



## Strathprints Institutional Repository

Elgamel, Sherif A.E.H. and Clemente, Carmine and Soraghan, J.J. (2010) *Radar matched filtering using the fractional fourier transform*. In: Sensor Signal Processing for Defence 2010 (SSPD 2010), 2010-09-29 - 2010-09-30, London, UK.

Strathprints is designed to allow users to access the research output of the University of Strathclyde. Copyright © and Moral Rights for the papers on this site are retained by the individual authors and/or other copyright owners. You may not engage in further distribution of the material for any profitmaking activities or any commercial gain. You may freely distribute both the url (<http://strathprints.strath.ac.uk/>) and the content of this paper for research or study, educational, or not-for-profit purposes without prior permission or charge.

Any correspondence concerning this service should be sent to Strathprints administrator: <mailto:strathprints@strath.ac.uk>

Elgamel, Sherif A.E.H. and Clemente, Carmine. and Soraghan, J.J. (2010) Radar matched filtering using the fractional Fourier transform. In: Sensor Signal Processing for Defence 2010 (SSPD 2010), 29-30 September, 2010 , London, UK.

<http://strathprints.strath.ac.uk/27167/>

Strathprints is designed to allow users to access the research output of the University of Strathclyde. Copyright © and Moral Rights for the papers on this site are retained by the individual authors and/or other copyright owners. You may not engage in further distribution of the material for any profitmaking activities or any commercial gain. You may freely distribute both the url (<http://strathprints.strath.ac.uk>) and the content of this paper for research or study, educational, or not-for-profit purposes without prior permission or charge. You may freely distribute the url (<http://strathprints.strath.ac.uk>) of the Strathprints website.

Any correspondence concerning this service should be sent to The Strathprints Administrator: [eprints@cis.strath.ac.uk](mailto:eprints@cis.strath.ac.uk)

# Radar Matched Filtering using the Fractional Fourier Transform

Sherif A. Elgamel<sup>#1</sup>, Carmine Clemente<sup>#2</sup>, John J. Soraghan<sup>#3</sup>

<sup>#</sup>*Department of Electronic and Electrical Engineering, University of Strathclyde  
204 George Street, G1 1XW, Glasgow, UK*

<sup>1</sup> sherifelgamel73@gmail.com

<sup>2</sup> carmine.clemete@eee.strath.ac.uk

<sup>3</sup> j.soraghan@eee.strath.ac.uk

**Abstract**—A matched filter is the optimal linear filter for maximizing the signal to noise ratio (SNR) in the presence of additive noise. Matched filters are commonly used in radar systems where the transmitted signal is known and may be used as a replica to be correlated with the received signal which can be carried out by multiplication in the frequency domain by applying Fourier Transform (FT). Fractional Fourier transform (FrFT) is the general case for the FT and is superior in chirp pulse compression using the optimum FrFT order. In this paper a matched filter is implemented for a chirp radar signal in the optimum FrFT domain. Mathematical formula for a received chirp signal in the frequency domain and a generalized formula in the fractional Fourier domain are presented in this paper using the Principle of Stationary Phase (PSP). These mathematical expressions are used to show the limitations of the matched filter in the fractional Fourier domain. The parameters that affect the chirp signal in the optimum fractional Fourier domain are described. The performance enhancement by using the matched filter in the fractional Fourier domain for special cases is presented.

## I. INTRODUCTION

Matched filters are commonly used in radar in which a known signal is transmitted, and the reflected signal is examined for common elements of the transmitted one. Radar matched filter correlates a known signal (replica of the transmitted signal) with an unknown signal (received signal) [1]. This is done by computing cross correlation of the received signal with the transmitted signal. This is achieved by convolving the incoming signal with a conjugated and time-reversed version of the transmitted signal which can be done using Fourier transform (FT) multiplication in the frequency domain before transforming back into the time domain.

