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# A Heuristic Approach for Big Bucket Multi-Level Production Planning Problems

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6 November 2007

## Abstract

Multi-level production planning problems in which multiple items compete for the same resources frequently occur in practice, yet remain daunting in their difficulty to solve. In this paper we propose a heuristic framework that can generate high quality feasible solutions quickly for various kinds of lot-sizing problems. In addition, unlike many other heuristics, it generates high quality lower bounds using strong formulations, and its simple scheme allows it to be easily implemented in the Xpress-Mosel modeling language. Extensive computational results from widely used test sets that include a variety of problems demonstrate the efficiency of the heuristic, particularly for challenging problems.

**Keywords:** Integer Programming, Production Planning, Heuristics, Relax-and-Fix, Strong Formulations

## 1 Introduction

Production planning problems occur often in manufacturing environments and have therefore interested researchers and practitioners for decades. The early MRP (Materials Requirement Planning) approach, while accounting for bill-of-material structures, follows decision rules that are too simple to achieve feasible plans (let alone high quality plans) consistently, particularly due to not taking capacity constraints into account. The more advanced systems of MRP-II (Manufacturing resource planning) and ERP (Enterprise Resource Planning) combine different methodologies such as MRP and aggregate planning, but they fail to account

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accurately for capacity. In this paper, we investigate multi-level capacitated production planning problems, for which MRP-II and ERP systems aim to find feasible production plans. Pochet and Wolsey [2006] give extensive details on the production planning, and Hoppe and Spearman [2000] explain MRP and ERP models in detail.

Since the seminal paper of Wagner and Whitin [1958] in 1958 on the uncapacitated lot-sizing problem, various forms of production planning problems have been investigated. Even though some special cases of lot-sizing problems can be solved in polynomial time (e.g. Florian and Klein [1971], Federgruen and Tzur [1991], Aggarwal and Park [1993] and van Hoesel and Wagelmans [1996]), real-world problems are much more complicated and computationally challenging, therefore more efficient methods are needed to tackle these difficulties. Moreover, the capacitated version of even the single-item lot-sizing problem is  $\mathcal{NP}$ -hard (Florian et al. [1980]), so these problems are also theoretically hard to solve. Given deterministic demand and other parameters such as inventory holding, production, and setup costs, lot-sizing problems seek a minimum-cost production plan that satisfies all the constraints of the finite horizon under consideration.

The solution approaches proposed so far for these problems vary from exact approaches based on mathematical programming to heuristic methods. Mathematical programming attempts to improve the formulations so that these can be solved faster and more efficiently, and in principle, these methods provide exact solutions with lower bounds. On the other hand, heuristics are practical methods to quickly discover good solutions, not necessarily the best, and these methods usually do not provide lower bounds that would guarantee solution quality.

## 1.1 Mathematical Programming Approaches

Mathematical programming provides tools for exact solution approaches for MIP problems, and we will give an overview of such approaches for production planning models. For production planning problems, polyhedral analysis has been useful in attempts to solve challenging models and to provide strong lower bounds. The first group of these exact methods strengthens the original formulation by adding valid inequalities. Barany et al. [1984a] propose valid inequalities that define the polytope of the single-item uncapacitated lot-sizing problem, which are also used for the strong formulation obtained in our framework. The capacitated problem with start-up costs is studied by Constantino [1996]. There are insightful polyhedral studies about some special cases of the single-item problem, see e.g. Pochet and Wolsey [1988], Pochet and Wolsey [1994] and Loparic et al. [2001]. Atamtürk and Muñoz [2004] investigate the bottleneck cover structure of single-item capacitated problems in their recent polyhedral study.

The second group of exact methods use extended reformulations of the model, which basically adds new variables to the problem to strengthen it. Krarup and Bilde [1977] provide a facility location reformulation of the problem, Eppen and Martin [1987] propose the shortest path reformulation which is equivalent to the facility location reformulation but

is smaller in size. Rardin and Wolsey [1993] define multicommodity reformulations for fixed charge network flow problems, which give stronger reformulations than the facility location reformulation for the multi-level lot-sizing problem. See also Belvaux and Wolsey [2001] and Wolsey [2002] for recent studies considering reformulation and modeling issues.

Although Pochet and Wolsey [1991] and Belvaux and Wolsey [2000] extend some of the single-item problem results to big-bucket problems, the current literature on strong formulations for big-bucket problems is limited. An exception is the study of multi-item problems in Miller et al. [2000] and Miller et al. [2003].

## 1.2 Production Planning Heuristics

Problem-specific heuristic algorithms have been proposed and used for complex, realistic lot-sizing problems, such as multi-level, capacitated problems. For a detailed study with comparisons of earlier lot-sizing heuristics, see Maes and van Wassenhove [1986]. Many heuristic approaches use decomposition ideas, and can be grouped as follows: 1) Lagrangian-based decomposition: Trigeiro et al. [1989] and Tempelmeier and Derstroff [1996] are primary examples for this group of heuristics. 2) Coefficient modification: Katok et al. [1998] propose a two-stage heuristic, where coefficient modification is used for finding an initial solution, and restricted LP relaxations of the second stage try to improve the initial solution. Van Vyve and Pochet [2004] propose a coefficient modification based heuristic algorithm to be used within branch-and-cut. 3) Forward scheme and Relax-and-fix: Afentakis and Gavish [1986] work on complicated BOM (Bills of Material) structures and use a forward scheme to obtain solutions while using Lagrangian relaxation for lower bounds. Belvaux and Wolsey [2000] solve practical lot-sizing problems using a special branch-and-cut system that employs relax-and-fix heuristics. Stadtler [2003] and Federgruen et al. [2007] primarily use the idea of “time windows” in their frameworks. Note that our approach also uses relax-and-fix idea. 4) Local search: A good example is the neighborhood search heuristic of Simpson and Erenguc [2005].

## 1.3 General MIP Heuristics

Real-world MIP problems are often computationally very difficult, hence researchers have often used heuristic approaches to tackle them. Heuristics can be classified as “Improvement Heuristics” (i.e. start with a solution and try to improve it) and “Constructive Heuristics” (i.e. start with no solution and try to find one).

Even though there are many problem-specific heuristics in the MIP literature, there are comparatively few general MIP heuristics. Recent general MIP heuristics include “Local Branching” by Fischetti and Lodi [2003], which uses the idea of branching on the neighborhoods of the current MIP solution, and “Relaxation Induced Neighborhood Search” (RINS) by Danna et al. [2005], which searches the neighborhood between the LP relaxation solution and the current MIP solution. The “Feasibility Pump” of Fischetti et al. [2005] provides a

framework designed for finding initial feasible solutions for very hard problems, and Balas et al. [2001] enumerate facets to find solutions for pure 0-1 problems. For a general review of MIP heuristics, such as LP-and-fix and relax-and-fix, refer to Pochet and Wolsey [2006], pp. 107-113.

