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Fractional Transformations of Generalised Functions

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A distributional theory of fractional transformations is developed. A constructive approach, based on the eigenfunction expansion method pioneered by A. H. Zemanian, is used to produce an appropriate space of test functions and corresponding space of generalised functions. The fractional transformations that are defined are shown to form an equicontinuous group of operators on the space of test functions and a weak* continuous group on the space of generalised functions. Integral representations for the fractional transformations are also obtained under certain conditions. The fractional Fourier transformation is considered as a particular case of our general theory.

Keywords: Fractional integral transforms; semigroups of operators; generalised functions.

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1. Introduction

In recent years there has been considerable interest in fractional versions of classical integral transforms, such as the Fourier transform; see [2] and [10]. This has been prompted by applications of the fractional Fourier transform (FrFT) to problems arising in signal processing, optics and quantum mechanics. Although the idea of a fractional power of the Fourier operator dates back to work by Wiener [12] in 1929, the development of a wide-ranging modern theory, including operational formulae, stems from a paper by Namias [9] which appeared in 1980.

The approach used by Namias relies primarily on eigenfunction expansions. For suitable functions ϕ, the Fourier transform F is defined by

$$(F\phi)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(y) e^{ixy} \, dy \,. \tag{1}$$

The integral in (1) exists pointwise for all $x \in \mathbb{R}$ when $\phi \in L^1(\mathbb{R})$ and is interpreted as

$$\int_{\mathbb{R}} \phi(y) e^{ixy} \, dy = \lim_{Y \to \infty} \int_{-Y}^{Y} \phi(y) e^{ixy} \, dy$$

for $\phi \in L^2(\mathbb{R})$, where lim denotes the limit in mean square. In the latter case, it is known that F is a homeomorphism on $L^2(\mathbb{R})$ and has eigenvalues

$$\mu_n = e^{in\pi/2}, \ n = 0, 1, \dots.$$

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with corresponding eigenfunctions

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-x^2/2} H_n(x), \tag{2}$$

where H_n is the Hermite polynomial of degree n. As $\{\psi_n\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R})$, it follows that

$$F\phi = \sum_n e^{in\pi/2} (\phi, \psi_n)_2 \psi_n \quad \forall \phi \in L^2(\mathbb{R})$$

and, by repeated application,

$$F^k\phi = \sum_n e^{in\pi k/2} (\phi, \psi_n)_2 \psi_n \quad \forall \phi \in L^2(\mathbb{R}), \quad k = 2, 3, \dots. \tag{3}$$

This led Namias to define a family of operators $\{F_\alpha\}_{\alpha \in \mathbb{R}}$ via the formula

$$F_\alpha \phi = \sum_n e^{in\alpha} (\phi, \psi_n)_2 \psi_n \quad \forall \phi \in L^2(\mathbb{R}). \tag{4}$$

Clearly, $F_\alpha = F$ when $\alpha = \pi/2$ and it is a routine matter to show that

$$F_\alpha F_\beta \phi = F_{\alpha+\beta} \phi, \quad \forall \phi \in L^2(\mathbb{R}), \ \forall \alpha, \beta \in \mathbb{R}.$$

Namias also gave an integral representation of $F_\alpha \phi$ and noted that

$$F_\alpha = e^{i\alpha E}, \text{ where } E = -\frac{1}{2}(D^2 - x^2 + 1), \ D = d/dx. \tag{5}$$

As Namias's innovative ideas and results were developed in a formal manner, they were later revisited by McBride and Kerr in [4], where a mathematically rigorous account is presented for the FrFT on the space S of test functions of rapid descent. It was pointed out in [4] that the integral formula obtained by Namias agreed with (4) only for certain values of α, not all. To rectify this, McBride and Kerr derived an alternative integral representation in the form

$$(\mathcal{F}_\alpha \phi)(x) = \frac{e^{i(\frac{\pi}{4}\hat{\alpha} - \alpha/2)}}{\sqrt{2\pi |\sin \alpha|}} e^{-i\frac{x^2}{2} \cot \alpha} \int_{-\infty}^{\infty} e^{\frac{ixy}{\sin \alpha}} e^{-i\frac{y^2}{2} \cot \alpha} \phi(y) \ dy, \tag{6}$$

where $\hat{\alpha} = \mathrm{sgn}(\sin \alpha)$, valid in the first instance for $0 < |\alpha| < \pi$. They noted that (6) reduces to the classical Fourier transform when $\alpha = \pi/2$ and to its inverse when $\alpha = -\pi/2$. With