The Fractional Fourier transform (FrFT) is a mathematical generalization of the ordinary FT, the latter being a special case of the first when the fractional angle equal  $90^\circ$  [2, 3]. FrFT converts a chirp signal in the optimal fractional Fourier domain to narrow, highly compact, sinc function. FT multiplication is used in Matched filter to enhance the signal to noise ratio. This paper investigates the concept of the matched filter in the FrFT domain.

The paper is organized as follows: The peak position of a chirp in the optimum FrFT domain is discussed in section

II. Section III uses the principle of stationary phase (PSP) to derive the chirp signal in the FT and FrFT domains respectively. The matched filter in the FT and FrFT domain are presented in section IV. A set of simulation results is presented in section V for different parameters. This section also includes a discussion on the chirp matched filtering in the FrFT domain. Section VI concludes the paper.

## II. FRACTIONAL FOURIER TRANSFORM

The fractional Fourier transform (FrFT) is the generalized formula for the Fourier transform that transforms a function into an intermediate domain between time and frequency. The signals with significant overlap in both the time and frequency domain may have little or no overlap in the fractional Fourier domain. The fractional Fourier transform of order  $a$  of an arbitrary function  $x(t)$ , with an angle  $\alpha$ , is defined as [4]:

$$X_\alpha(t_a) = \int_{-\infty}^{\infty} x(t) K_\alpha(t, t_a) dt \quad (1)$$

where  $K_\alpha(t, t_a)$  is the transformation Kernel,  $t_a$  is the transformation of  $t$  to the  $a^{\text{th}}$  order, and  $\alpha = a\pi/2$  with  $a \in \mathfrak{R}$ .  $K_\alpha(t, t_a)$  calculated from:

$$K_\alpha(t, t_a) = \begin{cases} \sqrt{\frac{1-j\cot\alpha}{2\pi}} e^{j\frac{t^2+t_a^2}{2}\cot\alpha - jt_a\csc\alpha} & \text{if } \alpha \text{ is not a multiple of } \pi \\ \delta(t-t_a) & \text{if } \alpha \text{ is a multiple of } 2\pi \\ \delta(t+t_a) & \text{if } \alpha+\pi \text{ is a multiple of } 2\pi \end{cases} \quad (2)$$

### A. The optimum FrFT order of chirp

The optimum order value,  $a_{opt}$  for a chirp signal may be written as [5]:

$$a_{opt} = -\frac{2}{\pi} \tan^{-1}\left(\frac{\delta f}{2\gamma \times \delta t}\right) \quad (3)$$

where  $\delta f$  is the frequency resolution ( $\delta f = F_s/N$ ),  $\delta t$  is the time resolution ( $\delta t = 1/F_s$ ),  $F_s$  is the sampling frequency, and  $\gamma$  is the chirp rate parameter.

The optimum FrFT order  $a_{opt}$  for the chirp can be

computed by applying (3) as:

$$a_{opt} = -\frac{2}{\pi} \tan^{-1} \left( \frac{F_s^2 \times T}{\Delta f \times L} \right) \quad (4)$$

where  $T$  is the chirp duration,  $L$  is the number of samples in the time received window, and  $\Delta f$  is the chirp bandwidth.

### B. Peak positioning of a chirp signal

The peak position  $P_p$  of a chirp signal in the FrFT domain is defined as [6]:

$$P_p = \sin(\alpha) \left[ \frac{F_{start}}{\Delta f} + \frac{\Delta f (L/M_T)}{2 \times \Delta f} \right] - \cos(\alpha) t_{st} \quad (5)$$

where  $M_T$  is the number of samples in the chirp signal with pulse width  $T$ , and  $t_{st}$  is the chirp start time sampling number. The peak position  $P_p$  of a chirp signal in the optimal FrFT domain can be computed by using (5) and parameters of the radar system as follows:

$$P_p = \sin(\alpha_{opt}) \left[ \frac{-(\Delta f/2)}{(F_s/L)} + \frac{\Delta f (L/M_T)}{2 \times (F_s/L)} \right] - \cos(\alpha_{opt}) t_{st} \quad (6)$$

### C. Shifting property in FrFT

The fractional Fourier transform is a shift variant transform. This property comes directly from the Fourier domain shift theorem [7]. In the fractional Fourier Transform case a shift in the function or a multiplication by exponential results in a mixture of both the operations if  $a$  (the transform order) is not an integer.