## 1.4 Organization of the Paper

The approach we propose in this paper is a decomposition-based framework that incorporates strengthened formulations and ideas used in general MIP heuristics. Our approach is simple and flexible enough to apply to a variety of production planning problems.

In Section 2, we give the basic formulation of the problem, and we also discuss some generalizations of the problem since our goal is to define a heuristic flexible enough to handle many kinds of problems. In Section 3, we discuss strengthened formulations for big bucket problems. Section 4 covers the general MIP heuristic methods that we use in our framework. Section 5 is devoted to the explanation of the framework of the proposed heuristic and any options that may be used. In Section 6, we present extensive computational results obtained using a number of published data sets. Finally, we conclude with future directions in Section 7.

## 2 Problem Formulation

We consider the general multi-level lot-sizing problem, where the objective is to minimize the total cost. In order to achieve the optimal plan, production and inventory quantities, as well as setup decisions, have to be determined. While doing so, capacities should not be exceeded and demand should be satisfied. Before providing the basic formulation of the problem, we will present the notation.

*Indices and Sets:*

$NT$	Number of periods
$NI$	Number of items
$NK$	Number of machines
$endp$	Set of all the end-items (items with external demand)
$\delta(i)$	Set of immediate successors of item $i$

*Variables:*

$x_t^i$	Production variable for item $i$ in period $t$
$y_t^i$	Setup variable for item $i$ in period $t$
$s_t^i$	Inventory holding variable for item $i$ in period $t$

*Parameters:*

$f^i$	Setup cost for item $i$ (per setup)
$h^i$	Inventory holding cost for item $i$ (per unit, per period)
$d_t^i$	Demand for end-product $i$ in period $t$
$d_{t,t'}^i$	Total demand for end-product $i$ from period $t$ to $t'$ , i.e., $d_{t,t'}^i = \sum_{\bar{t}=t}^{t'} d_{\bar{t}}^i$
$r^{ij}$	Number of items required of $i$ to produce one unit of $j$
$a_k^i$	Variable time necessary to produce one unit of $i$ on machine $k$
$ST_k^i$	Setup time for item $i$ on machine $k$
$C^k$	Capacity of machine $k$

Then, the formulation of the problem follows:

$$\min \sum_{t=1}^{NT} \sum_{i=1}^{NI} f^i y_t^i + \sum_{t=1}^{NT} \sum_{i=1}^{NI} h^i s_t^i \quad (1)$$

$$\text{s.t. } x_t^i + s_{t-1}^i - s_t^i = d_t^i \quad t \in [1, NT], i \in \text{endp} \quad (2)$$

$$x_t^i + s_{t-1}^i - s_t^i = \sum_{j \in \delta(i)} r^{ij} x_t^j \quad t \in [1, NT], i \in [1, NI] \setminus \text{endp} \quad (3)$$

$$\sum_{i=1}^{NI} (a_k^i x_t^i + ST_k^i y_t^i) \leq C^k \quad t \in [1, NT], k \in [1, NK] \quad (4)$$

$$x_t^i \leq M_t^i y_t^i \quad t \in [1, NT], i \in [1, NI] \quad (5)$$

$$y \in \{0, 1\}^{NT \times NI} \quad (6)$$

$$x \geq 0^{NT \times NI} \quad (7)$$

$$s \geq 0^{NT \times NI} \quad (8)$$

Here  $M_t^i$  is the maximum amount of item  $i$  that can be produced in  $t$ , and can be defined formally as follows:

$$M_t^i = \min(d_{t,NT}^i, \frac{C^k - ST_k^i}{a_k^i}) \quad i \in \text{endp}$$

$$M_t^i = \min(\sum_{j \in \text{endp}} r^{ij} d_{t,NT}^j, \frac{C^k - ST_k^i}{a_k^i}) \quad i \in [1, NI] \setminus \text{endp}$$

Constraints (2) and (3) ensure the production balance and demand satisfaction for end-items and other items respectively, (4) are the capacity constraints, (5) ensure that the setup variables are set to be 1 if there is positive production, and finally (6), (7) and (8) provide the integrality and nonnegativity requirements.

For an alternative formulation of the problem, we define echelon demand parameters  $D_t^i$  and echelon stock variables  $E_t^i$  as follows:

$$D_t^i = d_t^i + \sum_{j \in \delta(i)} r^{ij} D_t^j \quad t \in [1, NT], i \in [1, NI] \quad (9)$$

$$E_t^i = s_t^i + \sum_{j \in \delta(i)} r^{ij} E_t^j \quad t \in [1, NT], i \in [1, NI] \quad (10)$$

Note that echelon and original demands and stocks are equal for the end-items. Substituting (10) into (2) and (3) for  $s_t^i$ , and using the definition (9), we obtain the following equation, which can replace (2) and (3) in the original formulation:

$$x_t^i + E_{t-1}^i - E_t^i = D_t^i \quad t \in [1, NT], i \in [1, NI] \quad (11)$$

To satisfy (8), we can define the following constraint:

$$E_t^i \geq \sum_{j \in \delta(i)} r^{ij} E_t^j \quad t \in [1, NT], i \in [1, NI] \quad (12)$$

Finally, to eliminate the original inventory variables  $s$ , we define echelon inventory holding costs as  $h^{*j} = h^j - \sum_{i: j \in \delta(i)} r^{ij} h^i$ , and replace the objective function (1) with the following function:

$$\sum_{t=1}^{NT} \sum_{i=1}^{NI} f^i y_t^i + \sum_{t=1}^{NT} \sum_{i=1}^{NI} h^{*i} E_t^i \quad (13)$$

Hence, we can define the feasible region as  $X = \{(x, y, E) | (4) - (7), (11), (12), E \geq 0\}$ , which is the echelon stock reformulation of the original formulation and will be used in the remainder of the paper as the “basic formulation”. The problem can be then defined as:  $\min\{(13) | (x, y, E) \in X\}$ .

We could easily include the possibility of overtime in the problem statement by updating the capacity constraint (4) and adding overtime cost to the objective function. On the other hand, the possibility of backlogging can be incorporated into the model by defining backlogging variables  $b_t^i$ . Aside from the objective function, the only change in the initial formulation would be the replacement of (2) with

$$x_t^i + s_{t-1}^i - s_t^i + b_t^i - b_{t-1}^i = d_t^i \quad t \in [1, NT], i \in endp$$

Similarly, we can include backlogging into the echelon stock reformulation by replacing (11) with the following set of constraints:

$$x_t^i + E_{t-1}^i - E_t^i + \sum_{j \in endp} r^{ij} (b_t^j - b_{t-1}^j) = d_t^i \quad t \in [1, NT], i \in [1, NI]$$



Note also that the first term in the definition of big-M quantity used in (5) should be updated to  $D_{1,NT}^i$ . Other additional characteristics could be incorporated as well, and the interested reader should refer to Pochet and Wolsey [2006] for more detail on lot-sizing models. Some of the test problems we consider in Section 6 incorporate some of these factors.

