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Quantum state comparison and the universal-NOT operation

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We show that the form of the optimal universal-NOT operation for a single qubit can be determined by considering quantum-limited state comparison. Similarly, optimal state comparison can be derived from the properties of the universal-NOT. This points to the possibility of a fundamental link between these processes.

1 Introduction

The study of quantum information has identified unexpected resources for communications, computation and for metrology [1]. It is not possible, however, to do everything that we might like to. A notable example is the famous no-cloning theorem [2, 3], which tells us that it is not possible to copy, perfectly, a system prepared in an unknown state. This restriction was discovered in the process of attempting to show that nonlocal correlations, of the type associated with entangled states, do not allow superluminal communications. Imposing this no-communications condition, moreover, allows us to place bounds on the extent to which approximate cloning is possible [4]. It is indeed remarkable that this procedure leads to the quantum limit for the operation of a symmetric (that is state independent) cloning device [5]. Combining different phenomena in this way, in this case relativistic locality and quantum copying, is reassuring in that it demonstrates consistency, but it also tells us something about the way in which these apparently quite distinct ideas are related. There are further examples, of course, in particular the connection between the no-signaling theorem [6] and unambiguous state discrimination [7] and maximum confidence measurements [8, 9], and even the allowed forms of quantum dynamics [10].

In this paper we demonstrate a connection, not between relativity and quantum theory, but rather between two apparently quite distinct processes in quantum information theory. These are the extent to which we can determine whether or not two qubits have been prepared in the same state, and the extent to which it is possible to perform a universal-NOT operation. In the first of these we are given two qubits which have either been prepared in the same (pure) quantum state or different states. We are not told which states might have been prepared, but only that they are either the same or different. Given only this very limited information there is only one measurement we can reasonably perform and this, therefore, is the optimal one [11].

The existence of a universal-NOT operation, that is, one that replaces any pure qubit state by the one orthogonal to it, is necessarily an anti-unitary transformation [12] and is therefore forbidden [13–15]. We shall illustrate this important point with a simple example. As with cloning, the impossibility of performing the operation perfectly does not preclude achieving the task approximately. The optimal universal-NOT operation transforms any qubit pure state into the orthogonal state with probability $\frac{2}{3}$ and leaves it unchanged with probability $\frac{1}{3}$ [13–15].

We begin with brief reminders, first of quantum state comparison, and then of the universal-NOT operation, in which we also present a simple derivation of the optimal universal-NOT. This is followed, in section 4, by a simple rederivation of the optimal performance of a universal-NOT gate from the

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conditions for state comparison. We complete our comparison, in section 5, by showing that the universal-NOT operation also provides a tight bound on state comparison. These considerations suggest that there is a strong connection between these operations. We speculate on the origin of this connection.

2 State comparison

In quantum state comparison we are given two systems, each of which has been prepared in an unknown pure state. Our task is to determine, as well as is possible, whether they are the same or different [11]. It suffices, for our purposes, to consider a pair of qubits, the first of which is prepared in the general, but unknown, pure state $|\psi\rangle$ and the second of which is prepared either in the state $|\psi\rangle$ or in $|\psi^\perp\rangle$, the state orthogonal to it. This means that the two-qubit state is either $|\psi\rangle \otimes |\psi\rangle$ or $|\psi\rangle \otimes |\psi^\perp\rangle$. This situation is reminiscent of the problem addressed by Gisin and Popescu, in which two parallel or antiparallel spins are used to extract a direction in space [14]. The fact that there is a difference between these was, itself, an early indication that the ideal universal-NOT operation is not possible.

We are not given any information concerning the form of the state $|\psi\rangle$. This means that it can be any state of the form

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \tag{1}$$

where $|0\rangle$ and $|1\rangle$ are a pair of orthogonal states. The set of all possible states is obtained by intergration over the whole of the Bloch sphere, with θ varying between 0 and π and with ϕ taking all values between 0 and 2π. We can incorporate our ignorance of the state by performing this integration, with the result that the a priori density operator, if the states are the same, is

$$\begin{aligned}\hat{\rho}_{\text{same}} &= \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi| \\ &= \frac{1}{3}\hat{\mathrm{P}}_{\text{sym}},\end{aligned} \tag{2}$$

where $\hat{\mathrm{P}}_{\text{sym}} = \hat{\mathbb{I}} - |\Psi^-\rangle\langle\Psi^-|$ is the projector onto the space of symmetric states, $\mathbb{I}$ is to two-qubit identity operator, and