$$(\mathcal{F}_0 \phi)(x) = \phi(x), (\mathcal{F}_{\pm\pi} \phi)(x) = \phi(-x) \tag{7}$$

the definition of $\mathcal{F}_\alpha$ was extended to all $\alpha \in \mathbb{R}$ by periodicity. In [5], the resulting family of operators $\{\mathcal{F}_\alpha\}_{\alpha \in \mathbb{R}}$ was proved to be a C_0-group of unitary operators on $L^2(\mathbb{R})$, with infinitesimal generator $G = iG_1$, where G_1 is the self-adjoint realisation of the operator E given in (5) with domain

$$D(G_1) = \{\phi \in L^2(\mathbb{R}) : \phi' \in L^2(\mathbb{R}) \text{ and } E\phi \in L^2(\mathbb{R})\}.$$

This semigroup analysis provided a precise interpretation of the exponential formula (5).

More recently in [13], Zayed used the same aproach as Namias to produce a more general method for defining fractional versions of a wider class of transforms. His starting point was to consider a bounded linear operator A on some separable Hilbert space H, where A is assumed to have a complete orthonormal system of eigenvectors $\{\psi_n\}$ in H with corresponding eigenvalues $\{\mu_n\}$. It follows from the continuity of A that

$$A^k \phi = \sum_n \mu_n^k (\phi, \psi_n)_{_H} \psi_n, \ \ \forall \phi \in H, \ k = 1, 2, 3, \ldots$$

where $(\cdot, \cdot)_{_H}$ denotes the inner product in H and the series converges in H. On choosing an appropriate branch for the power function $f(z) = z^\alpha$, the fractional operator A_α was defined by

$$A_\alpha \phi = \sum_n \mu_n^\alpha (\phi, \psi_n)_{_H} \psi_n$$

provided the series converges in H.

Zayed then examined the specific case when $H = L_\rho^2(I)$, the Hilbert space consisting of all square integrable functions on I with respect to some measure ρ, where $I = (a, b)$, $-\infty \le a < b \le \infty$. Under the assumption that

$$\sum_{n=0}^\infty |\mu_n|^{2\alpha} |\psi_n(x)|^2 < \infty, \quad \forall x \in I,$$

the function k_α, defined by

$$k_\alpha(x, t) := \sum_{n=0}^\infty \mu_n^\alpha \psi_n(x) \overline{\psi_n(t)},$$

was shown to have the property that $k_\alpha(x, \cdot) \in L_\rho^2(I)$ for each fixed $x \in I$, and this led to the integral representation

$$(A_\alpha \phi)(x) = \int_I k_\alpha(x, t) \phi(t) d\rho(t).$$

When $\mu_n = e^{i\beta n}$, for some $\beta \in \mathbb{R}$, operators A_α, $\alpha \in \mathbb{R}$, were defined by the formula

$$(A_\alpha \phi)(x) = \lim_{r \to 1^-} \sum_{n=0}^\infty r^n e^{in\alpha} (\phi, \psi_n)_2 \psi_n(x).$$

In this case an integral representation for A_α was obtained under the weaker condition that

$$\sum_{n=0}^\infty z^n |\psi_n(x)|^2 < \infty, \quad \forall |z| < 1.$$

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Since

$$(A_\alpha \phi)(x) = \lim_{r \to 1^-} \sum_{n=0}^{\infty} r^n e^{in\alpha}(\phi, \psi_n)_2 \psi_n(x) = \sum_{n=0}^{\infty} e^{in\alpha}(\phi, \psi_n)_2 \psi_n(x)$$

whenever the series converges, it follows that

$$(A_\alpha \phi)(x) = \lim_{|z| \to 1^-} \int_I k_\alpha(x, t, z)\phi(t)d\rho(t) = \int_I k_\alpha(x, t)\phi(t)d\rho(t), \tag{8}$$

where $z = |z|e^{i\alpha} = re^{i\alpha}$,

$$k_\alpha(x, t, z) := \sum_{n=0}^{\infty} z^n \psi_n(x)\overline{\psi_n(t)},$$

$$k_\alpha(x, t) := \sum_{n=0}^{\infty} e^{in\alpha}\psi_n(x)\overline{\psi_n(t)}$$

and it is assumed that it is legitimate to take the limit inside the integral in (8). Zayed noted that, in terms of polar coordinates (r, θ), the kernel $k_\alpha(x, t)$ is the radial limit of $k_\alpha(x, t, z)$ as $|z| \to 1^-$ along the ray $\theta = \alpha$. Hence he referred to A_α as an angular transform. The operator A_α has properties similar to fractional Fourier transforms, which Zayed considered as a specific example.