### 3 Strengthening the Formulation

Next, we consider several ways to strengthen formulations, in order to provide high quality lower bounds. First, we consider the  $(\ell, S)$  inequalities proposed by Barany et al. [1984a] and generalized by Pochet and Wolsey [1991] to multi-level problems using echelon stocks, given as follows:

$$\sum_{t \in S} x_t^i \leq \sum_{t \in S} D_{t,\ell}^i y_t^i + E_\ell^i \quad \ell \in [1, NT], i \in [1, NI], S \subseteq [1, \ell] \quad (14)$$

Note that because of multiple levels, (14) uses echelon demand and echelon stock variables defined in the last section. For each item, although there are an exponential number of  $(\ell, S)$  inequalities, a simple polynomial separation algorithm exists (Barany et al. [1984a]). Note that  $(\ell, S)$  inequalities define the convex hull of the uncapacitated single-item lot-sizing problem, see Barany et al. [1984b].

Next we discuss the facility location reformulation of Krarup and Bilde [1977] originally proposed for single-item problems. The reformulation uses new variables  $u_{t,t'}^i$ . Each of these variables indicates the amount of item  $i$  that is produced in period  $t$  to satisfy demand in period  $t'$ . This reformulation can be applied to the problem by adding the following three constraints into the basic formulation:

$$u_{t,t'}^i \leq D_{t'}^i y_t^i \quad t \in [1, NT], t' \in [t, NT], i \in [1, NI] \quad (15)$$

$$\sum_{t=1}^{t'} u_{t,t'}^i = D_{t'}^i \quad t' \in [1, NT], i \in [1, NI] \quad (16)$$

$$x_{t'}^i \geq \sum_{t=t'}^{NT} u_{t,t'}^i \quad t' \in [1, NT], i \in [1, NI] \quad (17)$$

Note that the projection of the facility location reformulation onto the space of original variables gives the convex hull of the uncapacitated single-item lot-sizing problem, see Krarup and Bilde [1977].

Let  $X_{LS} = \{(x, y, E) | (4), (5), (7), (11), (12), (14), E \geq 0, 0 \leq y \leq 1\}$  and  $X_{FL} = \{(x, y, E, u) | (4), (5), (7), (11), (12), (15) - (17), E \geq 0, 0 \leq y \leq 1\}$ , i.e., the LP relaxations of the multi-level problem with all  $(\ell, S)$  inequalities and the echelon stock facility location reformulation, respectively.

**Proposition 1**  $\min\{(13) | (x, y, E) \in X_{LS}\} = \min\{(13) | (x, y, E, u) \in X_{FL}\}$

In words, adding  $(\ell, S)$  inequalities to the original formulation and using the echelon stock facility location reformulation provide the same lower bound for the multi-level lot-sizing problem. For the proof of the proposition, please refer to Akartunalı [2007]. The corollary below follows immediately.

**Corollary 1** *The shortest path reformulation of Eppen and Martin [1987] is equivalent to the echelon stock facility location formulation, and therefore all three formulations provide the same lower bounds for the multi-level lot-sizing problem.*

Using the fact that both shortest path and facility location reformulations define the convex hull of the single-item uncapacitated problem (see e.g. Pochet and Wolsey [2006]), this corollary can be proven with the same technique used for the proposition. Even though all three methods discussed above provide the same lower bounds, the echelon stock facility location and shortest path reformulations have the disadvantage of making the problem size much larger than the  $(\ell, S)$  inequalities, which are dynamically added to the formulation and deleted if inactive. Our experience, particularly from the companion paper Akartunalı and Miller [2007], shows that, for this reason,  $(\ell, S)$  inequalities allow for more efficient branch-and-bound and identifying of feasible solutions. Also note that a slightly modified family of  $(\ell, S)$  inequalities can be added to problems with backorders, as follows:

$$\sum_{t \in S} x_t^i \leq \sum_{t \in S} (D_{t,l}^i y_t^i + \sum_{j \in \text{endp}} r^{ij} b_{t-1}^j) + E_l^i \quad l \in [1, NT], i \in [1, NI], S \subseteq [1, l]$$

Here, observe that the separation algorithm for these inequalities has the same logic as the one described before. Finally, note that in the case of overtime, the original  $(\ell, S)$  inequalities are valid and will be added using the same separation algorithm.

There exist stronger inequalities and reformulations such as the multicommodity reformulation proposed by Rardin and Wolsey [1993], or multi-item single-period submodels used as by Miller et al. [2000], Miller et al. [2003]. However, we have found that using  $(\ell, S)$  inequalities alone seems to be the most effective way to strengthen the formulation without making the problem computationally inefficient, as discussed in a companion paper Akartunalı and Miller [2007].

## 4 MIP Heuristics

Here we discuss two techniques that we will use in our proposed framework.

### 4.1 LP-and-Fix

LP-and-fix is a simple technique, closely related to “diving” (see Pochet and Wolsey [2006]) and works as follows: First, we solve the LP relaxation ( $LPR$ ) of an  $MIP$ . Then, we check

all the integer variables in *LPR* and fix those having integral values. Finally, the restricted *MIP* with fixed variables is re-solved, with the hope of finding a solution to the original problem quickly. This scheme is employed in our heuristic framework both to provide an initial solution to be used as a cutoff value, and also throughout the algorithm to generate multiple production plans and to improve the best solution and hence the cutoff value.

## 4.2 Relax-and-Fix

Relax-and-fix is a heuristic method for problems with a special structure, and lot-sizing is well-suited for such a method since decisions made earlier in the horizon are more important than later ones. In this section, we discuss two previously developed heuristic methods that employ the “relax-and-fix” idea.

The first production planning tool that uses relax-and-fix is *bc-prod*, proposed by Belvaux and Wolsey [2000], which is a specialized branch-and-cut system for lot-sizing problems. The system first generates cutting planes, both default Xpress cuts and problem specific cuts, and then the relax-and-fix idea is applied using “time windows”, horizons under consideration, for which the length of the window is predefined. The basic description of the “time windows” idea is as follows: Except for the variables in the periods of the predefined time window, relax all the binary variables to be continuous, solve the problem, and using the solution obtained, fix the binary variables in the window. The next window is then processed in the same manner.

Another heuristic algorithm based on relax-and-fix is proposed by Stadtler [2003], which we will refer to as “SH” (Stadtler’s Heuristic) from now on. This approach uses time windows that overlap for better quality results. In such an approach, fixing variables in a window will occur only for the periods that do not overlap with the next window. Also, SH allows a period to be relaxed to continuous inside of the window, as well, because of using “bonuses”, which are calculated for each different problem. In this framework, each subproblem is formulated and strengthened separately. Figure 1 summarizes the idea of “time windows” in SH. A recent paper of Federgruen et al. [2007] also uses a relax-and-fix approach with time windows.