Define two functions  $f$  and  $g$  with  $f_a$  the fractional Fourier transform of  $f$  and  $g_a$  the fractional Fourier transform of  $g$ . For  $f \in L$  and  $\alpha = a\pi/2 \in \mathfrak{R}$  we have:

*Shift rule:* if  $g(x) = f(x + \tau)$  then [7]

$$g_a(t_a) = e^{i\tau \sin \alpha (t_a + \frac{1}{2}\tau \cos \alpha)} f_a(t_a + \tau \cos \alpha) \quad (7)$$

*Exponential rule:* if  $g(x) = e^{i\tau x} f(x)$  then [7],

$$g_a(t_a) = e^{i\tau \cos \alpha (t_a + \frac{1}{2}\tau \sin \alpha)} f_a(t_a + \tau \sin \alpha) \quad (8)$$

## III. PRINCIPLE OF STATIONARY PHASE

An integration technique known as the principle of stationary phase (PSP) [8] can be applied to oscillatory integrals of the form:

$$S(k_x) = \int_a^b s(x) e^{j\phi(x)} dx \quad (9)$$

where  $s(x)$  is a slowly varying function and  $\phi(x)$  changes by many cycles over the interval of integration. Under these conditions, contributions to the above integral over most of the  $x$  interval tend to cancel, and thus add little to the overall value of the integral [8]. An exception occurs for contributions at the stationary points of the phase  $\phi(x)$ , defined as those values of  $x$  for which

$$\frac{d}{dx} [\phi(x) = 0] \quad (10)$$

The PSP states that the integral of (9) has its greatest contributions from those values of  $x$  which are stationary points of  $\phi(x)$  on the interval  $[a, b]$ . The following two steps are employed to evaluate the integral (10) involving:

- Determine the location of the stationary point(s) of  $\phi(x)$ .
- Evaluate the integrand of (9) at the stationary point  $x^*$ .

If  $x^*$  is the only stationary point on  $[a, b]$ , then an approximation for the integral is

$$\int_a^b s(x) e^{j\phi(x)} dx \approx \left[ \frac{-\pi}{2\phi''(x^*)} e^{-j\frac{\pi}{2}} \right]^{\frac{1}{2}} s(x^*) e^{j\phi(x^*)} \quad (11)$$

For the radar signal processing of interest here, the factor in square brackets in (11), including magnitude and phase, is essentially constant ( $\phi''(x^*)$  is constant) and need not be considered it in the following analysis.

### A. PSP for chirp Fourier transform

An illustrative application of the PSP involves the computation of the Fourier transform of a received chirp waveform using PSP. The simplest form for a received signal  $s(t)$  as a result of a transmitted chirp may be described as

$$s(t) = \begin{cases} A e^{-j2\pi(\phi_0)} e^{j\pi(\frac{\Delta f}{T})(t - T_{start})^2} & T_{start} < t < T_{start} + T \\ 0 & elsewhere \end{cases} \quad (12)$$

where  $A$  is the received signal amplitude,  $\phi_0$  is a random phase shift,  $\Delta f$  is the chirp signal bandwidth,  $t$  denotes the time variable of the received chirp signal and  $T$  is the pulse duration.