As the classical theory of the FrFT has a number of limitations, due primarily to the fact that many common functions do not belong to either S or $L^2(\mathbb{R})$, several distributional versions have also been developed. In [14], Zayed described two approaches. The first is analytic and uses the so-called "embedding method" to define the FrFT on the space $\mathcal{E}'$ of distributions with compact support. The second is algebraic, and involves the theory of Boehmians. Prior to this, Kerr [6] followed the usual "adjoint method" for defining the Fourier transform on the space S' of tempered distributions to extend the FrFT to S'. Using the fact that $\mathcal{F}_\alpha$ is a homeomorphism on S, and defining the extended version $\widetilde{\mathcal{F}_\alpha}$ on S' to be the adjoint of $\mathcal{F}_\alpha$ on S, standard results on adjoints established that each $\widetilde{\mathcal{F}_\alpha}$ is a homeomorphism on S'.

Our aim in the current paper is to demonstrate how the Hilbert space eigenfunction-expansion approach used by Zayed can be adapted to produce a theory of fractional transforms defined on spaces of generalised functions that are obtained in a constructive manner. In contrast to [13], where the starting point is a *bounded* operator on an L^2 space, our strategy involves a symmetric, *unbounded* differential operator T that is defined on some subspace $\mathcal{A}$ of L^2. By assuming that T has a complete orthonormal system of smooth eigenfunctions in L^2 and using the elegant theory produced by Zemanian [15, Chapter 9] for representing generalised functions in terms of eigenfunction expansions, we are able to develop a mathematically rigorous and systematic eigenfunction-based approach for defining distributional versions of fractional transformations. Moreover, we establish connections with equicontinuous and weak*-continuous groups of operators defined on locally convex topological vector spaces. As a special case of our general theory, we shall recover, and also extend, the tempered distribution results obtained by Kerr in [6] for the FrFT.

We begin in Section 2 by recalling some of the results on fractional transformations given by Zayed. Then, together with some results on groups of operators, we develop a general theory of fractional transforms $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ on the Hilbert space

$L^2(I)$. We obtain a self-adjoint operator $\mathcal{T}$ in $L^2(I)$ such that $i\mathcal{T}$ generates a group of continuous operators $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ on $L^2(I)$, and then investigate conditions under which the fractional transform G_α can be represented as an integral transform.

In Section 3, we extend the L^2 theory of fractional transformations to spaces of test functions. In particular we concentrate on a particular Fréchet space $\mathcal{A}$ and show how $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ is an equicontinuous group of operators on $\mathcal{A}$. In this case we define a symmetric differential operator T which is a restriction of $\mathcal{T}$ and establish that iT is the infinitesimal generator of the equicontinuous group $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ on $\mathcal{A}$.

In Section 4, the results obtained in Section 3 for spaces of test functions are extended to the corresponding spaces of generalised functions. Here we obtain a weak*-continuous group of operators.

Finally, in Section 5, we consider a particular case and describe how the tempered distribution theory of the FrFT given by Kerr [6] can be obtained in a natural and constructive manner by using orthonormal series expansions.

2. Fractional Transforms in a Hilbert space

We consider the Hilbert space $L^2(I)$, where I is an open interval in $\mathbb{R}$. Given a complete orthonormal system of smooth eigenfunctions $\{\psi_n\}_{n=0}^\infty \subset L^2(I)$ and a sequence $\{\lambda_n\}_{n=0}^\infty$ of real numbers with $|\lambda_n| \to \infty$ as $n \to \infty$, we define an operator $\mathcal{T}$ by

$$\mathcal{T}\phi := \sum_n \lambda_n(\phi, \psi_n)_2 \psi_n, \tag{9}$$

$$D(\mathcal{T}) := \{\phi \in L^2(I) : \sum_n \lambda_n^2 |(\phi, \psi_n)_2|^2 < \infty\}. \tag{10}$$

Clearly $\psi_n \in D(\mathcal{T})$ and $\mathcal{T}\psi_n = \lambda_n \psi_n$ for each $n = 0, 1, 2 \ldots$.

