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# Almost Sure and Moment Exponential Stability of Euler–Maruyama Discretizations for Hybrid Stochastic Differential Equations

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## Abstract

Positive results are derived concerning the long time dynamics of numerical simulations of stochastic differential equation systems with Markovian switching. Euler–Maruyama discretizations are shown to capture almost sure and moment exponential stability for all sufficiently small timesteps under appropriate conditions.

**Key words:** Brownian motion, Euler–Maruyama, Markov chain, exponential stability.

## 1 Introduction

Stability analysis of numerical methods for stochastic differential equations (SDEs) has recently received a more and more attention. Stability analysis of numerical methods for ordinary differential equations (ODEs) is motivated by the question “for what choices of stepsize does the numerical method reproduce the characteristics of the test equation?” It was in this spirit that Mitsui and his coworkers [12, 18, 19] studied the asymptotic stability of numerical methods with respect to the linear test stochastic differential equation

$$dx(t) = \mu x(t)dt + \sigma x(t)dB(t). \quad (1.1)$$

To explain their results more precisely, let us recall the Euler–Maruyama (EM) method (see e.g. [11, 16]) applied to the SDE (1.1): Given a stepsize  $\Delta > 0$ , the discrete EM approximation  $X_k \approx x(k\Delta)$  is formed by setting  $X_0 = x(0)$  and, generally,

$$X_{k+1} = X_k(1 + \mu\Delta + \sigma\Delta B_k), \quad k = 0, 1, 2, \dots, \quad (1.2)$$

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where  $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$ . One of the main results of Mitsui *et al.* [12, 18, 19] is that the EM approximate solution is exponentially stable in mean square for a sufficiently small stepsize if the true solution of the SDE (1.1) is exponentially stable in mean square (namely,  $2\mu + \sigma^2 < 0$ ). This result was generalized by Higham, Mao and Stuart [8] to a multi-dimensional non-linear SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t), \quad (1.3)$$

where they showed that under the global Lipschitz condition of the coefficients  $f$  and  $g$ , the EM approximate solution to the SDE (1.2) is exponentially stable in mean square for a sufficiently small stepsize if and only if the true solution of the SDE is exponentially stable in mean square. They also showed that if the global Lipschitz condition does not hold, then the EM method cannot guarantee to preserve exponential mean-square stability, even for arbitrarily small stepsizes; see [8, Lemma 4.1].

On the other hand, relatively little is known about the ability of numerical methods for SDEs to reproduce almost sure asymptotic stability. It is well known that the test equation (1.1) is almost surely exponentially stable if  $\mu - \frac{1}{2}\sigma^2 < 0$ . The question is: Can the EM method reproduce this stability property? Higham [6] gave a partial answer to the question. Replacing the Brownian increments  $\Delta B_k$  by the independent random variables  $\xi_k$ , whose probability distributions are given by

$$\mathbb{P}(\xi_k = -\sqrt{\Delta}) = \mathbb{P}(\xi_k = \sqrt{\Delta}) = \frac{1}{2},$$

to give an alternative approximate solution

$$Y_{k+1} = Y_k(1 + \mu\Delta + \sigma\xi_k), \quad k = 0, 1, 2, \dots, \quad (1.4)$$

with  $Y_0 = x(0)$ , Higham [6] showed that  $Y_k$  will tend to zero exponentially with probability 1 provided the stepsize  $\Delta$  is sufficiently small and  $\mu - \frac{1}{2}\sigma^2 < 0$ . However, it is only recent that Higham, Mao and Yuan [9] give a full answer to the question. That is, they show that the EM solution (1.2) is almost surely exponentially stable for a sufficiently small  $\Delta$  if and only if the true solution of the SDE (1.1) is almost surely exponentially stable (i.e.,  $\mu - \frac{1}{2}\sigma^2 < 0$ ).

Recently, models that switch between different SDE systems according to an independent Markov chain have been proposed. These hybrid SDEs are designed to account for circumstances where an abrupt change may take place in the nature of a physical process. In particular, important examples arise in mathematical finance, where a market may switch between two or more distinct modes (nervous, confident, cautious, ...). For examples of such *regime switching* or *Markov-modulated dynamics* models, see, for example, [10, 21, 23] and the references therein.

Generally, hybrid SDEs cannot be solved analytically and hence numerical methods must be used. Although it is intuitively straightforward to adapt existing SDE methods to the hybrid case, the traditional numerical analysis issues associated with the resulting methods have only recently received attention. Finite time convergence analysis of an Euler–Maruyama type method is given in [23]. In this work, we consider long time dynamics. The issue that we address is: can a numerical method reproduce the stability behaviour of the underlying hybrid SDE? In particular, we focus on almost sure and

small moment exponential stability. In the general nonlinear case for (non-hybrid) SDEs it is known that the EM method cannot guarantee to preserve exponential mean-square stability, even for arbitrarily small stepsizes; see [8, Lemma 4.1]. Hence, in studying hybrid SDEs, we look for conditions under which positive results can be derived in the small stepsize setting. Our work therefore builds on the well known and highly informative analysis for deterministic problems and its more recent extension to SDEs [2, 4, 5, 6, 8, 12, 14, 18, 19]

## 2 Scalar Linear Hybrid SDEs

Throughout this paper, we let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  be a complete probability space with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual conditions (i.e. it is increasing and right continuous while  $\mathcal{F}_0$  contains all  $\mathbb{P}$ -null sets) and we let  $B(t)$  be a scalar Brownian motion defined on the probability space.

We let  $r(t)$ ,  $t \geq 0$ , be a right-continuous Markov chain on the probability space taking values in a finite state space  $\mathbb{S} = \{1, 2, \dots, N\}$  and independent of the Brownian motion  $B(\cdot)$ . The corresponding generator is denoted  $\Gamma = (\gamma_{ij})_{N \times N}$ , so that

$$\mathbb{P}\{r(t + \delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & : \text{ if } i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta) & : \text{ if } i = j, \end{cases}$$

where  $\delta > 0$ . Here  $\gamma_{ij}$  is the transition rate from  $i$  to  $j$  and  $\gamma_{ij} > 0$  if  $i \neq j$  while  $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$ . We note that almost every sample path of  $r(\cdot)$  is a right continuous step function with a finite number of sample jumps in any finite subinterval of  $\mathbb{R}_+ := [0, \infty)$ .

We will use  $|\cdot|$  to denote the Euclidean norm of a vector and the trace norm of a matrix. We will denote the indicator function of a set  $G$  by  $I_G$ . For  $x \in \mathbb{R}$ ,  $\text{int}(x)$  denotes the integer part of  $x$ .

We begin our study with the special but important case of scalar linear hybrid SDEs of the form

$$dx(t) = \mu(r(t))x(t)dt + \sigma(r(t))x(t)dB(t), \quad t \geq 0 \quad (2.1)$$

with initial data  $x(0) = x_0 \in \mathbb{R}$  and  $r(0) = r_0 \in \mathbb{S}$ . Here, to avoid complicated notations, we let  $B(t)$  be a scalar Brownian motion while  $\mu$  and  $\sigma$  are mappings from  $\mathbb{S} \rightarrow \mathbb{R}$ . The SDE (2.1) is known as the hybrid Brownian motion or the volatility-switching geometric Brownian motion. One motivation for studying this class is that sharp stability results can be derived, allowing us to test the efficiency of a numerical method. One more motivation is that it is a realistic model in mathematical finance [10] and hence the qualitative behaviour of numerical methods on this model is of independent interest.