The main advantage of *bc-prod* is that strong formulations provide good lower bounds for the problem, hence solution quality can be proven without extra computation. On the other hand, *bc-prod* does not have any overlapping windows, therefore it can miss considering the effect of future periods’ setup decisions. Also, it can require long computational times because of long windows, large numbers of inequalities added to the formulation and the fact that it solves the subproblem of each window to optimality. Even though SH provides comparatively good results for hard test instances, it should be noted that the implementation is not straightforward because of complex calculations, such as the bonuses. Moreover, as in many other heuristic frameworks, it does not generate lower bounds. In order to determine solution quality, lower bounds need to be generated separately.

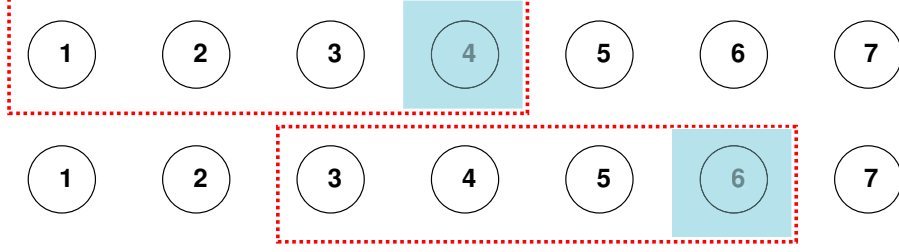


Figure 1: The first 2 “time windows” of SH, with window length of 4 periods, 2 periods overlapping and 1 period relaxed to be continuous

## 5 A Heuristic Framework

As in the last section, we use “time windows”, or simply “windows”, to refer to time intervals in which binary variables are forced to take binary values. For later periods, binary variables are relaxed to be continuous. The window has a “window length”, i.e., length of the time horizon considered in the subproblem. The first part of the window which does not overlap with the next window is called the “fixing interval”, since this is where the binary variables will be fixed once the subproblem is solved. Let  $\alpha$  and  $\beta$  indicate the window length and length of fixing interval, respectively. Note that in our approach the formulation is strengthened for the *entire* horizon, and not just within the window, as in SH.

We also use LP-and-fix idea to find feasible solutions for the original problem throughout the framework in order to provide multiple solutions and upper bounds (cutoff values) for later windows. Our computational experience is that using cutoff values significantly improves the solution process of many windows. Another difference from SH is that we use the objective function defined in Section 2 in each window. This is simpler to implement, and also helps us to gauge the effect of future setup decisions.

The structure of the general framework can be seen in the next page. We can describe the framework in words as follows: After generating violated  $(\ell, S)$  inequalities and deleting inactive ones, the LP relaxation solution of the original problem is used to start LP-and-fix to find a first feasible solution for the original problem. Note that it is not guaranteed that LP-and-fix will find a solution, except backlogging is allowed. If a solution is found, it will be used to initialize the cutoff value. After initialization, we start with the first window, i.e., solve the problem defined in Section 2 where all the binary variables are relaxed to continuous for any period beyond the scope of the window. If extra time is available, i.e., relax-and-fix is complete before the pre-set time limit, LP-and-fix is used in order to generate a feasible solution for the original problem. At the end of LP-and-fix, all the binary variables fixed beyond the window will be unfixed before processing the next window. The same procedure will be repeated for future windows.

Here, note that the heuristic provides us a lower bound to the original problem in the

**Input:** Lot-sizing problem  
**Output:** Multiple feasible solutions and lower bound for the problem  
 $(\ell, S)$  separation  
Initialize upper bound and cutoff bound using LP-and-fix  
**for**  $i=1$  **to** numwin  
    Relax the  $y$  variables after the window to continuous ( $t > (i - 1)\beta + \alpha$ )  
    Solve the sub-problem  
    Fix the  $y$  variables in the fixing interval ( $t \in [(i - 1)\beta + 1, i\beta]$ )  
    **if** extra time  
        Enforce binary restriction on  $y$  variables after the window  
        Fix all  $y$  variables having an integral value  
        Solve the partially fixed MIP  
        If solution found, reset cutoff value  
        Unfix all  $y$  variables after the window  
    **if** extra time at the end  
        Improve the solution

first window since we are solving a partial linear relaxation of the problem in which the formulation is already strengthened. The computational results in the next section show that this lower bound is often competitive with the best known methods.

In the interest of speed, we seek to avoid running a window for a long time to find the optimal solution but obtain a reasonable solution in limited time and move on to the next window. Therefore we set a maximum time for each window. This issue is discussed in more detail in section 6. Also note that, due to these time limits and due to heuristics' nature, neither LP-and-fix nor relax-and-fix is guaranteed to find a solution. However, as the results in the next section indicate, the framework has worked without facing a problem with the control parameters used.

One important question to ask for both relax-and-fix and LP-and-fix is “which variables to fix”. One basic concern for relax-and-fix is that the more we fix in a window, the higher probability we have for infeasibility in later windows. Hence, in our early tests, we tried to fix only 0's and only 1's, but no significant difference from fixing both 0's and 1's has been observed. On the other hand, for LP-and-fix, fixing all 0's and 1's appears to create problems occasionally, such as infeasibility and poor-quality solutions, hence only 1's are fixed in that part of the framework.

The heuristic framework depends on many parameters and options. These include the length of a window, whether we have to overlap consecutive windows, and how long the overlap should be. It is obvious that the shorter the length of the window, the easier it is to solve the related subproblem. However, this can deteriorate the solution quality since decisions become more myopic and since the number of windows grows and hence time

allocated to a window decreases as well. It is best to have a window length which neither takes too much time nor finds too poor solutions. When consecutive windows do not overlap, we deal with fewer windows, but this does not facilitate the consideration of setup decisions from later periods and therefore can result in bad solutions. After experimentation, we use the window length of 3, with an overlap of 1 period between windows. In this case, for example for a problem with  $NT = 10$ , there will be 5 windows to process.

We have found it advisable to allocate more time to earlier windows than to later windows because the problems are bigger in size and hence harder to solve, and also to employ LP-and-fix as often as possible in earlier periods in order to generate good solutions. After assigning a total time for a problem, this allocation process is achieved inside of the algorithm. We divided windows into four sets of the same size. Thus, if we have a total of 12 windows, the first three windows are in the first set, the next three windows are in the second set and so forth. Then we assign 1.75, 1.25, 0.75 and 0.25 times of the average time per window to those groups respectively. Thus, if we have a total of 180 seconds to process 12 windows, we allocate  $\frac{180}{12} * 1.75 = 26.25$  seconds to each of the first three windows,  $\frac{180}{12} * 1.25 = 18.75$  seconds to each of the next three windows, and so forth.

We also set a relative gap parameter so that if we obtain a solution with a duality gap less than this preset amount, the heuristic stops trying to complete the time assigned to this window and moves on either to the next window or starts LP-and-fix with the extra time if it is sufficient for that purpose. Our computational experience suggests that choosing gap values that are too small neither guarantees a better solution at the end nor allows LP-and-fix sufficient time to generate good solutions. In addition to these major parameters, there are some minor parameters used in the heuristic framework in order to increase its efficiency, such as minimum and maximum times to employ LP-and-fix. Also, note that we start employing LP-and-fix in the windows not after the first window but after the second window, because computational experience suggests that early LP-and-fix procedures take much more time to find a solution than later LP-and-fix procedures.