LEMMA 2.1 *The operator $\mathcal{T}$ is self-adjoint.*

Proof Let $\phi, \psi \in D(\mathcal{T})$. Then

$$(\mathcal{T}\phi, \psi)_2 = (\sum_n \lambda_n(\phi, \psi_n)_2 \psi_n, \psi)_2 = \sum_n \lambda_n(\phi, \psi_n)_2(\psi_n, \psi)_2$$

$$= (\phi, \sum_n \lambda_n(\psi, \psi_n)_2 \psi_n)_2 = (\phi, \mathcal{T}\psi)_2.$$

Hence $\mathcal{T}$ is a symmetric operator and it follows that

$$D(\mathcal{T}) \subset D(\mathcal{T}^*).$$

Now let $\psi \in D(\mathcal{T}^*)$. Then

$$(\mathcal{T}\phi, \psi)_2 = (\phi, \mathcal{T}^*\psi)_2, \quad \forall \phi \in D(\mathcal{T}).$$

Therefore

$$(\sum_n \lambda_n(\phi, \psi_n)_2 \psi_n, \psi)_2 = (\sum_n (\phi, \psi_n)_2 \psi_n, \mathcal{T}^*\psi)_2$$

and so

$$\sum_n \lambda_n (\phi, \psi_n)_2 \overline{(\psi, \psi_n)_2} = \sum_n (\phi, \psi_n)_2 \overline{(\mathcal{T}^*\psi, \psi_n)_2}.$$

Choose $\phi = \psi_0, \psi_1, \psi_2, \ldots$ to obtain

$$\lambda_n (\psi, \psi_n)_2 = (\mathcal{T}^*\psi, \psi_n)_2 \quad \forall n.$$

Consequently

$$\mathcal{T}^*\psi = \sum_n \lambda_n (\psi, \psi_n)_2 \psi_n = \mathcal{T}\psi,$$

which gives the required result that $\mathcal{T}$ is self-adjoint. ∎

It follows from Stone's theorem [3, p.32] that $i\mathcal{T}$ generates a (C_0)-unitary group $\{\exp(i\alpha\mathcal{T})\}_{\alpha \in \mathbb{R}}$ on the space $L^2(I)$.

If we now define a family of operators $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ on $L^2(I)$ by

$$G_\alpha \phi := \sum_n e^{i\lambda_n \alpha} (\phi, \psi_n)_2 \psi_n \quad \forall \phi \in L^2(I), \tag{11}$$

then we can prove the following result.

THEOREM 2.2 *Let $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ be defined by (11). Then $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ is a strongly continuous group of unitary operators on $L^2(I)$. Moreover, the infinitesimal generator $\mathcal{S}$ of $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ is given by $i\mathcal{T}$, where $\mathcal{T}$ is defined by (9) and (10).*

Proof It is straightforward to prove that $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ satisfies the algebraic properties of a group, that is $G_0 = I$ and $G_\alpha G_\beta = G_{\alpha+\beta} = G_\beta G_\alpha$ for all α, $\beta \in \mathbb{R}$. Moreover, for each $\phi \in L^2(I)$,

$$\|G_\alpha \phi - \phi\|_2^2 = \sum_n |(e^{i\lambda_n \alpha} - 1)|^2 |(\phi, \psi_n)_2|^2.$$

Clearly $|e^{i\lambda_n \alpha} - 1|^2 |(\phi, \psi_n)_2|^2 \to 0$ as $\alpha \to 0$ for each n. Also,

$$(e^{i\lambda_n \alpha} - 1)(e^{-i\lambda_n \alpha} - 1) = 2 - 2\cos(\lambda_n \alpha) \le 4, \quad \forall \alpha$$

and so

$$|e^{i\lambda_n \alpha} - 1|^2 |(\phi, \psi_n)_2|^2 \le 4|(\phi, \psi_n)_2|^2.$$

Since

$$\sum_n |(\phi, \psi_n)_2|^2 = \|\phi\|_2^2 < \infty,$$

it follows from the Weierstrass M-test [1, p.438] that

$$\|G_\alpha \phi - \phi\|_2^2 \to 0 \text{ as } \alpha \to 0.$$

Hence $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ is a strongly continuous group of operators on $L^2(I)$. In addition, for $\phi, \psi \in L^2(I)$,

$$(G_\alpha \phi, \psi)_2 = (\sum_n e^{i\lambda_n \alpha}(\phi, \psi_n)_2 \psi_n, \psi)_2 = \sum_n e^{i\lambda_n \alpha}(\phi, \psi_n)_2(\psi_n, \psi)_2$$

$$= (\phi, \sum_n e^{-i\lambda_n \alpha}(\psi, \psi_n)_2 \psi_n)_2 = (\phi, G_{-\alpha}\psi)_2,$$

and therefore $G_\alpha^* = G_{-\alpha} = G_\alpha^{-1}$, establishing that each G_α is unitary.