As a standing hypothesis, we assume moreover in this paper that the Markov chain is *irreducible*. This is equivalent to the condition that for any  $i, j \in \mathbb{S}$ , we can find  $i_1, i_2, \dots, i_k \in \mathbb{S}$  such that

$$\gamma_{i, i_1} \gamma_{i_1, i_2} \cdots \gamma_{i_k, j} > 0.$$

Note that  $\Gamma$  always has an eigenvalue 0. The algebraic interpretation of irreducibility is  $\text{rank}(\Gamma) = N - 1$ . Under this condition, the Markov chain has a unique stationary (prob-

ability) distribution  $\pi = (\pi_1, \pi_2, \dots, \pi_N) \in \mathbb{R}^{1 \times N}$  which can be determined by solving

$$\begin{cases} \pi \Gamma = 0 \\ \text{subject to } \sum_{j=1}^N \pi_j = 1 \text{ and } \pi_j > 0 \text{ for all } j \in \mathbb{S}. \end{cases}$$

It is known that the linear hybrid SDE (2.1) has the explicit solution

$$x(t) = x_0 \exp \left\{ \int_0^t [\mu(r(s)) - \frac{1}{2}\sigma^2(r(s))]ds + \int_0^t \sigma(r(s))dB(s) \right\}. \quad (2.2)$$

Making use of this explicit form we are able to discuss almost sure and moment exponential stability precisely. The following theorem gives a necessary and sufficient condition for the SDE (2.1) to be almost surely exponentially stable.

**Theorem 2.1** *The sample Lyapunov exponent of the SDE (2.1) is*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(|x(t)|) = \sum_{j=1}^N \pi_j (\mu_j - \frac{1}{2}\sigma_j^2) \quad a.s. \quad (2.3)$$

(for  $x_0 \neq 0$  of course). Hence the SDE (2.1) is almost surely exponentially stable if and only if

$$\sum_{j=1}^N \pi_j (\mu_j - \frac{1}{2}\sigma_j^2) < 0. \quad (2.4)$$

*Proof.* For any  $x_0 \neq 0$ , it follows from (2.2) that

$$\log(|x(t)|) = \log(|x_0|) + \int_0^t [\mu(r(s)) - \frac{1}{2}\delta^2(r(s))]ds + \int_0^t \delta(r(s))dB(s). \quad (2.5)$$

By the classical large number theorem of martingales (see e.g. [15, 16]),

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \delta(r(s))dB(s) = 0 \quad a.s.$$

while by the ergodic property of the Markov chain,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t [\mu(r(s)) - \frac{1}{2}\delta^2(r(s))]ds = \sum_{j=1}^N \pi_j (\mu_j - \frac{1}{2}\sigma_j^2) \quad a.s.$$

Dividing both sides of (2.5) by  $t$  and then letting  $t \rightarrow \infty$  we hence obtain the assertion (2.3). ■

The following theorem gives the sufficient and necessary condition for the SDE (2.1) to be  $p$ th moment exponentially stable. It should be pointed out that the proof for the  $p$ th moment exponential stability of a linear scalar (non-hybrid) SDE is rather simple (see e.g. [16]) while the proof below for the hybrid SDE becomes much more complicated.

**Theorem 2.2** *The  $p$ th moment Lyapunov exponent of the hybrid SDE (2.1) is*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^p) = \sum_{j=1}^N \pi_j p [\mu_j + \frac{1}{2}(p-1)\sigma_j^2] \quad (2.6)$$

(for  $x_0 \neq 0$  of course). Hence the SDE (2.1) is  $p$ th moment exponentially stable if and only if

$$\sum_{j=1}^N \pi_j [\mu_j + \frac{1}{2}(p-1)\sigma_j^2] < 0. \quad (2.7)$$

*Proof.* It is well known (see e.g. [1]) that almost every sample path of the Markov chain  $r(\cdot)$  is a right continuous step function with a finite number of sample jumps in any finite subinterval of  $\mathbb{R}_+ := [0, \infty)$ . Hence there is a sequence of finite stopping times  $0 = \tau_0 < \tau_1 < \dots < \tau_k \rightarrow \infty$  such that

$$r(t) = \sum_{k=0}^{\infty} r(\tau_k) I_{[\tau_k, \tau_{k+1})}(t), \quad t \geq 0.$$

For any integer  $z > 0$ , it then follows from (2.2) that

$$\begin{aligned} & |x(t \wedge \tau_z)|^p \\ &= |x_0|^p \exp \left\{ \int_0^{t \wedge \tau_z} [p\mu(r(s)) - \frac{1}{2}p\sigma^2(r(s))] ds + \int_0^{t \wedge \tau_z} p\sigma(r(s)) dB(s) \right\} \\ &= \xi(t \wedge \tau_z) \exp \left\{ - \int_0^{t \wedge \tau_z} \frac{1}{2}p^2\sigma^2(r(s)) ds + \int_0^{t \wedge \tau_z} p\sigma(r(s)) dB(s) \right\} \\ &= \xi(t \wedge \tau_z) \prod_{k=0}^{z-1} \zeta_k, \end{aligned}$$

where

$$\begin{aligned} \xi(t \wedge \tau_z) &= |x_0|^p \exp \left\{ \int_0^{t \wedge \tau_z} [p\mu(r(s)) + \frac{1}{2}p(p-1)\sigma^2(r(s))] ds \right\}, \\ \zeta_k &= \exp \left\{ - \frac{1}{2}p^2\sigma^2(r(t \wedge \tau_k))(t \wedge \tau_{k+1} - t \wedge \tau_k) \right. \\ &\quad \left. + p\sigma(r(t \wedge \tau_k))[B(t \wedge \tau_{k+1}) - B(t \wedge \tau_k)] \right\}. \end{aligned}$$

Let  $\mathcal{G}_t = \sigma(\{r(u)\}_{u \geq 0}, \{B(s)\}_{0 \leq s \leq t})$ , namely the  $\sigma$ -algebra generated by  $\{r(u)\}_{u \geq 0}$  and  $\{B(s)\}_{0 \leq s \leq t}$ . Compute

$$\begin{aligned} \mathbb{E}|x(t \wedge \tau_z)|^p &= \mathbb{E} \left( \xi(t \wedge \tau_z) \prod_{k=0}^{z-1} \zeta_k \right) = \mathbb{E} \left\{ \mathbb{E} \left( \xi(t \wedge \tau_z) \prod_{k=0}^{z-1} \zeta_k \middle| \mathcal{G}_{t \vee \tau_{z-1}} \right) \right\} \\ &= \mathbb{E} \left\{ \left[ \xi(t \wedge \tau_z) \prod_{k=0}^{z-2} \zeta_k \right] \mathbb{E} \left( \zeta_{z-1} \middle| \mathcal{G}_{t \vee \tau_{z-1}} \right) \right\}. \end{aligned} \quad (2.8)$$

Define, for  $i \in \mathbb{S}$ ,

$$\zeta_{z-1}(i) = \exp \left\{ -\frac{1}{2}p\sigma_i^2(t \wedge \tau_z - t \wedge \tau_{z-1}) + p\sigma_i[B(t \wedge \tau_z) - B(t \wedge \tau_{z-1})] \right\}.$$

By the exponential martingale formula (see e.g. [15, 16]), we have

$$\mathbb{E}\zeta_{z-1}(i) = 1, \quad i \in \mathbb{S}.$$

Then

$$\begin{aligned} \mathbb{E}\left(\zeta_{z-1} \middle| \mathcal{G}_{t \vee \tau_{z-1}}\right) &= \mathbb{E}\left(\sum_{i \in \mathbb{S}} I_{\{r(t \wedge \tau_{z-1})=i\}} \zeta_{z-1}(i) \middle| \mathcal{G}_{t \vee \tau_{z-1}}\right) \\ &= \sum_{i \in \mathbb{S}} I_{\{r(t \wedge \tau_{z-1})=i\}} \mathbb{E}\left(\zeta_{z-1}(i) \middle| \mathcal{G}_{t \vee \tau_{z-1}}\right). \end{aligned}$$

