} \frac{2}{p_j} \left\{1 + \mathfrak{Re}\left[\tanh\left(\frac{g_j}{2}\right)\right]\right\} + x_0 \tag{3.1}$$

and to note the relationship

$$p_2\hat{\xi}_1 - p_1\hat{\xi}_2 = \left(\frac{p_2^2 - p_1^2}{p_1 p_2}\right)\tau. \tag{3.2}$$

Also, without loss of generality, we choose $\xi_{j0} = \delta$ $(j = 1, 2)$ in (2.2).

First we consider the case $p_2 > p_1 > 0$ which corresponds to a loop–loop interaction. For this case our derivation is similar to the procedure given in Sections 3.2 and 3.3 in [7]; the main difference is that our starting point is the decomposed solution. From (3.2) and (2.20) we deduce that, with $\hat{\xi}_1$ fixed, $\hat{\xi}_2 \to \infty$ and $q_1 \to ie^{-2\delta}$ as $\tau \to -\infty$, and that $\hat{\xi}_2 \to -\infty$ and $q_1 \to i$ as $\tau \to \infty$. It follows from (2.12) and (3.1) that, as $t \to \mp\infty$,

$$u_1 \to \frac{2}{p_1}\,\mathrm{sech}(\hat{\xi}_1 \mp \delta), \tag{3.3}$$

$$\eta_1 \to \frac{\hat{\xi}_1}{p_1} - \frac{2}{p_1}\left[1 + \tanh(\hat{\xi}_1 \mp \delta)\right] - \frac{2}{p_2}(1 \pm 1) + x_0. \tag{3.4}$$

Hence, for $t \to \mp\infty$, u_1 is a function of η_1 via the parameter $\hat{\xi}_1$. From (2.15), as $t \to \mp\infty$, the crest of the soliton is located where $\hat{\xi}_1 = \pm\delta$ in the y-t plane; from

(3.4), this corresponds to

$$\eta_1 = \pm\frac{\delta}{p_1} - \frac{2}{p_1} - \frac{2}{p_2}(1 \pm 1) + x_0 \tag{3.5}$$

in the x-t plane. As the soliton is propagating in the negative x-direction (with speed $1/p_1^2$), it makes sense to define the phase shift as

$$\Delta_1 := \eta_1(t \to -\infty) - \eta_1(t \to \infty) = \frac{2\delta}{|p_1|} - \frac{4}{|p_2|}. \tag{3.6}$$

A similar calculation in which $\hat{\xi}_2$ is held fixed gives the following results corresponding to equations (3.3) – (3.6), respectively:

$$u_2 \to \frac{2}{p_2}\,\mathrm{sech}(\hat{\xi}_2 \pm \delta), \tag{3.7}$$

$$\eta_2 \to \frac{\hat{\xi}_2}{p_2} - \frac{2}{p_1}(1 \mp 1) - \frac{2}{p_2}\left[1 + \tanh(\hat{\xi}_2 \pm \delta)\right] + x_0, \tag{3.8}$$

$$\eta_2 = \mp\frac{\delta}{p_2} - \frac{2}{p_1}(1 \mp 1) - \frac{2}{p_2} + x_0, \tag{3.9}$$

$$\Delta_2 := \eta_2(t \to -\infty) - \eta_2(t \to \infty) = -\frac{2\delta}{|p_2|} + \frac{4}{|p_1|}. \tag{3.10}$$

(3.6) and (3.10) agree with (3.13a,b) in [7]. (The above calculation may be generalized to the case $N > 2$; the result agrees with (3.10) in [7].) It is straightforward to show that (3.6) and (3.10) also hold for the antiloop–antiloop interaction for which $-p_2 > -p_1 > 0$. A similar calculation yields the phase shifts for the antiloop–loop interaction for which $p_2 > -p_1 > 0$, and for the loop–antiloop interaction for which $-p_2 > p_1 > 0$, namely

$$\Delta_1 = -\frac{2\delta}{|p_1|} - \frac{4}{|p_2|}, \quad \Delta_2 = \frac{2\delta}{|p_2|} + \frac{4}{|p_1|}. \tag{3.11}$$

Before commenting on the phase-shift calculation for $N = 2$ in [4], it is useful to look at the one-soliton solution in [4]. In the notation of [4], this solution is

$$U(\eta) = 2\sqrt{\frac{1+c}{1-c}}\,\mathrm{sech}(\eta), \tag{3.12}$$

$$x = \frac{1}{2}(\sigma + \tau) - 2\sqrt{\frac{1+c}{1-c}}\,\tanh(\eta), \tag{3.13}$$

$$t = \frac{1}{2}(\sigma - \tau), \tag{3.14}$$

where $\eta = k(\sigma - c\tau) + \beta$, $k = 1/\sqrt{1-c^2}$, β is an arbitrary constant, c is a constant such that $|c| < 1$, and σ and τ are new independent variables. To express this solution in a convenient form for interpretation in the x-t plane, the solution pair U

and $x+vt$ should be parameterized in terms of the single variable η only, where the constant v is chosen suitably. We find that v has to be given by $v=(1+c)/(1-c)$ and then

$$x+vt=\sqrt{\frac{1+c}{1-c}}\,\eta-2\sqrt{\frac{1+c}{1-c}}\,\tanh(\eta) \tag{3.15}$$

with $v>0$. Equations (3.12) and (3.15) are equivalent to (2.15) and (2.17), respectively.

In [4], the two-loop solution is given as an approximate Moloney–Hodnett decomposition; this decomposition becomes exact only in the asymptotic limits $t\to\mp\infty$. In order to investigate the phase shifts, the authors in [4] considered $x+v_jt$ $(j=1,2)$ with $v_j=1/c_j$. (No doubt this choice of v_j was influenced by the corresponding step in [18,19] for the two-loop solution to the Vakhnenko equation in which $v_j=1/c_j$ is indeed the correct expression for v_j.) However, as indicated by our comments on the one-loop soliton solution, v_j should be taken to be $v_j=(1+c_j)/(1-c_j)$. This error in [4] led to an incorrect calculation for the phase shifts, and the erroneous claim that the asymptotic speeds of the two solitons in the negative x-direction are $1/c_1$ and $1/c_2$, respectively; the respective correct speeds are $(1+c_j)/(1-c_j)$ $(j=1,2)$.

4 Concluding comments

We have made observations and corrections regarding various aspects of the calculation of loop soliton solutions to the SPE, the aim being to complement the work in [4–7]. As far as we are aware, the Moloney–Hodnett decomposition associated with the bi-logarithmic bilinear transformation (2.4), and presented in Section 2, is new, as are the expressions for the phase-shifts for the interaction between two antiloops and between a loop and an antiloop as given in Section 3.

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