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle), \tag{3}$$

is the antisymmetric or singlet state. If the two systems are prepared in orthogonal states, then the same integration gives a different density operator:

$$\begin{aligned}\hat{\rho}_{\text{orth}} &= \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta |\psi\rangle\langle\psi| \otimes |\psi^\perp\rangle\langle\psi^\perp| \\ &= \frac{1}{6}\hat{\mathrm{P}}_{\text{sym}} + \frac{1}{2}\hat{\mathrm{P}}_{\text{antisym}},\end{aligned} \tag{4}$$

where $\hat{\mathrm{P}}_{\text{antisym}} = |\Psi^-\rangle\langle\Psi^-|$ is the projector onto the single antisymmetric state.

With the states written in the forms (2) and (4), it is clear that the only meaningful measurement we can perform is whether the two qubits are in their symmetric or antisymmetric subspaces [11]. This conclusion is true more generally and can be extended to apply to cases in which more than two qubits are provided [16]. We can treat the task of determining whether the states are the same or different as a minimum-error problem. If the a priori probability that the states are the same is P_{same} and the probability that they are orthogonal is $P_{\text{orth}} = 1 - P_{\text{same}}$, then the well-known necessary and sufficient conditions for minimum error [17–20] lead us to the operator

$$\hat{\Delta} = P_{\text{orth}}\,\hat{\rho}_{\text{orth}} - P_{\text{same}}\,\hat{\rho}_{\text{same}}. \tag{5}$$

The minimum-error discrimination between the states being the same or orthogonal is achieved by measuring in eigenbasis of $\hat{\Delta}$, and associating the positive eigenvalues with the states being orthogonal and the negative eigenvalues with them being the same. We note that if $P_{\text{orth}} > \frac{2}{3}$ then no measurement is required, and we minimize the probability of error simply by guessing that the states are orthogonal [21]. For smaller values of P_{orth} we simply need to perform a measurement to determine whether we are in the symmetric or antisymmetric subspaces and associate these two possibilities with the states being the same or different, respectively.

We are not restricted simply by the condition for minimum error, and our idea requires us, in particular, to consider maximum confidence measurements [8]. It suffices, for our purposes, to consider only the special case in which the two systems are equally likely to have been prepared in the same state, so that $P_{\text{orth}} = \frac{1}{2} = P_{\text{same}}$. It is straightforward to show that, in this case, the maximum-confidence and minimum-error measurements are the same and correspond to measuring the symmetry of the two-qubit system as embodied in the projectors $\hat{\text{P}}_{\text{sym}}$ and $\hat{\text{P}}_{\text{antisym}}$. This gives the following four possible joint probabilities[1]

$$\begin{aligned}
P(\text{antisym}, \text{same}) &= \text{Tr}\left(\hat{\text{P}}_{\text{antisym}}\,\hat{\rho}_{\text{same}}\right) P_{\text{same}} = 0,\\
P(\text{antisym}, \text{orth}) &= \text{Tr}\left(\hat{\text{P}}_{\text{antisym}}\,\hat{\rho}_{\text{orth}}\right) P_{\text{orth}} = \frac{1}{4},\\
P(\text{sym}, \text{same}) &= \text{Tr}\left(\hat{\text{P}}_{\text{sym}}\,\hat{\rho}_{\text{same}}\right) P_{\text{same}} = \frac{1}{2},\\
P(\text{sym}, \text{orth}) &= \text{Tr}\left(\hat{\text{P}}_{\text{sym}}\,\hat{\rho}_{\text{orth}}\right) P_{\text{orth}} = \frac{1}{4}.
\end{aligned} \tag{6}$$

A straightforward application of Bayes' rule leads to the conditional probabilities

$$\begin{aligned}
P(\text{same}|\text{antisym}) &= 0,\\
P(\text{orth}|\text{antisym}) &= 1,\\
P(\text{same}|\text{sym}) &= \frac{2}{3},\\
P(\text{orth}|\text{sym}) &= \frac{1}{3}.
\end{aligned} \tag{7}$$

These express the facts that if the two qubits are found to be in the antisymmetric state then they could not have been prepared in the same state, and that, if the a priori probabilities are equal then no measurement can determine that the states were the same (as opposed to being orthogonal) with a probability greater that $\frac{2}{3}$. The second of these will be important in establishing a link with the universal-NOT operation.