## 6 Computational Results

In order to provide diversified results and test the generality of our heuristic approach, we used the following various test sets from the literature for our computations:

- Test instances generated by Tempelmeier and Derstroff [1996] and Stadtler [2003]: These include overtime variables. Sets A+ and B+ involve problems with 10 items, 24 periods and 3 machines, and sets C and D involve problems with 40 items, 16 periods and 6 machines. Sets B+ and D include setup times. We chose the hardest instances of each data set for our computations, i.e., for each data set, we picked 10 assembly and 10 general instances with the highest duality gaps according to the results of Stadtler [2003].

- Multi-level LOTSIZELIB [1999] library instances: These include single-machine problems with big bucket capacities. Backlogging is allowed. The problems vary between 40 item, single end-item problems and 15 item, 3 end-item problems, with both assembly and general BOM structures. All problems have 12 periods.
- MULTI-LSB instances: We have generated 4 sets of test problems based on the problem described in the paper of Simpson and Erenguc [2005], each set having 30 instances with low, medium and high variability of demand. From now on, we will call these sets SET1, SET2, SET3, and SET4. These instances are different from the previous problems in that they take component commonality into consideration and hence consider setup variables for each family so that setup times are defined for each family of items instead of for each item. While keeping the original data such as BOM structures and holding costs, we removed the setup costs and added backlogging variables into the problem to obtain problems with a different nature from the other problems. The backlogging costs are set to the double of inventory holding costs for the first two sets, and 10 times the inventory holding costs for the last two sets. Except the problems in SET2, which considers a horizon of 24 periods, all the instances have 16 periods. The main difference between these three sets is that they have different resource utilization factors, and the rest of the data remain the same. All instances consider 78 items and have an assembly structure, and all instances allow backlogging in the last period. For more details about the instances, see Multi-LSB [2006] and Simpson and Erenguc [2005].

All the test instances were run on a PC with an Intel Pentium 4 2.53 GB processor and 1 GB of RAM. All the formulations and the heuristic algorithm were implemented using only Xpress Mosel, which is a high-level algebraic modeling language that is intuitive to use. In all the computational experiments, the Xpress-MP 2004C package and Mosel version 1.4.1 are used. We have also prepared a website web [2006] for this paper, where all the data sets and sample Mosel files can be accessed by other researchers for their testing purposes.

Different test sets will be discussed separately since we are using different benchmarks for each set, and different characteristics of each set make it more interesting to analyze each separately. We assigned a total time of 180 seconds for each instance of A+, B+, LOTSIZELIB, SET1 and SET2, and 500 seconds for each C, D, SET3 and SET4 instance because of the problem complexity. Also, a 0.5% duality gap in a window is successful for A+, B+, LOTSIZELIB, SET1 and SET2 instances, duality gaps of 2.5% for later windows and 4% for earlier windows for the C, D and SET4 instances, and finally 10% duality gap for SET3 instances. These percentages are set intuitively from the problem difficulties, e.g., if the default problem solved by Xpress augmented with  $(\ell, S)$  inequalities results with a 10% gap and if there are 10 windows, 1% will be assigned to each window.

For the test sets of Tempelmeier and Derstroff [1996] and Stadtler [2003], we ran an executable of SH (Stadtler's Heuristic) on our computer to provide fair comparisons. Another benchmark used for these instances is default Xpress augmented with the generation of

Table 1: Summary of results for Tempelmeier and Derstroff [1996] and Stadtler [2003] instances

Test Set	Best Solution found by			Average duality gap		
	SH	Xpress	Heuristic	SH	Xpress	Heuristic
A+	4	-	16	29.28%	25.28%	24.47%
B+	11	1	7	29.27%	34.21%	29.86%
C	11	-	9	30.69%	35.40%	28.33%
D	7	-	3	215.96%	364.57%	198.54%

$(\ell, S)$  inequalities. Inactive inequalities are deleted at the root node. The main results are summarized in Table 1: The first three columns show for how many instances of a particular test set any of these three methods found the best solution, and the last three columns show the average duality gaps of each method at each set. Note that since SH does not generate lower bounds, the  $(\ell, S)$  lower bound obtained at the root node is used for that purpose.

Table 2: Pairwise comparisons of the heuristics with benchmarks for Tempelmeier and Derstroff [1996] and Stadtler [2003] instances

Test Set	SH vs. Heuristic			Xpress vs. Heuristic			
	# S	# H	UB	# X	# H	UB	LB
A+	4	16	2.26%	-	20	5.77%	5.15%
B+	12	7	-1.63%	4	15	5.73%	4.08%
C	11	9	-0.22%	-	20	6.91%	1.26%
D	7	3	2.45%	-	10	74.94%	1.12%

Table 2 summarizes pairwise comparisons between our heuristic and the two benchmarks. The first two columns show how many times SH and our heuristic provide a better solution, respectively, and the next column shows the average difference between our heuristic’s and SH’s upper bounds, respectively, calculated as  $(\text{SH Bound} - \text{Heuristic Bound}) / \text{Heuristic Bound}$ . The last four columns are prepared in the same fashion, for the comparison between our heuristic and the default Xpress, and the last two columns are calculated as  $(\text{Xpress Bound} - \text{Heuristic Bound}) / \text{Heuristic Bound}$ . It is easy to observe that our heuristic generates good solutions compared to SH, with a good lower bound guarantee. On the other hand, as expected, default Xpress generates better lower bounds than our heuristic, but it is also notable that the harder the problem is, the better the lower bounds the heuristic generates compared to those generated by Xpress. Also note that the proposed heuristic method improves the lower bounds obtained at the root node with  $(\ell, S)$  inequalities on average by 1.11%. For detailed results for all the instances of these test sets, please refer to Appendix A.



Table 3: Computational Results on LOTSIZELIB [1999] Instances with time limit of 180 seconds

Instance	Optimal Solution	Xpress			Heuristic		
		UB	LB	Time	UB	LB	Time
LLIB B	3,965	3,965	3,957	180	3,973	3,915	23
LLIB C*	2,083	2,125	2,046	180	2,113	2,067	73
LLIB D*	6,482	6,852	4,723	180	7,002	4,714	180
LLIB E*	2,801	3,099	2,537	180	2,952	2,416	152
LLIB F*	2,429	2,458	2,180	180	2,429	2,099	99

\* indicates an instance that could not be solved to optimality by default Xpress in 900 seconds.