Noting that  $t \wedge \tau_z - t \wedge \tau_{z-1}$  is  $\mathcal{G}_{t \vee \tau_{z-1}}$ -measurable while  $B(t \wedge \tau_z) - B(t \wedge \tau_{z-1})$  is independent of  $\mathcal{G}_{t \vee \tau_{z-1}}$ , we have

$$\mathbb{E}\left(\zeta_{z-1}(i) \middle| \mathcal{G}_{t \vee \tau_{z-1}}\right) = \mathbb{E}\zeta_{z-1}(i) = 1,$$

whence

$$\mathbb{E}\left(\zeta_{z-1} \middle| \mathcal{G}_{t \vee \tau_{z-1}}\right) = 1.$$

Substituting this into (2.8) yields

$$\mathbb{E}|x(t \wedge \tau_z)|^p = \mathbb{E}\left[\xi(t \wedge \tau_z) \prod_{k=0}^{z-2} \zeta_k\right]. \quad (2.9)$$

Repeating this procedure implies

$$\mathbb{E}|x(t \wedge \tau_z)|^p = \mathbb{E}\xi(t \wedge \tau_z).$$

Letting  $z \rightarrow \infty$  we obtain

$$\mathbb{E}|x(t)|^p = \mathbb{E}\left\{|x_0|^p \exp\left[\int_0^t [p\mu(r(s)) + \frac{1}{2}p(p-1)\sigma^2(r(s))]ds\right]\right\}. \quad (2.10)$$

Now, by the ergodic property of the Markov chain (see e.g. [1]), we have

$$\begin{aligned} &\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t [p\mu(r(s)) + \frac{1}{2}p(p-1)\sigma^2(r(s))]ds \\ &= \sum_{j \in \mathbb{S}} \pi_j(p\mu_j + \frac{1}{2}p(p-1)\sigma_j^2) := \gamma \quad a.s. \end{aligned} \quad (2.11)$$

Let  $\varepsilon > 0$  be arbitrary. It follows from (2.10) that

$$\begin{aligned} &e^{-(\gamma-\varepsilon)t} \mathbb{E}|x(t)|^p \\ &= \mathbb{E}\left\{|x_0|^2 \exp\left[-(\gamma-\varepsilon)t + \int_0^t [p\mu(r(s)) + \frac{1}{2}p(p-1)\sigma^2(r(s))]ds\right]\right\}. \end{aligned}$$



By (2.11),

$$\lim_{t \rightarrow \infty} \exp \left[ -(\gamma - \varepsilon)t + \int_0^t [p\mu(r(s)) + \frac{1}{2}p(p-1)\sigma^2(r(s))]ds \right] = \infty \quad a.s.$$

Hence

$$\lim_{t \rightarrow \infty} e^{-(\gamma - \varepsilon)t} \mathbb{E}|x(t)|^p = \infty,$$

which implies

$$\mathbb{E}|x(t)|^p \geq e^{(\gamma - \varepsilon)t} \quad \text{for all sufficiently large } t,$$

whence

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^p) \geq \gamma - \varepsilon.$$

Similarly, we can show

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^p) \leq \gamma + \varepsilon.$$

Since  $\varepsilon$  is arbitrary, we must have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^p) = \gamma,$$

which is the required assertion (2.6). ■

We remark that for a single linear SDE of the form  $dx(t) = \mu x(t)dt + \sigma x(t)dB(t)$ , where  $\mu$  and  $\sigma$  are constants, Theorem 2.2 reproduces the well-known  $p$ th moment stability characterisation  $\mu + 0.5(p-1)\sigma^2 < 0$ . In the more general hybrid case (2.1), Theorem 2.2 tells us that the *appropriate average* of the quantity  $\mu_j + 0.5(p-1)\sigma_j^2$  over the states  $j$  of the Markov chain determines the stability. Intuitively, even though a numerical method such as the Euler–Maruyama can match the stability properties of a single linear SDE for sufficiently small  $\Delta > 0$ , it is much more demanding to ask a method to maintain this behaviour over all possible averages, especially those involving a mixture of individually stable and unstable problems. This gives further motivation for the focus in this work which we will discuss in the next section.

### 3 Numerical Exponential Stability

We now introduce the Euler-Maruyama method, which was shown in [23] to be strongly convergent. The method makes use of the following lemma (see [1]).

**Lemma 3.1** *Given  $\Delta > 0$ , let  $r_k^\Delta = r(k\Delta)$  for  $k \geq 0$ . Then  $\{r_k^\Delta, k = 0, 1, 2, \dots\}$  is a discrete Markov chain with the one-step transition probability matrix*

$$P(\Delta) = (P_{ij}(\Delta))_{N \times N} = e^{\Delta\Gamma}. \quad (3.1)$$

Given a fixed stepsize  $\Delta > 0$  and the one-step transition probability matrix  $P(\Delta)$  in (3.1), the discrete Markov chain  $\{r_k^\Delta, k = 0, 1, 2, \dots\}$  can be simulated as follows: Let

$r_0^\Delta = i_0$  and compute a pseudo-random number  $\xi_1$  from the uniform  $(0, 1)$  distribution. Define

$$r_1^\Delta = \begin{cases} i_1 & \text{if } i_1 \in \mathbb{S} - \{N\} \text{ such that } \sum_{j=1}^{i_1-1} P_{i_0,j}(\Delta) \leq \xi_1 < \sum_{j=1}^{i_1} P_{i_0,j}(\Delta), \\ N & \text{if } \sum_{j=1}^{N-1} P_{i_0,j}(\Delta) \leq \xi_1, \end{cases}$$

where we set  $\sum_{i=1}^0 P_{i_0,j}(\Delta) = 0$  as usual. In other words, we ensure that the probability of state  $s$  being chosen is given by  $\mathbb{P}(r_1^\Delta = s) = P_{i_0,s}(\Delta)$ . Generally, having computed  $r_0, r_1, r_2, \dots, r_k$ , we compute  $r_{k+1}$  by drawing a uniform  $(0, 1)$  pseudo-random number  $\xi_{k+1}$  and setting

$$r_{k+1}^\Delta = \begin{cases} i_{k+1} & \text{if } i_{k+1} \in \mathbb{S} - \{N\} \text{ such that } \sum_{j=1}^{i_{k+1}-1} P_{r_k^\Delta,j}(\Delta) \leq \xi_{k+1} < \sum_{j=1}^{i_{k+1}} P_{r_k^\Delta,j}(\Delta), \\ N & \text{if } \sum_{j=1}^{N-1} P_{r_k^\Delta,j}(\Delta) \leq \xi_{k+1}. \end{cases}$$

This procedure can be carried out independently to obtain more trajectories.

Having explained how to simulate the discrete Markov chain, we now define the Euler-Maruyama (EM) approximation for the linear hybrid SDE (2.1). The discrete approximation  $X_k \approx x(t_k)$ , with  $t_k = k\Delta$ , is formed by setting  $X_0 = x_0$ ,  $r_0^\Delta = i_0$  and, generally,

$$X_{k+1} = X_k [1 + \mu(r_k^\Delta)\Delta + \sigma(r_k^\Delta)\Delta B_k], \quad k \geq 1, \quad (3.2)$$

where  $\Delta B_k = B(t_{k+1}) - B(t_k)$ . In words,  $r_k^\Delta$  defines which of the  $N$  linear SDEs is currently active, and we apply the EM to this SDE. Compared with the numerical analysis of standard SDEs, a new source of error arises in the method (3.2): the switching can only occur at discrete time points  $\{t_k\}$ , whereas for the underlying continuous-time problem (2.1) the Markov chain can produce a switch at any point in time.

The following theorem shows that the EM method matches the  $p$ th moment exponential stability of the linear hybrid SDE, for sufficiently small  $\Delta$ .