The Fourier transform of the signal  $s(t)$  can be written as

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t} dt \quad (13)$$

Using the Principle of Stationary Phase (PSP), the phase of integrand

$$\phi(t, f) = -2\phi_0 + \left(\frac{\Delta f}{T}\right)(t - \tau)^2 - 2t f \quad (14)$$

where  $\tau = T_{start} + \frac{T}{2}$ . To obtain the stationary point

$$\frac{\partial \phi(t, f)}{\partial t} = 0 \quad (15)$$

The stationary point is

$$t^* = f \frac{T}{\Delta f} + \tau \quad (16)$$

Using (12) and (16) in (11)

$$S(f) = A e^{-j2\pi\phi_0} \text{rect}\left(\frac{t^*}{T}\right) e^{j\pi(\frac{\Delta f}{T})(t^*)^2} e^{-j2\pi f t^*} \quad (17)$$

Letting  $B = A e^{-j2\pi\phi_0}$ , then the formula of the received signal in Fourier domain may be written as:

$$S(f) = \text{Brect}\left(\frac{f}{\Delta f}\right) e^{-j\pi \frac{T}{\Delta f} f^2} e^{-j2\pi \tau f} \quad (18)$$

The *rect* function correctly indicates the nominal bandwidth of the chirp waveform and the exponential function correctly identifies the quadratic relationship across this bandwidth. The result in (18) is an approximation formula however it is a useful engineering approximation especially in coherent radar problems emphasizing phase rather than magnitude.

### B. PSP for chirp FrFT

The radar received signal  $s(t)$  in the FrFT domain  $S^a(f_t)$  is calculated from (1) as

$$S_a(t_a) = \int_{-\infty}^{\infty} s(t) k_a(t, t_a) dt \quad (19)$$

where the FrFT kernel  $k_a(t, t_a)$  is calculated from (2) as

$$k_a(t, t_a) = \exp[j\pi((t^2 u_o'^2 + t_a^2 u_o'^2) \cot \alpha - 2t t_a \csc \alpha)] \quad (20)$$

where  $u_o'$ ,  $u_o$  represent the normalized factors for  $t$  and  $t_a$  respectively.

Using PSP, the phase of integrand can be described as

$$\begin{aligned} \phi(t, t_a) &= 2(\phi_o) + \left(\frac{\Delta f}{T}\right)(t - \tau)^2 \\ &+ t_a^2 u_o'^2 \cot \alpha + t^2 u_o'^2 \cot \alpha - 2t t_a \csc \alpha \end{aligned} \quad (21)$$

The stationary point  $t^*$  is calculated as

$$t^* = \frac{f_t + \frac{\Delta f}{T} \tau \sin \alpha}{\frac{\Delta f}{T} \sin \alpha + u_o'^2 \cos \alpha} \quad (22)$$

and  $S(f_t)$  in FrFT domain is written as

$$S(t_a) = A \text{rect}\left(\frac{t^* - \tau}{T}\right) e^{-j2\pi \phi_o} e^{j\pi \left(\frac{\Delta f}{T}\right)(t^* - \tau)^2} k(t^*, t_a) \quad (23)$$

Letting  $B = A e^{-j2\pi \phi_o}$ ,

$$\begin{aligned} S(t_a) &= \text{Brect}\left(\frac{t^* - \tau}{T}\right) e^{j\pi \left(\frac{\Delta f}{T} + \cot \alpha u_o'^2\right) t^{*2}} \\ &\times e^{j\pi \left(\frac{-2\tau \Delta f}{T} - 2t_a \csc \alpha\right) t^*} e^{j\pi \left(\frac{\tau^2 \Delta f}{T} + \cot \alpha t_a^2 u_o'^2\right)} \end{aligned} \quad (24)$$

Letting  $D = \frac{\Delta f}{T}$ , the chirp signal representation in the FrFT domain may be written as :