For the infinitesimal generator of $\{G_\alpha\}_{\alpha \in \mathbb{R}}$, we show first that if $\phi \in D(\mathcal{T})$ then $\phi \in D(\mathcal{S})$ and

$$\lim_{\alpha \to 0} \left\| \frac{G_\alpha \phi - \phi}{\alpha} - i\mathcal{T}\phi \right\|_2 = 0. \tag{12}$$

Consider

$$\lim_{\alpha \to 0} \left\| \sum_n (\frac{e^{i\lambda_n \alpha} - 1}{\alpha} - i\lambda_n)(\phi, \psi_n)_2 \psi_n \right\|_2^2 = \lim_{\alpha \to 0} \sum_n \left| \frac{e^{i\lambda_n \alpha} - 1}{\alpha} - i\lambda_n \right|^2 |(\phi, \psi_n)_2|^2.$$

Let

$$g_\alpha(n) = \begin{cases} (\frac{e^{i\lambda_n \alpha}-1}{\alpha} - i\lambda_n)(\phi, \psi_n)_2 & \text{when } \alpha \neq 0 \\ 0 & \text{when } \alpha = 0. \end{cases}$$

Arguing as above, we obtain

$$|g_\alpha(n)|^2 \leq 4|\lambda_n|^2 |(\phi, \psi_n)_2|^2 \quad \forall n, \forall \alpha.$$

Hence, by the Weierstrass M-test,

$$\lim_{\alpha \to 0} \sum_n |g_\alpha(n)|^2 = \sum_n \lim_{\alpha \to 0} |g_\alpha(n)|^2 = 0,$$

and therefore (12) is satisfied. Consequently $\phi \in D(\mathcal{S})$ and $i\mathcal{T}\phi = \mathcal{S}\phi$ for all $\phi \in D(\mathcal{T}) \subset D(\mathcal{S})$.

For the converse, suppose that $\phi \in D(\mathcal{S})$. Then

$$\lim_{\alpha \to 0} \left\| \frac{G_\alpha \phi - \phi}{\alpha} - \mathcal{S}\phi \right\|_2^2 = 0.$$

Also $\mathcal{S}\phi \in L^2(I)$ and can be written as $\mathcal{S}\phi = \sum_n (\mathcal{S}\phi, \psi_n)_2 \psi_n$. Therefore

$$\left\| \frac{G_\alpha \phi - \phi}{\alpha} - \mathcal{S}\phi \right\|_2^2 = \sum_n \left| \frac{e^{i\lambda_n \alpha} - 1}{\alpha}(\phi, \psi_n)_2 - (\mathcal{S}\phi, \psi_n)_2 \right|^2 \to 0 \quad \text{as } \alpha \to 0.$$

Thus

$$\lim_{\alpha \to 0} \frac{e^{i\lambda_n \alpha} - 1}{\alpha}(\phi, \psi_n)_2 = (\mathcal{S}\phi, \psi_n)_2 \qquad \forall n,$$

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that is

$$\mathcal{S}\phi = \sum_n i\lambda_n(\phi, \psi_n)_2\psi_n = i\mathcal{T}\phi, \qquad \forall \phi \in D(\mathcal{S}).$$

Hence the result follows. ∎

Our aim now is to investigate conditions under which the fractional transform G_α, defined by (11), has an equivalent representation in the form of an integral transform. Following Zayed's approach [13], we define the operator $G_{\alpha,r}$ as

$$G_{\alpha,r}\phi := \sum_{n=0}^{\infty} r^n e^{i\lambda_n \alpha}(\phi, \psi_n)_2\psi_n, \qquad (13)$$

where $0 < r \le 1$, and so $G_\alpha := G_{\alpha,1}$. Clearly, $G_{\alpha,r} \in B(L^2(I))$ for each $\alpha \in \mathbb{R}$ and $r \in (0, 1]$, since $\|G_{\alpha,r}\phi\|_2^2 \le \|\phi\|_2^2$ for all $\phi \in L^2(I)$. Moreover

$$G_{\alpha,r}\phi \to G_\alpha\phi \text{ in } L^2(I) \text{ as } r \to 1^-. \qquad (14)$$

This leads immediately to the following result.

COROLLARY 2.3 *For each fixed $\phi \in L^2(I)$, there exists $\{r_j\}_{j=1}^{\infty}$, with $r_j \to 1^-$ as $j \to \infty$, such that*

$$(G_\alpha\phi)(x) = \lim_{j\to\infty}(G_{\alpha,r_j}\phi)(x),$$

for almost all $x \in I$.