For the LOTSIZELIB instances, on the other hand, default Xpress with  $(\ell, S)$  inequalities has been the primary benchmark. The reason that we did not test SH on these problems is that it is not designed for problems with backlogging and significant modification would be necessary. Also note that these instances already have known optimal solutions, hence the solution quality is more transparent. Table 3 shows results of the LOTSIZELIB instances: The first column indicates the optimal solution, then the next three columns show respectively upper and lower bounds obtained by default Xpress and in how many seconds, and finally the last three columns present results of our heuristic. As expected, default Xpress generates better lower bounds generally, however, the heuristic provides comparably good solutions, better solutions in 3 out of 5 instances, and in less time.

Finally, we analyze the 4 sets of MULTI-LSB. These instances vary from easy to very hard problems, as can be seen in detailed results in Appendix A. Here, we summarize the results in Table 4 separately for each different set and also separately for different problem difficulties, as follows: Problems with a duality gap less than 10% after the default time with Xpress augmented with  $(\ell, S)$  inequalities are called “easy”, problems with a duality gap more than 50% are called “hard”, and the rest are called “moderate”.

Table 4 is organized as follows: For “Upper Bounds” and “Lower Bounds”, “#X” and “#H” indicate how many times in that set Xpress and the proposed heuristic found a better bound, respectively. Note that cases in which both methods generate exactly the same upper bound are not included in these numbers. Finally, the last two columns indicate the average difference between the bounds in percentage, and these are calculated by the expression  $(\text{Xpress Bound} - \text{Heuristic Bound}) / \text{Heuristic Bound}$ . As the summary indicates, the heuristic finds generally better solutions for the problem than Xpress does. Similar to previous results, Xpress default lower bounds are often better than the heuristic lower bounds, although it is interesting to observe that in SET3, i.e., the hardest test instances of all our computations, and in the hard instances of SET4, the heuristic generates lower bounds that are competitive with those of Xpress. This seems to be due to the fact that, for the hardest problems, the problems’ difficulty prevents Xpress from making more than minuscule improvements to the lower bound during the branch-and-bound process.

Table 4: Summary of results for MULTI-LSB instances

Test Set	Class	Upper Bounds		Lower Bounds		Ave. Difference	
		# X	# H	# X	# H	UB	LB
SET1	Easy	1	10	11	1	0.67%	2.24%
	Moderate	0	18	5	13	7.22%	-0.11%
SET2	Easy	0	17	13	3	1.24%	0.49%
	Moderate	0	13	2	10	4.66%	-0.50%
SET3	Hard	0	30	15	15	28.48%	-0.10%
SET4	Easy	2	1	3	0	-1.02%	6.34%
	Moderate	1	9	9	1	2.78%	5.84%
	Hard	0	17	4	13	9.35%	-3.88%

\*“Easy” class of SET1 and “hard” class of SET3 has each one instance where both methods found the exactly same solution, and one easy and one moderate instance of SET2 had equal lower bounds for both methods, which are not included in the numbers above.

To summarize this section, the proposed heuristic generates good solutions for different kinds of lot-sizing problems when compared to the default Xpress solver augmented with  $(\ell, S)$  inequalities; Xpress generated a better solution than the heuristic only in 10 of 194 instances, and these were close to the heuristic’s solutions. The comparative efficiency of the heuristic is especially noticeable for harder problems, such as set D, SET3 and SET4, where the heuristic improves default Xpress solutions significantly. On the other hand, the proposed heuristic generates results competitive with those generated by SH for the test sets A+, B+, C and D. The advantages of the proposed heuristic over SH are its flexibility in being applicable to problems with different characteristics, its ability to generate lower bounds, its ease of implementation, and the fact that it generates multiple solutions.

## 7 Conclusion

We have presented a heuristic framework that is designed for easy implementation and is flexible enough to handle a variety of production planning problems. The heuristic finds both good solutions and competitive lower bounds. Moreover, computational results indicate that the heuristic is particularly effective on the most difficult problems, where effectiveness is measured by comparison with our benchmarks.

We have also provided further evidence that using  $(\ell, S)$  inequalities is a good method to strengthen the formulation of big-bucket problems. Computational experience suggests that they should be used within any default solver. Moreover, when combined with an effective heuristic framework, they can help to find good solutions as well.

Even though mathematical programming provides tools for exact solutions, extended reformulations generally result in big problems that cannot be solved efficiently. A recent

study of Van Vyve and Wolsey [2005] suggests that partial reformulations can be applied in order to have high quality lower bounds while preventing the problem size from growing too much. Similar approaches may be effective for hard lot-sizing problems as well, and this is left for future research.

While the proposed framework for finding good solutions to lot-sizing problems is comparably efficient, many of the instances discussed in the computational results remain challenging. As we discuss in detail in our companion paper Akartunalı and Miller [2007], big bucket lot-sizing problems still need a deeper analysis and better understanding of the polyhedral structure to improve lower bounds and provide better solutions; polyhedral analysis focused on their particular structure is an important future research area.

Another interesting question to be answered is whether this simple approach can be applied to other challenging MIP problems or not. We know that production planning problems have a special structure in which early decisions affect later decisions, but similar structures may be applicable to other problems as well, e.g. scheduling and facility location problems.

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## Appendix A: Detailed Results

Instance	SH UB	LB from $l, S$	Xpress UB	Xpress LB	Heuristic UB	Heuristic LB
AG501130	153,418	116,183	157,433	133,715	154,515	119,146
AG501131	146,500	107,829	157,450	116,004	145,225	109,714
AG501132	156,822	118,677	160,678	127,827	154,191	121,740
AG501141	175,619	133,424	186,362	140,206	171,895	134,421
AG501142	195,110	145,508	197,582	154,895	192,582	148,911
AG502130	168,707	122,353	170,059	146,383	167,927	128,101
AG502131	145,322	109,085	157,623	115,833	146,361	111,001
AG502141	179,371	134,971	188,933	141,121	173,640	136,353
AG502232	124,015	97,032	123,304	103,715	121,108	97,632
AG502531	129,080	102,340	140,294	108,749	129,640	103,506
AK501131	128,095	96,968	132,020	102,978	123,366	99,020
AK501132	124,499	101,699	126,919	106,109	123,473	103,077
AK501141	180,695	134,805	186,662	139,754	170,897	136,428
AK501142	174,701	134,880	176,504	138,587	161,262	135,875
AK501432	110,851	92,533	114,088	99,194	109,249	93,546
AK502130	129,307	102,222	133,429	110,583	127,889	103,949
AK502131	126,394	93,369	122,273	98,627	115,819	94,969
AK502132	119,175	96,312	121,712	100,679	118,319	97,233
AK502142	146,616	127,792	164,357	131,234	147,729	129,034
AK502432	110,094	88,980	110,896	94,459	105,415	89,609
BG511132	140,155	108,772	166,747	115,116	137,637	110,466
BG511142	159,769	133,158	183,318	138,130	167,262	133,880
BG512131	139,839	104,054	151,552	112,311	138,752	105,804
BG512142	199,051*	142,917	219,790	150,058	227,996	143,848
BG521132	138,133	108,324	145,975	115,055	146,709	110,024
BG521142	156,694	131,363	179,427	136,920	157,802	132,604
BG522130	154,581	113,540	164,913	131,120	156,075	121,578
BG522132	147,894	113,382	185,786	120,032	162,389	115,158
BG522142	186,268	137,126	201,118	142,692	202,851	138,077
BK511131	123,699	92,602	128,861	97,923	120,303	94,411
BK511132	125,658	95,323	120,227	99,889	115,416	95,938
BK511141	162,629*	125,307	183,541	129,896	172,762	126,769

**Note:** \* indicates instances that could not be run on our computer with SH's executable, in which case their published results are used.