$$\begin{aligned} S(t_a) &= \text{Brect}\left(\frac{t_a - \tau u_o'^2 \cos \alpha}{\Delta f \sin \alpha + u_o'^2 T \cos \alpha}\right) \\ &\times e^{j\pi \left(\frac{(D + \cot \alpha u_o'^2)}{(D \sin \alpha + u_o'^2 \cos \alpha)^2} + \frac{-2 \csc \alpha}{D \sin \alpha + u_o'^2 \cos \alpha}\right) t_a^2} \\ &\times e^{j\pi \left(\frac{2D^2 \tau \sin \alpha + 2D \tau \cos \alpha u_o'^2}{(D \sin \alpha + u_o'^2 \cos \alpha)^2} + \frac{-4D\tau}{D \sin \alpha + u_o'^2 \cos \alpha}\right) t_a} \\ &\times e^{j\pi \left(\frac{(D\tau \sin \alpha)^2 (D + \cot \alpha u_o'^2)}{(D \sin \alpha + u_o'^2 \cos \alpha)^2} + \frac{-2D^2 \tau^2 \sin \alpha}{D \sin \alpha + u_o'^2 \cos \alpha} + \tau^2 D\right)} \end{aligned} \quad (25)$$

Eq (25) can be considered as the general transformation of the received chirp signal (12) to the FrFT domain with angle  $\alpha$ . For a special case  $\alpha = 90^\circ$  the signal

representation in FrFT  $S(t_a)$  in (25) can be shown to be the same representation as in (18).

## IV. MATCHED FILTER

The matched filter in both the frequency and the optimum fractional Fourier domains are discussed in this section.

### C. Matched Filter in Frequency domain

The matched filter  $M(f)$  in the frequency domain for  $S(f)$  is designed as the complex conjugate of the exponential part of the signal  $S(f)$  in (18)

$$M(f) = [e^{-j\pi \left(\frac{T}{\Delta f}\right) f^2}]^H \quad (26)$$

where  $H$  is the Hermissian. The radar return chirp signal after the matched filter is

$$S(f) = \text{Brect}\left(\frac{f}{\Delta f}\right) \quad (27)$$

To transform  $S(f)$  to the time domain an inverse Fourier transform formula is used as

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi f t} df \quad (28)$$

Therefore  $s(t)$  can be written as

$$s(t) = B \int_{-\infty}^{\infty} \text{rect}\left(\frac{f}{\Delta f}\right) e^{j2\pi f (t - \tau)} df \quad (29)$$

$$s(t) = B \text{sinc}(\Delta f (t - \tau)) \quad (30)$$

### D. Matched Filter in FrFT domain

The shape of the signal in the FrFT domain in (25) is characterized by  $\text{rect}\left(\frac{t_a - \tau u_o'^2 \cos \alpha}{\Delta f \sin \alpha + u_o'^2 T \cos \alpha}\right)$  in which the

position of the signal in the  $a^{\text{th}}$  FrFT domain depends on  $\tau u_o'^2 \cos \alpha$ . This dependence means the FrFT of the same chirp signal with the same chirp width changes position in the FrFT depending on two parameters  $\tau$  and  $\alpha$ . So the chirp shape in the FrFT depends on the start time

$T_{\text{start}}$  because  $\tau = T_{\text{start}} + \frac{T}{2}$ , the chirp bandwidth  $\Delta f$  and

the chirp duration  $T$  because  $\alpha = -\tan^{-1}\left(\frac{F_s^2 T}{\Delta f L}\right)$

As a result the matched filter implementation in the FrFT domain requires knowledge about the start time  $T_{\text{start}}$  which is an unknown parameter (all other parameters are known) in our case because it depends on the target range  $R_t$  from

$$T_{\text{start}} = \frac{2R_t}{c} \quad (31)$$

where  $c$  is the speed of light with approximate value  $3 \times 10^8$ . While in the case of frequency domain transformation (18), the signal is characterized by

$rect\left(\frac{f}{\Delta f}\right)$  this depends only on the chirp bandwidth

$\Delta f$  which is known information from the transmitted chirp waveform generator. In the special case when  $T_{start}$  is known or equal to zero, it is easy to relocate the replica signal in the optimum FrFT.

## V. SIMULATION RESULTS

A chirp radar is simulated with pulse width 100 microseconds and a pulse repetition interval of 1.6 milliseconds. The incoming base band signals are sampled at 1 MHz. Also it is assumed that the radar operating range is 1:402 range bins. The desired targets are known to exist at range bins = 50 and 250 respectively with target signal to noise ratio (SNR) set to 60 dB and a Doppler frequency of 150 Hz. FrFT matched filter design dependence parameters that were described in Section IV are now presented..