Proof This is a consequence of a standard result that if a sequence $\{\phi_n\}$ converges in $L^2(I)$ to ϕ, then there exists a subsequence $\{\phi_{n_k}\}$ that converges pointwise almost everywhere to ϕ. ∎

We now assume that, for each $x \in I$,

$$\sum_{n=0}^{\infty} r^{2n}|\psi_n(x)|^2 < \infty \quad \forall r \in (0,1), \qquad (15)$$

which will enable us to obtain an integral representation of $G_{\alpha,r}$, for $\alpha \in \mathbb{R}$ and $0 < r < 1$.

LEMMA 2.4 *If (15) holds, then*

$$(G_{\alpha,r}\phi)(x) = \int_I k_{\alpha,r}(x,y)\phi(y)dy \qquad 0 < r < 1, \phi \in L^2(I), \qquad (16)$$

where

$$k_{\alpha,r}(x,y) = \sum_{n=0}^{\infty} r^n e^{i\lambda_n \alpha}\psi_n(x)\overline{\psi_n(y)}. \qquad (17)$$

Proof If (15) holds, then, for each $x \in I$, $\sum_{n=0}^{\infty} r^n e^{i\lambda_n \alpha}\psi_n(x)\overline{\psi_n}$ converges in $L^2(I)$

to a function which we denote by $k_{\alpha,r}(x,\cdot)$. Now, for each $\phi \in L^2(I)$ and $x \in I$,

$$(\phi, \overline{k_{\alpha,r}(x,\cdot)})_2 = (\phi, \sum_{n=0}^{\infty} r^n e^{-i\lambda_n \alpha} \overline{\psi_n(x)} \psi_n)_2 = \sum_{n=0}^{\infty} r^n e^{i\lambda_n \alpha} (\phi, \psi_n)_2 \psi_n(x).$$

Therefore,

$$(G_{\alpha,r}\phi)(x) = \int_I k_{\alpha,r}(x,y)\phi(y)dy = \int_I (\sum_{n=0}^{\infty} r^n e^{i\lambda_n \alpha} \psi_n(x)\overline{\psi_n(y)})\phi(y)dy.$$

∎

COROLLARY 2.5 *If (15) holds, then for each $\phi \in L^2(I)$ there exists $\{r_j\}$, with $r_j \to 1^-$ as $j \to \infty$, such that*

$$(G_\alpha \phi)(x) = \lim_{j\to\infty}(G_{\alpha,r_j}\phi)(x) = \lim_{j\to\infty}\int_I k_{\alpha,r_j}(x,y)\phi(y)dy$$

almost everywhere in I.

Proof The proof follows from Corollary 2.3. ∎

Note that, if $k_{\alpha,r}(x,y)$ converges pointwise to $k_{\alpha,1}(x,y)$ as $r \to 1^-$ and *the limit can be taken inside the integral*, then we arrive at

$$(G_\alpha \phi)(x) = \int_I k_{\alpha,1}(x,y)\phi(y)dy, \ \phi \in L^2(I).$$

We shall show in Section 5 that this procedure is valid in the case of the FrFT.

3. Fractional Transforms on Test Functions

In this section, we develop a general theory of fractional transforms on spaces of test functions that are constructed in a systematic manner. Central to this are differential operators $T : L^2(I) \supset D(T) \to L^2(I)$ of the type introduced by Zemanian in [15, Chapter 9]. Therefore throughout the following discussion T will be a differential expression of the form

$$T = \theta_0 D^{n_1} \theta_1 D^{n_2} \ldots D^{n_\nu} \theta_\nu \tag{18}$$

where $D = d/dt$, the n_k are positive integers, and the θ_k are smooth functions on I. Moreover, it is assumed that T satisfies the symmetry condition

$$T = \overline{\theta_\nu}(-D)^{n_\nu} \ldots (-D)^{n_2} \overline{\theta_1}(-D)^{n_1} \overline{\theta_0}, \tag{19}$$

has eigenfunctions $\{\psi_n\}_{n=0}^{\infty}$ that form a complete orthonormal basis for $L^2(I)$ and where the corresponding real eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ are such that $|\lambda_n| \to \infty$ as $n \to \infty$. We shall take the domain, $D(T)$, of T to be

$$D(T) := \left\{ \phi \in C^\infty(I) : T^k\phi \in L^2(I), (T^k\phi, \psi_n)_2 = (\phi, T^k\psi_n)_2, \forall \ k,n = 0,1,\ldots \right\} \tag{20}$$

where T^0 is the identity operator on $L^2(I)$.