Instance	SH UB	LB from $l, S$	Xpress UB	Xpress LB	Heuristic UB	Heuristic LB
BK512131	113,996	90,733	121,357	94,459	113,536	92,058
BK512132	115,697	90,814	118,918	95,929	112,809	91,346
BK521131	118,217	92,350	127,362	97,784	121,292	94,164
BK521132	117,423	94,257	119,690	99,359	118,464	94,957
BK521142	153,805*	124,988	157,186	128,203	154,258	125,480
BK522131	116,340	90,532	119,591	95,153	111,339	91,742
BK522142	154,286	119,559	148,471	122,836	156,557	119,625
CG501120	1,252,308	1,011,260	1,383,799	1,037,547	1,296,913	1,027,177
CG501131	614,303	472,421	652,793	480,446	614,971	478,437
CG501141	777,831	627,035	854,196	631,554	803,432	628,114
CG501121	1,261,525	945,696	1,376,403	964,910	1,247,493	959,756
CG502221	889,548	724,648	1,005,102	730,536	911,616	728,105
CG501132	842,734	561,827	851,427	619,623	849,500	606,568
CG501222	858,289	697,129	983,752	702,660	877,364	699,021
CG501142	1,146,638	754,238	1,169,704	847,853	1,150,718	824,887
CG501122	1,814,877	1,161,383	1,832,565	1,274,366	1,787,833	1,281,687
CG502222	873,858	704,096	1,024,136	709,326	899,970	708,597
CK501120	176,187	141,900	187,551	145,740	179,099	143,260
CK501221	123,206	101,028	133,710	101,409	123,066	101,105
CK501121	173,083	131,993	185,519	134,662	169,804	132,840
CK502221	123,657	101,478	133,305	102,193	122,596	101,899
CK501222	122,485	97,937	132,502	98,522	123,298	98,096
CK501422	128,558	101,864	134,032	104,802	124,315	102,150
CK502222	119,965	98,052	134,820	98,625	122,302	98,282
CK501122	210,824*	153,861	212,018	162,312	206,646	155,485
CK501132	98,363*	75,257	101,602	78,690	98,248	75,782
CK501142	120,722*	90,218	122,122	92,471	115,918	90,673
DG512141	736,181	609,464	1,841,593	613,816	759,136	615,992
DG512131	581,932	465,272	2,052,160	496,633	596,395	469,460
DG012132	3,625,599*	554,595	3,661,662	559,893	3,160,347	555,689
DG012142	3,815,898*	756,588	3,730,720	761,229	3,121,762	756,588
DG012532	1,194,003*	554,167	1,558,743	558,186	1,228,463	555,032
DG012542	1,413,475*	756,062	1,652,806	759,010	1,420,613	756,062
DG512132	2,909,628*	512,330	5,570,581	518,927	3,010,696	514,682
DG512142	3,779,026*	678,733	5,655,268	688,424	3,583,354	682,205
DG512532	584,491	509,567	1,031,986	516,101	597,404	512,147
DG512542	767,428	674,241	1,342,964	683,455	796,831	677,189

**Note:** \* indicates instances that could not be run on our computer with Stadler's executable, in which case their published results are used.



Instance	LB from $l, S$	Xpress UB	Xpress LB	Heuristic UB	Heuristic LB	Improvement in UB
SET1.01	17,888	25,873	18,355	22,781	18,889	13.57%
SET1.02	23,534	30,317	23,934	28,624	24,134	5.91%
SET1.03	21,227	26,943	21,638	26,349	21,676	2.25%
SET1.04	22,232	30,236	22,698	26,337	23,175	14.80%
SET1.05	21,446	27,064	21,782	25,621	21,994	5.63%
SET1.06	22,974	29,548	23,187	26,741	23,636	10.50%
SET1.07	20,360	27,316	20,772	24,693	21,125	10.62%
SET1.08	25,582	32,361	25,893	29,810	26,249	8.56%
SET1.09	16,321	24,015	16,780	21,146	17,013	13.57%
SET1.10	17,998	23,985	18,243	22,863	18,945	4.91%
SET1.11	11,080	13,010	12,374	12,956	11,407	0.42%
SET1.12	24,721	27,377	25,211	26,985	25,238	1.45%
SET1.13	20,782	24,003	21,581	23,129	21,195	3.78%
SET1.14	22,264	25,931	22,388	25,720	22,745	0.82%
SET1.15	12,401	15,023	13,070	14,121	12,575	6.39%
SET1.16	15,122	17,542	16,030	17,559	15,387	-0.10%
SET1.17	20,468	24,284	20,576	23,404	20,864	3.76%
SET1.18	11,075	13,272	11,858	12,300	11,456	7.90%
SET1.19	13,276	18,091	14,044	17,448	13,342	3.69%
SET1.20	14,101	19,433	14,236	17,167	14,612	13.20%
SET1.21	10,159	12,428	11,003	12,421	10,392	0.06%
SET1.22	38,040	40,419	38,556	40,158	38,040	0.65%
SET1.23	29,331	30,687	29,622	30,606	29,355	0.26%
SET1.24	28,858	32,282	30,272	32,174	29,250	0.34%
SET1.25	51,371	53,650	51,720	53,009	51,371	1.21%
SET1.26	39,379	42,236	39,979	41,442	39,488	1.92%
SET1.27	40,838	43,929	41,424	43,320	40,918	1.41%
SET1.28	39,846	40,993	40,993	40,993	40,144	0.00%
SET1.29	23,155	25,633	23,698	25,606	23,232	0.11%
SET1.30	68,989	71,095	69,899	70,868	68,989	0.32%

**Note:** Only one of these 30 SET1 instances resulted in optimal solution in that allowed time. “Improvement in UB” column indicates (Xpress Upper Bound - Heuristic Upper Bound)/Heuristic Upper Bound.