**Theorem 3.2** *For any given pair of constants  $p > 0$  and  $\varepsilon > 0$ , there is a positive constant  $\Delta^* = \Delta^*(p, \varepsilon)$  such that if the stepsize  $\Delta < \Delta^*$ , then the EM approximation (3.2) obeys*

$$\limsup_{k \rightarrow \infty} \frac{1}{k\Delta} \log \left( \mathbb{E}|X_k|^p \right) \leq \sum_{j \in \mathbb{S}} \pi_j p \left[ \mu_j + \frac{1}{2}(p-1)\sigma_j^2 \right] + \varepsilon. \quad (3.3)$$

Hence, the EM approximation (3.2) is  $p$ th moment exponentially stable for sufficiently small  $\Delta$  if

$$\sum_{j \in \mathbb{S}} \pi_j \left[ \mu_j + \frac{1}{2}(p-1)\sigma_j^2 \right] < 0.$$

*Proof.* For any integer  $z$ , it follows from (3.2) that

$$|X_{z+1}|^p = |x_0|^p \prod_{k=0}^z Y_k, \quad \text{where } Y_k = |1 + \mu(r_k^\Delta)\Delta + \sigma(r_k^\Delta)\Delta B_k|^p.$$

Using the  $\sigma$ -algebra  $\mathcal{G}_t$  defined in the proof of Theorem 2.2, we compute

$$\mathbb{E}|X_{z+1}|^p = \mathbb{E} \left( \mathbb{E} \left[ |x_0|^p \prod_{k=0}^z Y_k \middle| \mathcal{G}_{t_z} \right] \right) = \mathbb{E} \left( |x_0|^p \prod_{k=0}^{z-1} Y_k \mathbb{E} \left[ Y_z \middle| \mathcal{G}_{t_z} \right] \right). \quad (3.4)$$

Writing

$$\begin{aligned}
Y_z &= |1 + \mu(r_z^\Delta)\Delta + \sigma(r_z^\Delta)\Delta B_z|^p \\
&= ([1 + \mu(r_z^\Delta)\Delta + \sigma(r_z^\Delta)\Delta B_z]^2)^{p/2} \\
&= (1 + 2[\mu(r_z^\Delta)\Delta + \sigma(r_z^\Delta)\Delta B_z] + [\mu(r_z^\Delta)\Delta + \sigma(r_z^\Delta)\Delta B_z]^2)^{p/2},
\end{aligned}$$

and recalling the fundamental inequality

$$(1 + u)^{p/2} \leq 1 + \frac{p}{2}u + \frac{p(p-2)}{8}u^2 + \frac{p(p-2)(p-4)}{2^3 \times 3!}u^3, \quad u \geq -1, \quad (3.5)$$

we have

$$\begin{aligned}
Y_z &\leq 1 + \frac{p}{2} (2[\mu(r_z^\Delta)\Delta + \sigma(r_z^\Delta)\Delta B_z] + [\mu(r_z^\Delta)\Delta + \sigma(r_z^\Delta)\Delta B_z]^2) z \\
&+ \frac{p(p-2)}{8} (2[\mu(r_z^\Delta)\Delta + \sigma(r_z^\Delta)\Delta B_z] + [\mu(r_z^\Delta)\Delta + \sigma(r_z^\Delta)\Delta B_z]^2)^2 \\
&+ \frac{p(p-2)(p-4)}{48} (2[\mu(r_z^\Delta)\Delta + \sigma(r_z^\Delta)\Delta B_z] + [\mu(r_z^\Delta)\Delta + \sigma(r_z^\Delta)\Delta B_z]^2)^3. \quad (3.6)
\end{aligned}$$

Making use of the properties of the normally distributed random variable  $\Delta B_z$  that

$$\mathbb{E}(\Delta B_z^{2n}) = \Delta^n (2n-1)!! \quad \text{and} \quad \mathbb{E}(\Delta B_z^{2n-1}) = 0, \quad n = 1, 2, 3, \dots,$$

we can compute

$$\begin{aligned}
\mathbb{E}\left[2[\mu(r_z^\Delta)\Delta + \sigma(r_z^\Delta)\Delta B_z] \middle| \mathcal{G}_{t_z}\right] &= 2\mu(r_z^\Delta)\Delta + 2\sigma(r_z^\Delta)\mathbb{E}\left[\Delta B_z \middle| \mathcal{G}_{t_z}\right] \\
&= 2\mu(r_z^\Delta)\Delta + 2\sigma(r_z^\Delta)\mathbb{E}(\Delta B_z) \\
&= 2\mu(r_z^\Delta)\Delta
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[[\mu(r_z^\Delta)\Delta + \sigma(r_z^\Delta)\Delta B_z]^2 \middle| \mathcal{G}_{t_z}\right] \\
&= \mathbb{E}\left[\mu^2(r_z^\Delta)\Delta^2 + 2\mu(r_z^\Delta)\Delta\sigma(r_z^\Delta)\Delta B_z + \sigma^2(r_z^\Delta)\Delta B_z^2 \middle| \mathcal{G}_{t_z}\right] \\
&= \mu^2(r_z^\Delta)\Delta^2 + \sigma^2(r_z^\Delta)\Delta.
\end{aligned}$$

Similarly, we can compute the conditional expectations of the other terms in the right-hand side of (3.6) to obtain

$$\mathbb{E}\left[Y_z \middle| \mathcal{G}_{t_z}\right] \leq 1 + p[\mu(r_z^\Delta) + \frac{1}{2}(p-1)\sigma^2(r_z^\Delta)]\Delta + C_1\Delta^2,$$

where  $C_1$  is a constant independent of  $\Delta$  and  $z$ . It then follows from (3.4) that

$$\mathbb{E}|X_{z+1}|^p \leq \mathbb{E}\left[|x_0|^p (1 + p[\mu(r_z^\Delta) + \frac{1}{2}(p-1)\sigma^2(r_z^\Delta)]\Delta + C_1\Delta^2) \prod_{k=0}^{z-1} Y_k\right].$$

We compute furthermore that

$$\mathbb{E}|X_{z+1}|^p \leq \mathbb{E}\left(|x_0|^p (1 + p[\mu(r_z^\Delta) + \frac{1}{2}(p-1)\sigma^2(r_z^\Delta)]\Delta + C_1\Delta^2) \prod_{k=0}^{z-2} Y_k \mathbb{E}\left[Y_{z-1} \middle| \mathcal{G}_{t_{z-1}}\right]\right).$$

But, in the same way as before, we can show that

$$\mathbb{E}\left[Y_{z-1}\left|\mathcal{G}_{t_{z-1}}\right.\right] \leq 1 + p\left[\mu(r_{z-1}^\Delta) + \frac{1}{2}(p-1)\sigma^2(r_{z-1}^\Delta)\right]\Delta + C_1\Delta^2.$$

Hence

$$\mathbb{E}|X_{z+1}|^p \leq \mathbb{E}\left(|x_0|^p \left[ \prod_{k=z-1}^z \left(1 + p\left[\mu(r_k^\Delta) + \frac{1}{2}(p-1)\sigma^2(r_k^\Delta)\right]\Delta + C_1\Delta^2\right) \right] \prod_{k=0}^{z-2} Y_k\right).$$

Repeating this procedure we obtain

$$\mathbb{E}|X_{z+1}|^p \leq \mathbb{E}\left(|x_0|^p \prod_{k=0}^z \left(1 + p\left[\mu(r_k^\Delta) + \frac{1}{2}(p-1)\sigma^2(r_k^\Delta)\right]\Delta + C_1\Delta^2\right)\right).$$

Re-write this as

$$\mathbb{E}|X_{z+1}|^p \leq \mathbb{E}\left\{|x_0|^p \exp\left[\sum_{k=0}^z \log\left(1 + p\left[\mu(r_k^\Delta) + \frac{1}{2}(p-1)\sigma^2(r_k^\Delta)\right]\Delta + C_1\Delta^2\right)\right]\right\},$$

whence

$$\begin{aligned} & e^{-(\lambda+\varepsilon)(1+z)\Delta} \mathbb{E}|X_{z+1}|^p \\ & \leq \mathbb{E}\left\{|x_0|^p \exp\left[-(\lambda+\varepsilon)(1+z)\Delta\right.\right. \\ & \quad \left.\left. + \sum_{k=0}^z \log\left(1 + p\left[\mu(r_k^\Delta) + \frac{1}{2}(p-1)\sigma^2(r_k^\Delta)\right]\Delta + C_1\Delta^2\right)\right]\right\}. \end{aligned} \quad (3.7)$$