Note that $D(T)$ is dense in $L^2(I)$, since $C_0^\infty(I) \subset D(T)$, and each eigenfunction $\psi_n \in D(T)$. Moreover, it follows from [15, p.255] that T is symmetric on $D(T)$ and, since

$$T\phi = \sum_n (T\phi, \psi_n)_2 \psi_n = \sum_n (\phi, T\psi_n)_2 \psi_n$$

$$= \sum_n (\phi, \lambda_n \psi_n)_2 \psi_n = \sum_n \lambda_n (\phi, \psi_n)_2 \psi_n$$

$$= \mathcal{T}\phi \quad \forall \phi \in D(T),$$

we deduce that $(T, D(T))$ is a restriction of the operator $(\mathcal{T}, D(\mathcal{T}))$ defined via (9) and (10).

For such a differential operator T, the Fréchet space $\mathcal{A}$ introduced by Zemanian in [15, Chapter 9] can be defined as follows.

DEFINITION 3.1 *The space $\mathcal{A}$ is the vector space $D(T)$ equipped with the topology generated by the countable multinorm $\{\beta_k\}_{k=0}^\infty$, where*

$$\beta_k(\phi) := \|T^k \phi\|_2 < \infty, \quad k = 0, 1, 2, \dots.$$

It follows from the definition of β_k that T^r is a continuous linear operator on $\mathcal{A}$ for each $r = 1, 2, 3, \dots$.

Now consider the family of operators $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ defined by (11). Our aim is to show that $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ is an equicontinuous group on the space $\mathcal{A}$. First we shall establish that each G_α is well defined as a continuous linear mapping from $\mathcal{A}$ into $\mathcal{A}$. For this we require the following result.

LEMMA 3.2

    (i) *The series $\sum_n (\phi, \psi_n)_2 \psi_n$ converges to ϕ in the topology of $\mathcal{A}$ for each $\phi \in \mathcal{A}$.*

    (ii) *Let $\{a_n\}$ be a sequence of complex numbers. Then $\sum_n a_n \psi_n$ converges in $\mathcal{A}$ if and only if $\sum_n |\lambda_n|^{2k} |a_n|^2$ converges for every non-negative integer k.*

Proof See [15, Lemmas 9.3-2, 9.3-3]. ∎

LEMMA 3.3 *The operator G_α is continuous on $\mathcal{A}$ for all $\alpha \in \mathbb{R}$.*

Proof Let $\phi \in \mathcal{A}$. Then

$$\sum_n |e^{i\lambda_n \alpha} (\phi, \psi_n)_2|^2 |\lambda_n|^{2k} = \sum_n |(\phi, \psi_n)_2|^2 |\lambda_n|^{2k} < \infty \quad \forall k = 0, 1, 2, \dots.$$

Therefore, it follows from Lemma 3.2 that $G_\alpha \phi \in \mathcal{A}$. To complete the proof we shall use the result

$$T^k G_\alpha \phi = G_\alpha T^k \phi, \qquad \forall \phi \in \mathcal{A}, \ \alpha \in \mathbb{R}, \ k = 1, 2, \dots, \tag{21}$$

which can easily be deduced from the linearity and continuity of T^k on $\mathcal{A}$. From (21) we obtain, for each $\phi \in \mathcal{A}$ and $k = 0, 1, 2, \dots$,

$$\beta_k(G_\alpha \phi) = \|T^k G_\alpha \phi\|_2 = \|G_\alpha T^k \phi\|_2$$

$$= \|T^k \phi\|_2 = \beta_k(\phi), \quad \text{(since } G_\alpha \text{ is an isometry on } L^2(I)\text{)}.$$

This shows that $G_\alpha \in L(\mathcal{A})$, the space of continuous linear operators on $\mathcal{A}$. ∎

THEOREM 3.4 *The family of continuous operators $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ is an equicontinuous group on $\mathcal{A}$.*

Proof From Lemma 3.3, $\{G_\alpha\}_{\alpha \in \mathbb{R}} \subset L(\mathcal{A})$. The algebraic conditions for $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ to be a group are satisfied, as in Theorem 2.2, because $\mathcal{A} \subset L^2(I)$. Moreover, for $k = 0, 1, 2, \ldots,$

$$
\begin{aligned}
\beta_k(G_\alpha \phi - \phi) &= \|T^k G_\alpha \phi - T^k \phi\|_2 \\
&= \|G_\alpha(T^k \phi) - T^k \phi\|_2 \quad \text{(from (21))} \\
&\to 0 \quad \text{as} \quad \alpha \to 0 \quad \text{(from Theorem 2.2)}.
\end{aligned}
$$