### A. FrFT Matched Filter dependence on $T_{start}$

The chirp signal for the target at range bin 50 is shown in Fig. 1. It starts at range bin 50 with width 100 bins (equivalent to 100 microseconds). In a similar fashion the target at range bin 250 starts at range bin 250 with width 100 bins.

Both target signals are filtered using a 200 kHz Gaussian band pass filter and the FFT of the output is computed. The magnitudes of the spectra are illustrated in Fig. 2. As expected the spectra only depends on the chirp band width  $\Delta f$ . The fractional Fourier transform (FrFT) of the target signals at range bins 50 and 250 are shown in Fig. 3. The optimal order  $a_{opt}$  for the chirp signal calculated from (3) is 1.7061 and the absolute value of the FrFT of the signal is a spike at a peak position at sample number 293 (as seen in the zoomed figure) which could also be determined mathematically from (6). The target signal at range bin 250 shown in Fig. 3 appears at sample 114 in the FrFT domain. From these results the FrFT dependence on start time  $T_{start}$  even when the chirp signal has the same chirp width is evident.

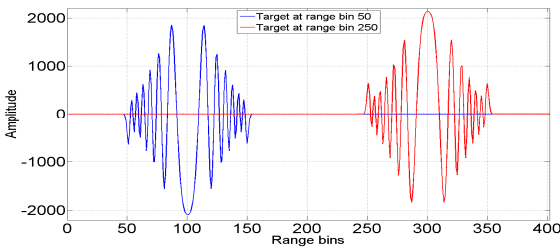


Fig. 1 Chirp signals for different targets ranges

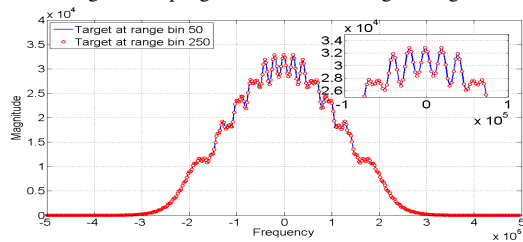


Fig. 2 FFT for different targets ranges

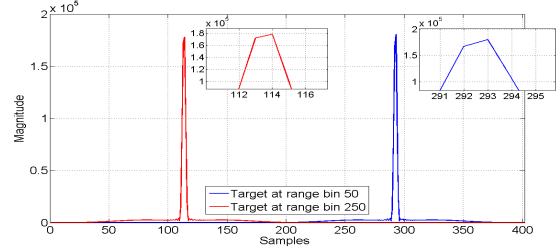


Fig. 3 FrFT for different targets ranges

### B. FrFT Matched Filter dependence on $\Delta f$

Two different target chirp signals with band width of 200 kHz and 500 kHz respectively are considered next. These target chirp signals start at the same time and are completely overlapped in time as seen in Fig. 4. The FFT of both signals is seen in Fig. 5. As expected they also overlap in the frequency domain. In Fig. 6, the two spikes at samples 2125 and 2285 (in the zoomed figures) demonstrate the dependence of the signal on the chirp band width  $\Delta f$  in the optimum FrFT domain. From Fig. 6 we can observe that if the two chirp signals completely overlap in time and frequency they can be separable in the FrFT domain.