But, by the ergodic property of the Markov chain as well as the elementary inequality

$$\log(1+x) \leq x, \quad \forall x > -1, \quad (3.8)$$

we derive that

$$\begin{aligned} & \lim_{z \rightarrow \infty} \frac{1}{1+z} \sum_{k=0}^z \log\left(1 + p\left[\mu(r_k^\Delta) + \frac{1}{2}(p-1)\sigma^2(r_k^\Delta)\right]\Delta + C_1\Delta^2\right) \\ & = \sum_{j \in \mathbb{S}} \pi_j \log\left(1 + p\left[\mu_j + \frac{1}{2}(p-1)\sigma_j^2\right]\Delta + C_1\Delta^2\right) \\ & \leq \sum_{j \in \mathbb{S}} \pi_j \left(p\left[\mu_j + \frac{1}{2}(p-1)\sigma_j^2\right]\Delta + C_1\Delta^2\right) \quad a.s. \end{aligned}$$

Hence, for any  $\varepsilon > 0$ , when  $\Delta$  is sufficiently small, we have

$$\lim_{z \rightarrow \infty} \frac{1}{(1+z)\Delta} \sum_{k=0}^z \log\left(1 + p\left[\mu(r_k^\Delta) + \frac{1}{2}(p-1)\sigma^2(r_k^\Delta)\right]\Delta + C_1\Delta^2\right) < \lambda + \varepsilon \quad a.s.$$

where  $\lambda = \sum_{j \in \mathbb{S}} \pi_j p\left[\mu_j + \frac{1}{2}(p-1)\sigma_j^2\right]$ . This implies that

$$\lim_{z \rightarrow \infty} \left[-(\lambda+\varepsilon)(1+z)\Delta + \sum_{k=0}^z \log\left(1 + p\left[\mu(r_k^\Delta) + \frac{1}{2}(p-1)\sigma^2(r_k^\Delta)\right]\Delta + C_1\Delta^2\right)\right] = -\infty$$

almost surely. Using this and the well-known Fatou lemma, we obtain easily from (3.7) that

$$\lim_{z \rightarrow \infty} e^{-(\lambda+\varepsilon)(1+z)\Delta} \mathbb{E}|X_{z+1}|^p = 0,$$

and the required result (3.3) follows immediately.  $\blacksquare$

**Theorem 3.3** For any given constant  $\varepsilon > 0$ , there is a positive constant  $\Delta^* = \Delta^*(\varepsilon)$  such that if the stepsize  $\Delta < \Delta^*$ , then the EM approximation (3.2) obeys

$$\limsup_{k \rightarrow \infty} \frac{1}{k\Delta} \log(|X_k|) \leq \sum_{j \in \mathbb{S}} \pi_j [\mu_j - \frac{1}{2}\sigma_j^2] + \varepsilon \quad a.s. \quad (3.9)$$

Hence, the EM approximation (3.2) is almost surely exponentially stable for sufficiently small  $\Delta$  if

$$\sum_{j \in \mathbb{S}} \pi_j [\mu_j - \frac{1}{2}\sigma_j^2] < 0.$$

*Proof.* Choose  $p = p(\varepsilon) > 0$  sufficiently small for

$$p \sum_{j \in \mathbb{S}} \pi_j \sigma_j^2 < \frac{1}{2}\varepsilon. \quad (3.10)$$

For the pair of  $p$  and  $\varepsilon$ , by Theorem 3.2, there is a positive constant  $\Delta^* = \Delta^*(\varepsilon)$  such that if the stepsize  $\Delta < \Delta^*$ , then the EM approximation (3.2) obeys

$$\limsup_{k \rightarrow \infty} \frac{1}{k\Delta} \log \left( \mathbb{E}|X_k|^p \right) \leq \sum_{j \in \mathbb{S}} \pi_j p [\mu_j + \frac{1}{2}(p-1)\sigma_j^2] + \frac{1}{4}p\varepsilon. \quad (3.11)$$

But, by (3.10)

$$\sum_{j \in \mathbb{S}} \pi_j p [\mu_j + \frac{1}{2}(p-1)\sigma_j^2] + \frac{1}{4}p\varepsilon \leq \sum_{j \in \mathbb{S}} \pi_j p [\mu_j - \frac{1}{2}\sigma_j^2] + \frac{1}{2}p\varepsilon.$$

So, whenever  $\Delta < \Delta^*$ ,

$$\limsup_{k \rightarrow \infty} \frac{1}{k\Delta} \log \left( \mathbb{E}|X_k|^p \right) \leq \sum_{j \in \mathbb{S}} \pi_j p [\mu_j - \frac{1}{2}\sigma_j^2] + \frac{1}{2}p\varepsilon. \quad (3.12)$$

Now, let us fix any  $\Delta < \Delta^*$  and denote  $\lambda = \sum_{j \in \mathbb{S}} \pi_j [\mu_j - \frac{1}{2}\sigma_j^2]$ . It then follows from (3.12) that there is a constant  $M > 0$  such that

$$\mathbb{E}|X_k|^p \leq M e^{p(\lambda + \frac{3}{4}\varepsilon)k\Delta} \quad \forall k = 1, 2, \dots. \quad (3.13)$$

By the well-known Chebyshev inequality, we have

$$\mathbb{P}\left\{|X_k| > e^{(\lambda + \varepsilon)k\Delta}\right\} \leq e^{-p(\lambda + \varepsilon)k\Delta} \mathbb{E}|X_k|^p \leq M e^{-\frac{1}{4}p\varepsilon k\Delta} \quad \forall k = 1, 2, \dots.$$

Applying the well-known Borel–Cantelli lemma (see e.g. [16]), we obtain that for almost all  $\omega \in \Omega$ ,

$$|X_k| \leq e^{(\lambda + \varepsilon)k\Delta} \quad (3.14)$$

holds for all but finitely many  $k$ . Hence, there exists a  $k_0(\omega)$ , for almost all  $\omega \in \Omega$ , for which (3.14) holds whenever  $k \geq k_0$ . Consequently, for almost all  $\omega \in \Omega$ ,

$$\frac{1}{k\Delta} \log(|X_k|) \leq \lambda + \varepsilon$$

whenever  $k \geq k_0$ . Therefore

$$\limsup_{k \rightarrow \infty} \frac{1}{k\Delta} \log(|X_k|) \leq \lambda + \varepsilon \quad a.s.$$

which is the desired assertion (3.9). ■

## 4 Multi-Dimensional Nonlinear Hybrid SDEs

Let us now consider a more general  $n$ -dimensional hybrid SDE

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t) \quad (4.1)$$

on  $t \geq 0$  with initial data  $x(0) = x_0 \in \mathbb{R}^n$  and  $r(0) = r_0 \in \mathbb{S}$ . We assume that

$$f : \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^n$$

are sufficiently smooth for the existence and uniqueness of the solution (see e.g. [17, 22]). We still keep the Brownian motion to be a scalar one to avoid the notation becoming too complicated, but we will state the multi-dimensional Brownian motion case in the next section. To analyse, we begin by imposing the linear growth assumption

$$|f(x, i)| \vee |g(x, i)| \leq K|x|, \quad \forall (x, i) \in \mathbb{R}^n \times \mathbb{S}. \quad (4.2)$$

This implies

$$f(0, i) = 0 \quad \text{and} \quad g(0, i) = 0 \quad \forall i \in \mathbb{S}, \quad (4.3)$$

so equation (4.1) admits the zero solution,  $x(t) \equiv 0$ , whose stability is the issue under consideration. We will be concerned with pathwise convergence of the solution  $x(t)$  of (4.1) to the zero solution, as  $t \rightarrow \infty$ , and the preservation of this property under discretisation. We also note that condition (4.2) ensures that, with probability one, the solution will never reach the origin whenever  $x_0 \neq 0$ ; see, for example, [17, Lemma 2.1].