3 Universal-NOT

A perfect universal-NOT operation would transform *every* single-qubit state $|\psi\rangle$ into the corresponding orthogonal state $|\psi^{\perp}\rangle$. That this is not possible follows directly from the observation that such a transformation is anti-unitary and that such transformations cannot be realized [13–15].

Anti-unitarity [12] is, perhaps, not the most familiar of concepts in quantum theory and a simple demonstration might help to convey the main idea. Let us start by supposing that an ideal and universal-NOT

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[1] Note that these probabilities are precisely the same if we use the pure states $|\psi\rangle \otimes |\psi\rangle$ and $|\psi\rangle \otimes |\psi^{\perp}\rangle$ in place of the averaged states $\hat{\rho}_{\text{same}}$ and $\hat{\rho}_{\text{orth}}$.

operation can be performed and note that the effect of this on an eigenstate of any of the Pauli operators $\hat{\sigma}_x$, $\hat{\sigma}_y$, or $\hat{\sigma}_z$, will be to change the sign of the eigenvalue. The fully antisymmetric state $|\Psi^-\rangle$ is a simultaneous eigenstate of the three operators $\hat{\sigma}_x \otimes \hat{\sigma}_x$, $\hat{\sigma}_y \otimes \hat{\sigma}_y$, and $\hat{\sigma}_z \otimes \hat{\sigma}_z$, with the eigenvalue in each case being -1. Applying an ideal universal-NOT operation to the first qubit in the state $|\Psi^-\rangle$ would necessarily produce a new eigenstate of the operators $\hat{\sigma}_x \otimes \hat{\sigma}_x$, $\hat{\sigma}_y \otimes \hat{\sigma}_y$, and $\hat{\sigma}_z \otimes \hat{\sigma}_z$, but this time with the eigenvalue in each case being $+1$. It is straightforward to show that no such state exists. If we multiply our three operators together then we find

$$(\hat{\sigma}_x \otimes \hat{\sigma}_x)(\hat{\sigma}_y \otimes \hat{\sigma}_y)(\hat{\sigma}_z \otimes \hat{\sigma}_z) = -\hat{\mathbb{I}}. \tag{8}$$

It necessarily follows that at least one of the eigenvalues of our three operators must be -1 and hence that the ideal universal-NOT operation cannot be realized.

The best we can do for a symmetric universal-NOT (that is one that acts equally well for all pure states) is that found by Bužek, Hillery and Werner [13, 15]. This operation, which we denote $\mathcal{N}$, performs the transformation

$$\mathcal{N}(|\psi\rangle\langle\psi|) = \frac{2}{3}|\psi^\perp\rangle\langle\psi^\perp| + \frac{1}{3}|\psi\rangle\langle\psi|, \tag{9}$$

for all single-qubit states $|\psi\rangle$. Note that the fidelity of this optimal universal-NOT operation [22], that is the probability of success in generating the state $|\psi^\perp\rangle$, is $\frac{2}{3}$. It is worth noting that the optimal state-independent cloning machine [5] realizes, as a byproduct, this universal-NOT operation outputted as the state of the required third qubit [15]. The appearance of the fraction $\frac{2}{3}$ as the maximum of the probability in determining that two states are the same, in state comparison, and as the maximum fidelity of the universal-NOT operation, is an important pointer to the connection between them.

We conclude by presenting a simple proof of the optimality of the universal-NOT operation. Our staring point is to write the ideal, but unphysical, universal-NOT operation as a (purely mathematical) transformation of an initial density operator $\hat{\rho}$:

$$\hat{\rho} \to \frac{1}{2}\hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + \frac{1}{2}\hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + \frac{1}{2}\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z - \frac{1}{2}\hat{\rho}. \tag{10}$$

It is a straightforward exercise to confirm that this transforms any pure-state density operator $\hat{\rho} = |\psi\rangle\langle\psi|$ into the orthogonal state $|\psi^\perp\rangle\langle\psi^\perp|$. That this is an unphysical transformation is evident in that it cannot be expressed in the language of effects and operations [19, 23]. The problem is the minus sign in the last term, the presence of which violates complete positivity [24]. We can resolve this difficulty, and so arrive at physically-allowed transformation, by reducing to zero the coefficient in front of the negative term $(-\hat{\rho})$ and also increasing the coefficients in front of the remaining three terms, so as to preserve the trace. The resulting optimal universal-NOT operation then has the form

$$\hat{\rho} \to \frac{1}{3}\hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + \frac{1}{3}\hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + \frac{1}{3}\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z, \tag{11}$$

the action of which produces the operation (9).