### C. FrFT Matched Filter dependence on pulse duration

Consider a received radar signal that comprises two chirps with the same start time  $T_{start}$  but with different pulse width duration are seen in Fig. 7. Both chirps start at range bin 150 with pulse width duration equal to 100 micro sec. and 120 micro sec respectively. These two signals have the same chirp bandwidth so the FFT of both signals are identical as seen in Fig. 8. The FrFT of both signals are seen in Fig. 9 in sample 203 and 194 (shown in the zoomed

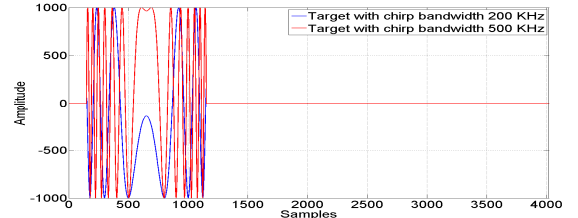


Fig. 4 Chirp signals for different targets bandwidth

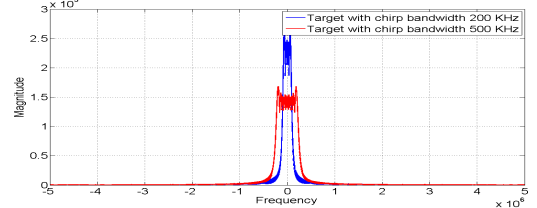


Fig. 5 FFT of different targets bandwidth

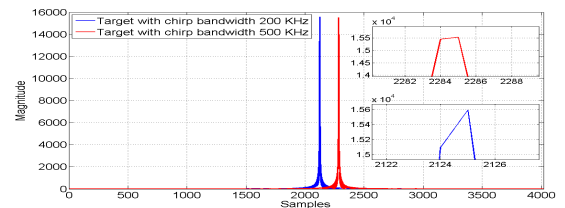


Fig. 6 FrFT of different targets bandwidth

for the chirps with pulse width duration equal to 100 micro sec. and 120 micro sec. respectively. Thus we can observe the dependence of the matched filter design in FrFT domain on the chirp pulse with duration.

#### D. Performance enhancement using FrFT matched filter

It is clear from the mathematical model for matched filter in FrFT domain (26) and the simulation results (section V A) that to obtain a matched filter in the FrFT domain the delay of the received chirp is required.

Considering a low noise signal interference level then the time delay  $T_{start}$ , can be estimated through two methods: The first one using the FrFT of the received chirp using (6), by knowing  $P_p$  we can get  $t_{st}$  that is then used to determine  $T_{start}$ . The second one uses the traditional matched filter, the matched filter output peak existing at  $T_{start}$  of the chirp pulse. Using one of these methods to estimate  $T_{start}$  that used to design the matched filter in the FrFT domain.

FT and FrFT matched filters outputs for two chirp signals with duration 1 sec. and chirp bandwidth 30 Hz starting at zero and 7.5 sec are shown in Fig. 10 ( $T_{start} = 0$ ) and Fig. 11 (estimated  $T_{start}$ ) respectively. In these figures an improvement of the chirp compressed width using the FrFT matched filter rather than the FT matched filter for the applications with for  $T_{start} = 0$  or estimated  $T_{start}$  is observed. In this case the FrFT matched filter output is narrower