### 4.1 Exponential stability of true solutions

We begin by giving sufficient conditions for almost sure exponential stability of the hybrid SDE.

**Theorem 4.1** *Let (4.2) hold. For each  $i \in \mathbb{S}$ , let*

$$\lambda_i := \sup_{x \in \mathbb{R}^n, x \neq 0} \left( \frac{\langle x, f(x, i) \rangle + \frac{1}{2}|g(x, i)|^2}{|x|^2} - \frac{\langle x, g(x, i) \rangle^2}{|x|^4} \right) < \infty. \quad (4.4)$$

*Then the solution of (4.1) obeys*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |x(t)| \leq \sum_{i \in \mathbb{S}} \pi_i \lambda_i \quad a.s. \quad (4.5)$$

*In particular, if  $\sum_{i \in \mathbb{S}} \pi_i \lambda_i < 0$ , then the hybrid SDE (4.1) is almost surely exponentially stable.*

*Proof.* The assertion (4.5) holds when  $x_0 = 0$  so we need to consider the case when  $x_0 \neq 0$ . In this case, as pointed out above, the solution will never reach the origin with probability one. We can therefore apply the Itô formula to show that

$$\begin{aligned} \log(|x(t)|^2) &= \log(|x(0)|^2) + M(t) \\ &+ \int_0^t 2 \left( \frac{\langle x(s), f(x(s), r(s)) \rangle + \frac{1}{2}|g(x(s), r(s))|^2}{|x(s)|^2} - \frac{\langle x(s), g(x(s), r(s)) \rangle^2}{|x(s)|^4} \right) ds, \end{aligned}$$

where

$$M(t) = \int_0^t \frac{2\langle x(s), g(x(s), r(s)) \rangle}{|x(s)|^2} dB(s).$$

By (4.4), we have

$$\log(|x(t)|^2) \leq \log(|x(0)|^2) + M(t) + \int_0^t 2\lambda_{r(s)} ds. \quad (4.6)$$

From the condition  $|g(x, i)| \leq K|x|$ , it is straightforward to show that

$$\lim_{t \rightarrow \infty} \frac{M(t)}{t} = 0 \quad a.s.$$

Moreover, by the ergodic property of the Markov chain, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \lambda_{r(s)} ds = \sum_{i \in \mathbb{S}} \pi_i \lambda_i = -\lambda. \quad (4.7)$$

Dividing both sides of (4.6) by  $2t$  and then letting  $t \rightarrow \infty$  we obtain (4.5). ■

## 4.2 Exponential stability of EM solutions

Let us now define the Euler-Maruyama (EM) approximation for the hybrid SDE (4.1). The discrete approximation  $X_k \approx x(t_k)$ , with  $t_k = k\Delta$ , is formed by setting  $X_0 = x_0$ ,  $r_0^\Delta = i_0$  and, generally,

$$X_{k+1} = X_k + f(X_k, r_k^\Delta)\Delta + g(X_k, r_k^\Delta)\Delta B_k, \quad (4.8)$$

where  $\Delta B_k = B(t_{k+1}) - B(t_k)$ .

**Theorem 4.2** *Let (4.2) and (4.4) hold and assume that*

$$-\lambda := \sum_{i \in \mathbb{S}} \pi_i \lambda_i < 0. \quad (4.9)$$

*Then for any  $\varepsilon \in (0, \lambda)$  there is a constant  $\Delta^* \in (0, 1)$  such that for any  $0 < \Delta < \Delta^*$  the EM approximation (4.8) satisfies*

$$\limsup_{k \rightarrow \infty} \frac{1}{k\Delta} \log(|X_k|) \leq -(\lambda - \varepsilon) \quad a.s. \quad (4.10)$$

*Proof.* We compute from (4.8) that

$$\begin{aligned} |X_{k+1}|^2 &= |X_k|^2 + 2\langle X_k, f(X_k, r_k^\Delta)\Delta + g(X_k, r_k^\Delta)\Delta B_k \rangle \\ &\quad + |f(X_k, r_k^\Delta)\Delta + g(X_k, r_k^\Delta)\Delta B_k|^2 \\ &= |X_k|^2(1 + \xi_k), \end{aligned}$$

where

$$\xi_k = \frac{1}{|X_k|^2} [2\langle X_k, f(X_k, r_k^\Delta)\Delta + g(X_k, r_k^\Delta)\Delta B_k \rangle + |f(X_k, r_k^\Delta)\Delta + g(X_k, r_k^\Delta)\Delta B_k|^2]$$

if  $X_k \neq 0$ , otherwise it is set to  $-1$ . Clearly,  $\xi_k \geq -1$ . For any  $p \in (0, 1)$ , by inequality (3.5), we have

$$\begin{aligned} |X_{k+1}|^p &= |X_k|^p (1 + \xi_k)^{p/2} \\ &\leq |X_k|^p \left( 1 + \frac{p}{2} \xi_k + \frac{p(p-2)}{8} \xi_k^2 + \frac{p(p-2)(p-4)}{2^3 \times 3!} \xi_k^3 \right). \end{aligned}$$

Recalling the definition of the  $\sigma$ -algebra  $\mathcal{G}_t$  in the proof of Theorem 2.2, we compute the conditional expectation

$$\begin{aligned} \mathbb{E}(|X_{k+1}|^p | \mathcal{G}_{k\Delta}) &\leq |X_k|^p \mathbb{E} \left( 1 + \frac{p}{2} \xi_k + \frac{p(p-2)}{8} \xi_k^2 + \frac{p(p-2)(p-4)}{2^3 \times 3!} \xi_k^3 \middle| \mathcal{G}_{k\Delta} \right) \\ &= |X_k|^p \mathbf{1}_{\{X_k \neq 0\}} \mathbb{E} \left( 1 + \frac{p}{2} \xi_k + \frac{p(p-2)}{8} \xi_k^2 + \frac{p(p-2)(p-4)}{2^3 \times 3!} \xi_k^3 \middle| \mathcal{G}_{k\Delta} \right), \end{aligned} \quad (4.11)$$

where  $\mathbf{1}_A$  denotes the indicator function for  $A$ . Now,

$$\begin{aligned} \mathbf{1}_{\{X_k \neq 0\}} \mathbb{E}(\xi_k | \mathcal{G}_{k\Delta}) &= \mathbf{1}_{\{X_k \neq 0\}} \mathbb{E} \left( \frac{1}{|X_k|^2} [2\langle X_k, f(X_k, r_k^\Delta) \Delta + g(X_k, r_k^\Delta) \Delta B_k \rangle \right. \\ &\quad \left. + |f(X_k, r_k^\Delta) \Delta + g(X_k, r_k^\Delta) \Delta B_k|^2] \middle| \mathcal{G}_{k\Delta} \right). \end{aligned}$$