4 From state comparison to the universal-NOT

We can show how our bound on state comparison leads to the optimal operation of the universal-NOT by considering a pure state of three qubits, which we label a, b, and c. Let qubit a be prepared in our unknown pure state $|\psi\rangle$ and the remaining two be in the antisymmetric Bell state $|\Psi^-\rangle$, so that the combined state vector is $|\psi\rangle_a|\Psi^-\rangle_{bc}$. The state $|\Psi^-\rangle$ is perfectly anti-correlated in that a measurement of any component of spin on the two component qubits will give the result $+1$ for one and -1 for the other. The antisymmetry of this Bell state means that, apart from an unimportant phase factor, we can write it in terms of our unknown state $|\psi\rangle$ and the state orthogonal to it:

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle \otimes |\psi^\perp\rangle - |\psi^\perp\rangle \otimes |\psi\rangle). \tag{12}$$

It follows that the reduced density operator for our ab-system is an equally weighted mixture of the states $|\psi\rangle_a|\psi\rangle_b$ and $|\psi\rangle_a|\psi^\perp\rangle_b$:

$$\begin{aligned}\hat{\rho}_{ab} &= \mathrm{Tr}_c \left(|\psi\rangle\langle\psi| \otimes |\Psi^-\rangle\langle\Psi^-|\right) \\ &= \frac{1}{2}|\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi| + \frac{1}{2}|\psi\rangle\langle\psi| \otimes |\psi^\perp\rangle\langle\psi^\perp|. \end{aligned} \tag{13}$$

This is precisely the a priori state that faces us when trying to perform the state discrimination problem, described in section 2, with the probabilities P_{same} and P_{orth} equal.

The optimal measurement to determine whether qubits a and b are in the same or different states is, as we have seen, to measure whether they are are in the symmetric or antisymmetric state spaces. If they are found in the symmetric state space then we should assume that the states were the same and, because of the anti-correlation properties of the state $|\Psi^-\rangle$, that the NOT operation has been realized on qubit c.[1] We have seen that the probability that the qubits a and b were indeed in the same state, given that our measurement shows them to be in the symmetric subspace, cannot exceed $\frac{2}{3}$, and that they will be different with probability $\frac{1}{3}$. It follows immediately that qubit c is left in the state

$$\hat{\rho}_c = \frac{2}{3}|\psi^\perp\rangle\langle\psi^\perp| + \frac{1}{3}|\psi\rangle\langle\psi|, \tag{14}$$

and that this is the closest we can get to transforming qubit c into the state $|\psi^\perp\rangle$. It also corresponds, of course, the optimal universal-NOT operation. If we could identify two qubits, equally likely to be in the same or orthognal states, as being in the same space with a probability of greater than $\frac{2}{3}$ then we would be able to realize a universal-NOT operation with a fidelity of greater than $\frac{2}{3}$.

If our measurement reveals qubits a and b to be in the antisymmetic space then it immediately follows that qubit c is left in the state $|\psi\rangle$. In such cases we can repeat the whole process and continue to do so until we get a result corresponding to the symmetric subspace. In this way we can, eventually, enforce the universal-NOT operation.

5 From the universal-NOT to state comparison

Having established that the optimal performance of the universal-NOT operation can be determined from state comparison, we show, in this section, that we can also obtain optimal state comparison from the properties of the universal-NOT. To this end, let us suppose, once again, that we have two quibts prepared, with equal probability either in the same, unknown, state or in orthogonal states so that the combined state vector is $|\psi\rangle \otimes |\psi\rangle$ or $|\psi\rangle \otimes |\psi^\perp\rangle$. We have seen that no measurement is possible, the outcome of which will allow us to infer that the states are equal with a probability of greater than $\frac{2}{3}$. Let us further suppose that we can perform a universal-NOT operation $\mathcal{N}_p$ that succeeds with probability p, so that

$$\begin{aligned}\mathcal{N}_p\left(|\psi\rangle\langle\psi|\right) \otimes |\psi\rangle\langle\psi| &= p|\psi^\perp\rangle\langle\psi^\perp| \otimes |\psi\rangle\langle\psi| + (1-p)|\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|, \\ \mathcal{N}_p\left(|\psi\rangle\langle\psi|\right) \otimes |\psi^\perp\rangle\langle\psi^\perp| &= p|\psi^\perp\rangle\langle\psi^\perp| \otimes |\psi^\perp\rangle\langle\psi^\perp| + (1-p)|\psi\rangle\langle\psi| \otimes |\psi^\perp\rangle\langle\psi^\perp|. \end{aligned} \tag{15}$$

[1] This situation is reminiscent, of course, of the process of teleportation of the state of a qubit [25].