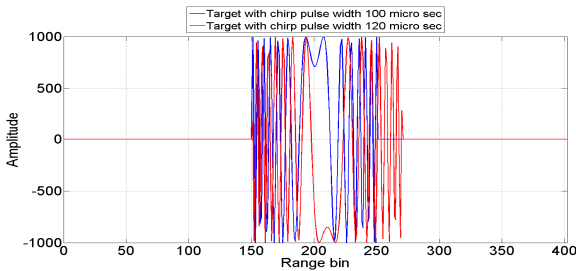


Fig. 7 Chirp signals for different targets pulse duration

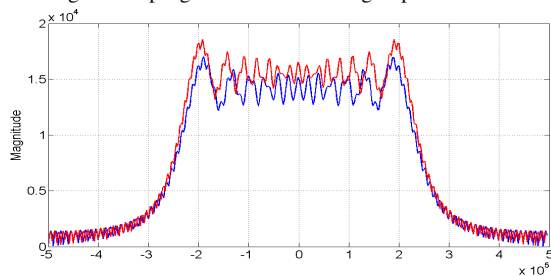


Fig. 8 FFT for different targets pulse duration

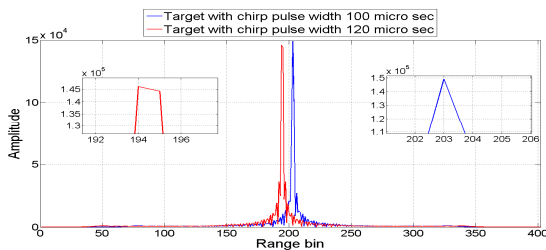


Fig. 9 FrFT for different targets pulse duration

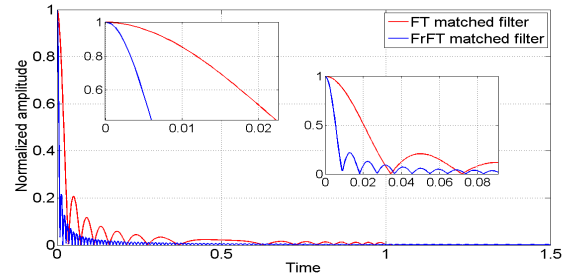


Fig. 10 FT and FrFT Matched filters for  $T_{start} = 0$

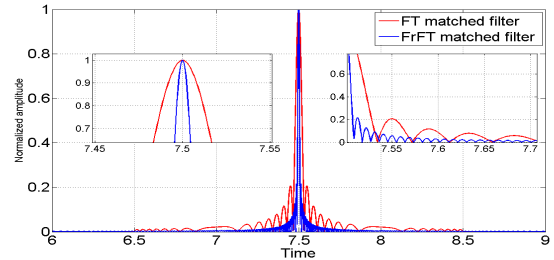


Fig. 11 FT and FrFT Matched filters for known  $T_{start}$

than the FT matched filter. At the  $-3\text{dB}$  point the ratio between the width of the FT matched filter out to that of the FrFT matched filter out is approximately 3.706. A significant reduction in side lobe using the FrFT matched filter is also observed in Figs 10 and 11.

## VI. CONCLUSION

In this paper matched filters in the FT domain and the FrFT domain were investigated. The limitation of using matched filter in the optimum FrFT for a chirp was shown to be the dependence of the chirp shape in this domain on the chirp start time and chirp band width. In the special case where specific parameters can be estimated then a significant enhancement results when using matched filter in the optimum FrFT domain compared to the conventional FT approach.

## REFERENCES

- [1] S. A. Elgamel and J. J. Soraghan, "Enhanced Monopulse Radar Tracking Using Filtering In Fractional Fourier Domain," in *IEEE International Radar Conference* Washington DC, USA, 2010.
- [2] H. M. Ozaktas, O. Arikan, M. A. Kutay, and G. Bozdogat, "Digital computation of the fractional Fourier transform," *Signal Processing, IEEE Transactions on*, vol. 44, pp. 2141-2150, 1996.
- [3] H. M. Ozaktas, G. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform: with Applications in Optics and Signal Processing*: John wiley & Sons Ltd, January 2001.
- [4] C. Candan, M. A. Kutay, and H. M. Ozaktas, "The discrete fractional Fourier transform," *Signal Processing, IEEE Transactions on*, vol. 48, pp. 1329-1337, 2000.
- [5] C. Capus and K. Brown, "Short-Time fractional fourier methods for the time-frequency representation of chirp signals," *The Journal of the Acoustical Society of America*, vol. 113(6), pp. 3253-63, 2003.
- [6] R. Jacob, T. Thomas, and A. Unnikrishnan, "Applications of Fractional Fourier Transform in Sonar Signal Processing," *IETE Journal of Research*, vol. 55, pp. 16-27, 2009.
- [7] A. Bultheel and H. Martinez Sulbaran, "A shattered survey of the fractional Fourier transform," <http://www.cs.kuleuven.be/~nalag/papers/ade/frft/index.html>, 2003.
- [8] E. Leith, "Review of 'Systems and Transforms With Applications to Optics' (Papoulis, A.; 1968)," *Information Theory, IEEE Transactions on*, vol. 18, pp. 451-452, 1972.