Since  $\Delta B_k$  is independent of  $\mathcal{G}_{k\Delta}$ , we have  $\mathbb{E}(\Delta B_k | \mathcal{G}_{k\Delta}) = \mathbb{E}(\Delta B_k) = 0$  and  $\mathbb{E}((\Delta B_k)^2 | \mathcal{G}_{k\Delta}) = \mathbb{E}((\Delta B_k)^2) = \Delta$ . It then easy to obtain that

$$\begin{aligned} &\mathbf{1}_{\{X_k \neq 0\}} \mathbb{E}(\xi_k | \mathcal{G}_{k\Delta}) \\ &= \mathbf{1}_{\{X_k \neq 0\}} \left( \frac{1}{|X_k|^2} [2\langle X_k, f(X_k, r_k^\Delta) \Delta \rangle + |f(X_k, r_k^\Delta)|^2 \Delta^2 + |g(X_k, r_k^\Delta)|^2 \Delta] \right) \\ &\leq \mathbf{1}_{\{X_k \neq 0\}} \left( \frac{1}{|X_k|^2} [2\langle X_k, f(X_k, r_k^\Delta) \Delta \rangle + |g(X_k, r_k^\Delta)|^2] \Delta + K^2 \Delta^2 \right), \end{aligned} \quad (4.12)$$

where (4.2) has been used. Similarly, we can show that

$$\mathbf{1}_{\{X_k \neq 0\}} \mathbb{E}(\xi_k^2 | \mathcal{G}_{k\Delta}) \geq \frac{4}{|X_k|^4} \langle X_k, g(X_k, r_k^\Delta) \rangle^2 \Delta - c_K \Delta^2 \quad (4.13)$$

and

$$\mathbf{1}_{\{X_k \neq 0\}} \mathbb{E}(\xi_k^3 | \mathcal{G}_{k\Delta}) \leq c_K \Delta^2, \quad (4.14)$$

where  $c_K > 0$  is a constant dependent only on  $K$ . Substituting (4.12), (4.13) and (4.14) into (4.11) and then using (4.4) and (4.2) we derive that

$$\begin{aligned} &\mathbb{E}(|X_{k+1}|^p | \mathcal{G}_{k\Delta}) \\ &\leq |X_k|^p \mathbf{1}_{\{X_k \neq 0\}} \left( 1 + \frac{p}{2|X_k|^2} [2\langle X_k, f(X_k, r_k^\Delta) \Delta \rangle + |g(X_k, r_k^\Delta)|^2] \Delta \right. \\ &\quad \left. + \frac{p(p-2)}{2|X_k|^4} \langle X_k, g(X_k, r_k^\Delta) \rangle^2 \Delta + C \Delta^2 \right) \\ &= |X_k|^p \mathbf{1}_{\{X_k \neq 0\}} \left\{ 1 + p \Delta \left( \frac{\langle X_k, f(X_k, r_k^\Delta) \rangle + \frac{1}{2}|g(X_k)|^2}{|X_k|^2} - \frac{\langle X_k, g(X_k, r_k^\Delta) \rangle^2}{|X_k|^4} \right) \right. \\ &\quad \left. + \frac{p^2 \Delta \langle X_k, g(X_k, r_k^\Delta) \rangle^2}{2|X_k|^4} + C \Delta^2 \right\} \\ &\leq |X_k|^p \left( 1 + p \Delta [\lambda_{r_k^\Delta} + \frac{1}{2} p K^2] + C \Delta^2 \right), \end{aligned} \quad (4.15)$$



where  $C = C(K, p) > 0$  is a constant independent of  $\Delta$ .

Now, set  $\hat{\lambda} = \max_{i \in \mathbb{S}} |\lambda_i|$ . For any given  $\varepsilon \in (0, \lambda)$ , choose  $p \in (0, 1)$  sufficiently small for  $pK^2 < \frac{1}{2}\varepsilon$  and then choose  $\Delta^* \in (0, 1)$  sufficiently small for  $p\hat{\lambda}\Delta^* < 1$  and  $C\Delta^* < \frac{1}{4}p\varepsilon$ . It then follows from (4.15) that for any  $\Delta < \Delta^*$

$$\mathbb{E}(|X_{k+1}|^p | \mathcal{G}_{k\Delta}) \leq |X_k|^p \left(1 + p\Delta [\lambda_{r_k^\Delta} + \frac{1}{2}\varepsilon]\right). \quad (4.16)$$

Since this holds for all  $k \geq 0$ , we further compute

$$\begin{aligned} \mathbb{E}(|X_{k+1}|^p | \mathcal{G}_{(k-1)\Delta}) &\leq \mathbb{E}(|X_k|^p | \mathcal{G}_{(k-1)\Delta}) \left(1 - p\Delta [\lambda_{r_k^\Delta} - \frac{1}{2}\varepsilon]\right) \\ &\leq |X_{k-1}|^p \prod_{z=k-1}^k \left(1 + p\Delta [\lambda_{r_z^\Delta} + \frac{1}{2}\varepsilon]\right). \end{aligned}$$

Repeating this procedure we obtain

$$\mathbb{E}(|X_{k+1}|^p | \mathcal{G}_0) \leq |x_0|^p \prod_{z=0}^k \left(1 + p\Delta [\lambda_{r_z^\Delta} + \frac{1}{2}\varepsilon]\right).$$

Taking expectations on both sides yields

$$\mathbb{E}(|X_{k+1}|^p) \leq |x_0|^p \mathbb{E} \exp \left[ \sum_{z=0}^k \log \left(1 + p\Delta [\lambda_{r_z^\Delta} + \frac{1}{2}\varepsilon]\right) \right]. \quad (4.17)$$

However, by the ergodic property of the Markov chain and inequality (3.8), we compute

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{1+k} \sum_{z=0}^k \log \left(1 + p\Delta [\lambda_{r_z^\Delta} + \frac{1}{2}\varepsilon]\right) &= \sum_{i \in \mathbb{S}} \pi_i \log \left(1 + p\Delta [\lambda_i + \frac{1}{2}\varepsilon]\right) \\ &\leq p\Delta \sum_{i \in \mathbb{S}} \pi_i [\lambda_i + \frac{1}{2}\varepsilon] \\ &= p\Delta (-\lambda + \frac{1}{2}\varepsilon) \quad a.s. \end{aligned}$$

This yields

$$\lim_{k \rightarrow \infty} \left[ p\Delta (\lambda - \varepsilon)(k+1) + \sum_{z=0}^k \log \left(1 + p\Delta [\lambda_{r_z^\Delta} + \frac{1}{2}\varepsilon]\right) \right] = -\infty \quad a.s. \quad (4.18)$$

However, it follows from (4.17)

$$\begin{aligned} &e^{p\Delta(\lambda-\varepsilon)(k+1)} \mathbb{E}(|X_{k+1}|^p) \\ &\leq |x_0|^p \mathbb{E} \exp \left[ p\Delta(\lambda - \varepsilon)(k+1) + \sum_{z=0}^k \log \left(1 + p\Delta [\lambda_{r_z^\Delta} + \frac{1}{2}\varepsilon]\right) \right]. \end{aligned}$$

By the well-known Fatou lemma and property (4.18), we hence derive that

$$\begin{aligned} &\limsup_{k \rightarrow \infty} \left[ e^{p\Delta(\lambda-\varepsilon)(k+1)} \mathbb{E}(|X_{k+1}|^p) \right] \\ &\leq |x_0|^p \mathbb{E} \left( \limsup_{k \rightarrow \infty} \exp \left[ p\Delta(\lambda - \varepsilon)(k+1) + \sum_{z=0}^k \log \left(1 + p\Delta [\lambda_{r_z^\Delta} + \frac{1}{2}\varepsilon]\right) \right] \right) \\ &= 0. \end{aligned}$$

Hence, there is an integer  $k_0$  such that

$$\mathbb{E}(|X_k|^p) \leq e^{pk\Delta(\lambda-\varepsilon)}, \quad \forall k \geq k_0.$$

This implies that

$$\mathbb{P}\{|X_k|^p > k^2 e^{-pk\Delta(\lambda-\varepsilon)}\} \leq \frac{1}{k^2}, \quad \forall k \geq k_0.$$

By the Borel–Cantelli lemma, we see that for almost all  $\omega \in \Omega$

$$|X_k|^p \leq k^2 e^{-pk\Delta(\lambda-\varepsilon)} \tag{4.19}$$

holds for all but finitely many  $k \geq k_0$ . Hence, there exists a  $k_1(\omega) \geq k_0$ , for all  $\omega \in \Omega$  excluding a  $\mathbb{P}$ -null set, for which (4.19) holds whenever  $k \geq k_1$ . Consequently, for almost all  $\omega \in \Omega$ ,

$$\frac{1}{k\Delta} \log(|X_k|) \leq -(\lambda - \varepsilon) + \frac{2 \log(k)}{pk\Delta}$$

whenever  $k \geq k_1$ . Letting  $k \rightarrow \infty$  we obtain the assertion (4.10). ■