Our task is then to find an upper bound on the allowed value of p using state comparison.

We can perform a state comparison measurement on the states (15) by determining whether the two qubits are in the symmetric of antisymmetric state spaces. For equal equal probabilities ($P_{\mathrm{orth}} = P_{\mathrm{same}}$) we find the joint probabilities

$$\begin{aligned} P_p(\mathrm{antisym}, \mathrm{same}) &= \frac{p}{4}, \\ P_p(\mathrm{antisym}, \mathrm{orth}) &= \frac{1-p}{4}, \\ P_p(\mathrm{sym}, \mathrm{same}) &= \frac{1}{2} - \frac{p}{4}, \\ P_p(\mathrm{sym}, \mathrm{orth}) &= \frac{1}{4} + \frac{p}{4}. \end{aligned} \tag{16}$$

It is best, because we have performed a (hopefully) optimal universal-NOT operation, to associate antisymmetric outcomes with the qubits having been prepared in the same state. The confidence with which we can make this association is limited by the conditional probabilities, calculated using Bayes' rule:

$$\begin{aligned} P_p(\mathrm{same}|\mathrm{orth}) &= p, \\ P_p(\mathrm{orth}|\mathrm{same}) &= 1 - p. \end{aligned} \tag{17}$$

It is not possible, however, to determine that the states were the same with a probability of greater than $\frac{2}{3}$ and it follows that $p \leq \frac{2}{3}$. This is precisely the value associated with the universal-NOT and so we have established that optimal state comparison provides the limiting behaviour of the universal-NOT.

6 Conclusion and speculation

It is pleasing when apparently distinct ideas turn out to be related and even more so when they are very strongly connected. We have shown that the, superficially quite distinct, tasks of comparing the states of two quantum systems and of inverting the unknown state of a qubit are related in this way. It is possible to derive the optimal forms of either process by assuming that of the other.

We conclude by speculating on the origins of the strong connection between optimal state comparison and the universal-NOT. A possible clue to this lies in the prominent part played by the antisymmetric state $|\Psi^-\rangle$ throughout our analysis. This state is uniquely defined by the requirement that the results of measurements of any spin component on both of our qubits qubits are perfectly anti-correlated. By contrast, there is no state for which such all such spin measurements are perfectly correlated. It is this difference that is responsible for the fact that we can determine, with certainty, that two unknown pure states are different but not that they are the same. We showed, in section 3, that this difference provides a straightforward proof of the non-existence of an ideal universal-NOT operation and, in section 4, that consideration of this same state leads to the best possible approximation to this ideal. If the perfectly correlated analogue of the state $|\Psi^-\rangle$ did exist then it would be possible to know, with certainty, that the states of two qubits were the same, and also to perform an ideal universal-NOT operation on a single qubit.

We can test this idea by considering a simpler system in which our qubits are prepared in states restricted to a single great circle on the Bloch sphere. We may, for example, take these to be the real states [26], those in (1) with $\phi = 0$ or π. A rotation through π about an axis perpendicular to the plane of the great circle will transform a state $|\psi\rangle$ into $|\psi^\perp\rangle$. For the real states, this is achieved by means of the unitary transformation $\hat{\sigma}_y$. Thus if we restrict ourselves to the real qubit-states then the ideal universal-NOT operation *is possible*. For this restriction, it is also possible to do something in state comparison that is impossible for more general states. If, once again, we have two qubits prepared in the state $|\psi\rangle \otimes |\psi\rangle$ or

$|\psi\rangle \otimes |\psi^{\perp}\rangle$, where now $|\psi\rangle$ is a real state, then the projector

$$|\Psi^{+}\rangle\langle\Psi^{+}| = \frac{1}{2}\left(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\right)\left(\langle 0| \otimes \langle 1| + \langle 1| \otimes \langle 0|\right) \tag{18}$$

has a zero expectation value for the state $|\psi\rangle \otimes |\psi^{\perp}\rangle$. Hence if we perform a measurement and obtain the result corresponding to this projector, then we know for certain that the states are the same rather than orthogonal. This conclusion would not be possible without the restriction to the states on a single great circle on the Bloch sphere. For the real states, $|\Psi^{+}\rangle$ is the perfectly correlated analogue of the state $|\Psi^{-}\rangle$ that does not exist for more general states.

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