Instance	LB from $l, S$	Xpress UB	Xpress LB	Heuristic UB	Heuristic LB	Improvement in UB
SET2.01	46,116	57,604	46,341	55,039	46,591	4.66%
SET2.02	47,780	59,307	47,780	57,825	48,159	2.56%
SET2.03	40,551	52,438	40,551	49,147	40,814	6.70%
SET2.04	36,347	47,828	36,347	44,656	36,808	7.10%
SET2.05	45,395	57,950	45,395	55,650	45,784	4.13%
SET2.06	45,902	57,791	45,902	54,361	45,902	6.31%
SET2.07	52,825	63,216	52,825	61,140	53,108	3.40%
SET2.08	48,033	60,193	48,198	56,444	48,632	6.64%
SET2.09	37,553	46,376	37,651	44,523	37,943	4.16%
SET2.10	38,751	50,334	38,751	49,481	39,181	1.72%
SET2.11	65,210	70,623	65,382	69,177	65,648	2.09%
SET2.12	62,792	68,277	62,918	66,914	62,792	2.04%
SET2.13	34,778	41,254	34,978	40,114	34,987	2.84%
SET2.14	62,907	68,070	63,545	67,201	62,907	1.29%
SET2.15	59,079	62,046	59,432	61,616	59,079	0.70%
SET2.16	75,682	81,490	75,695	79,576	75,682	2.41%
SET2.17	36,809	43,455	37,095	41,484	36,925	4.75%
SET2.18	77,873	84,288	77,873	83,200	78,087	1.31%
SET2.19	54,981	59,724	55,055	59,010	55,484	1.21%
SET2.20	119,568	124,272	119,568	122,974	119,568	1.06%
SET2.21	22,281	24,517	22,520	24,459	22,281	0.24%
SET2.22	51,279	54,090	51,687	53,690	51,279	0.74%
SET2.23	29,793	35,851	29,921	33,969	29,793	5.54%
SET2.24	65,891	69,365	66,277	68,727	65,891	0.93%
SET2.25	75,627	79,274	76,035	78,266	75,627	1.29%
SET2.26	60,952	63,998	61,564	63,558	60,977	0.69%
SET2.27	53,016	56,100	53,983	54,797	53,016	2.38%
SET2.28	44,545	46,884	45,293	46,733	44,549	0.32%
SET2.29	93,631	97,474	93,735	96,281	93,631	1.24%
SET2.30	68,324	72,783	68,860	71,919	68,573	1.20%

**Note:** None of these 30 SET2 instances resulted in optimal solution in that allowed time.

Instance	LB from $l, S$	Xpress UB	Xpress LB	Heuristic UB	Heuristic LB	Improvement in UB
SET3_01	65,668	274,297	74,333	209,129	70,207	31.16%
SET3_02	82,342	337,279	88,125	243,511	88,681	38.51%
SET3_03	74,209	329,113	79,993	235,198	81,830	39.93%
SET3_04	78,282	340,701	86,922	240,339	85,499	41.76%
SET3_05	76,607	331,850	84,114	227,758	83,975	45.70%
SET3_06	79,093	295,468	82,420	235,642	87,790	25.39%
SET3_07	72,979	313,851	78,187	237,218	78,976	32.30%
SET3_08	88,610	332,507	93,913	251,628	96,111	32.14%
SET3_09	64,180	274,953	70,971	216,025	70,301	27.28%
SET3_10	66,878	285,525	73,164	229,242	75,040	24.55%
SET3_11	42,946	184,964	48,991	152,962	48,565	20.92%
SET3_12	86,047	279,693	97,536	217,497	94,746	28.60%
SET3_13	74,643	270,744	84,968	224,670	83,763	20.51%
SET3_14	85,209	296,748	93,522	225,657	95,835	31.50%
SET3_15	40,715	222,557	44,794	167,494	46,252	32.87%
SET3_16	46,548	216,952	53,472	162,616	50,979	33.41%
SET3_17	71,555	241,420	81,449	212,399	79,287	13.66%
SET3_18	39,533	206,693	45,876	112,468	46,879	83.78%
SET3_19	47,495	204,225	54,267	154,981	55,749	31.77%
SET3_20	58,189	257,030	64,603	191,639	64,141	34.12%
SET3_21	44,182	209,485	54,340	150,758	53,149	38.95%
SET3_22	130,235	367,229	138,697	292,199	134,729	25.68%
SET3_23	96,810	282,565	106,346	240,643	108,870	17.42%
SET3_24	105,300	330,999	114,461	292,996	113,949	12.97%
SET3_25	203,044	387,648	209,797	349,975	210,315	10.76%
SET3_26	145,184	349,324	156,009	323,870	162,088	7.86%
SET3_27	145,420	389,594	158,120	343,486	156,422	13.42%
SET3_28	145,227	280,123	154,258	254,008	155,123	10.28%
SET3_29	79,813	276,899	88,871	207,127	90,600	33.69%
SET3_30	274,018	489,154	284,240	431,136	278,826	13.46%

**Note:** These 30 SET3 instances have an average duality gap of 236% in the allowed time.

Instance	LB from $l, S$	Xpress UB	Xpress LB	Heuristic UB	Heuristic LB	Improvement in UB
SET4_01	16,353	66,394	22,119	58,720	24,809	13.07%
SET4_02	31,541	83,818	39,072	82,496	41,655	1.60%
SET4_03	24,864	76,954	31,191	73,740	33,577	4.36%
SET4_04	27,786	78,073	33,786	73,651	36,553	6.00%
SET4_05	25,450	75,596	33,836	67,874	34,995	11.38%
SET4_06	30,632	80,132	38,733	79,781	40,893	0.44%
SET4_07	22,650	77,171	29,248	65,736	30,514	17.40%
SET4_08	40,532	100,638	46,045	88,388	49,106	13.86%
SET4_09	13,490	58,504	19,061	57,070	22,860	2.51%
SET4_10	15,542	62,611	22,729	59,319	25,077	5.55%
SET4_11	12,802	35,384	17,589	28,989	17,446	22.06%
SET4_12	43,341	84,540	49,504	78,062	50,961	8.30%
SET4_13	28,152	61,670	34,355	53,833	36,046	14.56%
SET4_14	56,174	85,443	64,907	82,406	64,257	3.69%
SET4_15	14,628	27,634	18,706	26,980	15,797	2.42%
SET4_16	17,171	36,115	23,792	35,280	22,606	2.37%
SET4_17	29,001	55,101	34,116	54,515	36,832	1.07%
SET4_18	19,184	26,279	24,679	26,596	22,950	-1.19%
SET4_19	10,724	40,237	16,512	31,974	15,037	25.84%
SET4_20	18,718	43,396	26,400	39,983	24,035	8.54%
SET4_21	15,812	26,322	22,061	25,899	18,149	1.63%
SET4_22	91,715	123,711	96,535	120,166	93,603	2.95%
SET4_23	55,058	79,912	60,799	76,857	57,446	3.97%
SET4_24	58,919	86,332	64,856	85,119	64,305	1.43%
SET4_25	171,987	201,717	176,087	202,501	173,205	-0.39%
SET4_26	110,570	145,290	117,138	142,090	117,379	2.25%
SET4_27	101,114	146,827	105,713	139,874	103,814	4.97%
SET4_28	112,892	126,027	125,250	129,789	115,176	-2.90%
SET4_29	51,149	71,618	59,369	68,320	56,930	4.83%
SET4_30	241,678	270,746	248,344	267,976	241,712	1.03%

**Note:** These 30 SET4 instances have an average duality gap of 79% in the allowed time.