## 5 Multi-Dimensional Brownian Motion Case

So far we have restricted ourselves to the case of a scalar Brownian motion in order to keep our notations relatively simple. However, the theory developed above can certainly be generalised to the multi-dimensional Brownian motion case. Accordingly, let  $B(t) = (B^1(t), \dots, B^m(t))^T$  be an  $m$ -dimensional Brownian motion. Consider a more general  $n$ -dimensional hybrid SDE

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t) \tag{5.1}$$

on  $t \geq 0$  with initial data  $x(0) = x_0 \in \mathbb{R}^n$  and  $r(0) = r_0 \in \mathbb{S}$ , where

$$f : \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^{n \times m}.$$

Assume that both  $f$  and  $g$  are sufficiently smooth for the existence and uniqueness of the solution (see e.g. [17, 22]). As before, we impose the linear growth assumption

$$|f(x, i)| \vee |g(x, i)| \leq K|x|, \quad \forall (x, i) \in \mathbb{R}^n \times \mathbb{S}. \tag{5.2}$$

The EM approximation  $X_k \approx x(t_k)$ , with  $t_k = k\Delta$ , is formed by setting  $X_0 = x_0$ ,  $r_0^\Delta = i_0$  and, generally,

$$X_{k+1} = X_k + f(X_k, r_k^\Delta)\Delta + g(X_k, r_k^\Delta)\Delta B_k, \tag{5.3}$$

where  $\Delta B_k = B(t_{k+1}) - B(t_k)$ . The following two theorems can be proved in the same way as in the previous section.

**Theorem 5.1** *Let (5.2) hold. For each  $i \in \mathbb{S}$ , let*

$$\lambda_i := \sup_{x \in \mathbb{R}^n, x \neq 0} \left( \frac{\langle x, f(x, i) \rangle + \frac{1}{2}|g(x, i)|^2}{|x|^2} - \frac{|x^T g(x, i)|^2}{|x|^4} \right) < \infty. \tag{5.4}$$

Then the solution of (5.1) obeys

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |x(t)| \leq \sum_{i \in \mathbb{S}} \pi_i \lambda_i \quad \text{a.s.} \quad (5.5)$$

In particular, if  $\sum_{i \in \mathbb{S}} \pi_i \lambda_i < 0$ , then the hybrid SDE (5.1) is almost surely exponentially stable.

**Theorem 5.2** *Let (5.2) and (5.4) hold and assume that*

$$-\lambda := \sum_{i \in \mathbb{S}} \pi_i \lambda_i < 0. \quad (5.6)$$

Then for any  $\varepsilon \in (0, \lambda)$  there is a constant  $\Delta^* \in (0, 1)$  such that for any  $0 < \Delta < \Delta^*$  the EM approximation (5.3) satisfies

$$\limsup_{k \rightarrow \infty} \frac{1}{k\Delta} \log(|X_k|) \leq -(\lambda - \varepsilon) \quad \text{a.s.} \quad (5.7)$$

To close our paper, let us consider the important class of multi-dimensional linear hybrid SDEs driven by multi-dimensional Brownian motions of the form

$$dx(t) = A(r(t))x(t)dt + \sum_{z=1}^m G_z(r(t))x(t)dB^z(t) \quad (5.8)$$

on  $t \geq 0$  with initial value  $x(0) = x_0 \in \mathbb{R}^n$ , where  $A$  and  $G_z$  are all mappings from  $\mathbb{S} \rightarrow \mathbb{R}^{n \times n}$ . For convenience, we will write  $A(i) = A_i$  and  $G_z(i) = G_{zi}$ . Equation (5.8) corresponds to

$$f(x, i) = A_i x \quad \text{and} \quad g(x, i) = (G_{1i}x, \dots, G_{mi}x), \quad (x, i) \in \mathbb{R}^n \times \mathbb{S},$$

in (5.1). Accordingly,

$$|g(x, i)|^2 = \sum_{z=1}^m |G_{zi}x|^2 \quad \text{and} \quad |x^T g(x, i)|^2 = \sum_{z=1}^m |x^T G_{zi}x|^2.$$

Let  $\lambda_{\max}(\cdot)$  and  $\lambda_{\min}(\cdot)$  denote the maximum and minimum eigenvalues of a symmetric matrix, respectively, and  $\|\cdot\|$  the operator norm of a matrix. Note, for  $i \in \mathbb{S}$  and  $z = 1, 2, \dots, m$ ,

$$\langle x, A_i x \rangle = \frac{1}{2} \langle x, (A_i + A_i^T)x \rangle \leq \frac{1}{2} \lambda_{\max}(A_i + A_i^T) |x|^2$$

and

$$|G_{zi}x|^2 \leq \|G_{zi}\|^2 |x|^2.$$

Moreover, if  $G_{zi} + G_{zi}^T$  is either non-positive definite or non-negative definite,

$$\langle x, G_{zi}x \rangle^2 \geq \frac{1}{4} [|\lambda_{\max}(G_{zi} + G_{zi}^T)| \wedge |\lambda_{\min}(G_{zi} + G_{zi}^T)|]^2 |x|^4.$$

We hence observe that if every  $G_{zi} + G_{zi}^T$  is either non-positive definite or non-negative definite,

$$\begin{aligned} & \frac{\langle x, A_i x \rangle + \frac{1}{2} \sum_{z=1}^m |G_{zi} x|^2}{|x|^2} - \frac{\sum_{z=1}^m |x^T G_{zi} x|^2}{|x|^4} \\ & \leq \frac{1}{2} \lambda_{\max}(A_i + A_i^T) + \sum_{z=1}^m \frac{1}{2} \|G_{zi}\|^2 \\ & - \frac{1}{4} \sum_{z=1}^m [|\lambda_{\max}(G_{zi} + G_{zi}^T)| \wedge |\lambda_{\min}(G_{zi} + G_{zi}^T)|]^2. \end{aligned}$$

By Theorem 5.1 we reach the following useful corollary.

**Corollary 5.3** *Assume that every  $G_{zi} + G_{zi}^T$  ( $i \in \mathbb{S}$  and  $1 \leq z \leq m$ ) is either non-positive definite or non-negative definite and define*

$$\begin{aligned} \lambda_i & := \frac{1}{2} \lambda_{\max}(A_i + A_i^T) + \sum_{z=1}^m \frac{1}{2} \|G_{zi}\|^2 \\ & - \frac{1}{4} \sum_{z=1}^m [|\lambda_{\max}(G_{zi} + G_{zi}^T)| \wedge |\lambda_{\min}(G_{zi} + G_{zi}^T)|]^2. \end{aligned}$$

If

$$-\lambda := \sum_{i \in \mathbb{S}} \pi_i \lambda_i < 0, \quad (5.9)$$

then the linear hybrid SDE (5.8) is almost surely exponentially stable.

The EM method applying to the linear hybrid SDE (5.8) forms approximation  $X_k \approx x(t_k)$ , with  $t_k = k\Delta$ , by setting  $X_0 = x_0$ ,  $r_0^\Delta = i_0$  and, generally,

$$X_{k+1} = X_k + A(r_k^\Delta) X_k \Delta + \sum_{z=1}^m G_z(r_k^\Delta) X_k \Delta B_k^z, \quad (5.10)$$

where  $\Delta B_k^z = B^z(t_{k+1}) - B^z(t_k)$ . By Theorem 5.2 we have the following conclusion.

**Corollary 5.4** *Under the conditions of Corollary 5.3, for any  $\varepsilon \in (0, \lambda)$  there is a constant  $\Delta^* \in (0, 1)$  such that for any  $0 < \Delta < \Delta^*$  the EM approximation (5.10) satisfies*

$$\limsup_{k \rightarrow \infty} \frac{1}{k\Delta} \log(|X_k|) \leq -(\lambda - \varepsilon) \quad \text{a.s.} \quad (5.11)$$

This shows that the EM method recovers the almost sure exponential stability of multi-dimensional linear hybrid SDEs very well indeed.

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