Hence $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ is a strongly continuous group of continuous linear operators on $\mathcal{A}$. It follows from the argument used in Lemma 3.3 that for each k

$$
\beta_k(G_\alpha \phi) = \beta_k(\phi) \quad \forall \alpha \in \mathbb{R}, \phi \in \mathcal{A}. \tag{22}
$$

For equicontinuity, it is sufficient to verify that for each β_k there exists a continuous seminorm q_k on $\mathcal{A}$ such that

$$
\beta_k(G_\alpha \phi) \leq q_k(\phi) \quad \forall \alpha \in \mathbb{R}, \phi \in \mathcal{A}.
$$

But this follows from (22), as we can simply take $q_k = \beta_k$. Hence $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ is equicontinuous. ∎

The following theorem shows that the infinitesimal generator R of the group $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ on $\mathcal{A}$ is the operator $iT \in L(\mathcal{A})$.

THEOREM 3.5 *Let $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ be the equicontinuous group on the space $\mathcal{A}$ defined by (11), where the series now converges in $\mathcal{A}$. Then the associated infinitesimal generator is $R = iT$, where $T \in L(\mathcal{A})$ is defined by (18) - (20).*

Proof To prove that iT is the generator R of $\{G_\alpha\}_{\alpha \in \mathbb{R}}$, we need to show that $D(R) = D(T) = \mathcal{A}$ and $R\phi = iT\phi$, for all $\phi \in \mathcal{A}$. Clearly $D(R) \subseteq D(T)$, since iT is defined on all of $\mathcal{A}$. To prove the reverse inclusion, let $\phi \in D(T) = \mathcal{A}$ and let k be any non-negative integer. Then $\sum_n |\lambda_n|^{2k} |(\phi, \psi_n)_2|^2 < \infty$. Also,

$$
\begin{aligned}
\left[\beta_k \left(\frac{G_\alpha \phi - \phi}{\alpha} - iT\phi \right) \right]^2 &= \left\| T^k \left(\frac{G_\alpha \phi - \phi}{\alpha} \right) - iT^k T\phi \right\|_2^2 \\
&= \left\| \frac{G_\alpha(T^k \phi) - T^k \phi}{\alpha} - iT(T^k \phi) \right\|_2^2 \quad \text{(from (21))} \\
&\to 0 \quad \text{by Theorem 2.2.}
\end{aligned}
$$

Consequently for each $\phi \in D(T) = \mathcal{A}$, we have

$$
iT\phi = \lim_{\alpha \to 0} \frac{G_\alpha \phi - \phi}{\alpha},
$$

and so $\phi \in D(R)$. Moreover, $R\phi = iT\phi$ for all $\phi \in \mathcal{A}$ and the stated result follows. ∎

We now return to to the operators $G_{\alpha,r}$ defined by (13) and examine their properties on the space $\mathcal{A}$.

LEMMA 3.6 *For $\phi \in \mathcal{A}$, the series given by (13) converges in $\mathcal{A}$. Also $G_{\alpha,r} \in L(\mathcal{A})$ and $G_{\alpha,r}\phi \to G_\alpha\phi$ in $\mathcal{A}$ as $r \to 1^-$.*

Proof That $G_{\alpha,r} \in L(\mathcal{A})$ follows from the fact that $\beta_k(G_{\alpha,r}\phi) \le \beta_k(\phi)$ for all $\phi \in \mathcal{A}$ and $k = 0, 1, 2, \dots$. Also,

$$T^k G_{\alpha,r}\phi = G_{\alpha,r}T^k\phi, \qquad \forall \phi \in \mathcal{A}, \ \alpha \in \mathbb{R}, \ r \in (0,1], \ k = 1, 2, \dots,$$

and therefore

$$\beta_k(G_\alpha\phi - G_{\alpha,r}\phi) = \|G_\alpha T^k \phi - G_{\alpha,r}T^k\phi\|_2 \to 0 \ \text{ as } \ r \to 1^-, \forall k = 0, 1, 2, \dots \ .$$

∎

Note that convergence in $\mathcal{A}$ implies convergence in $\mathcal{E}$, where $\mathcal{E}$ is the usual space of C^∞ test functions; see [15, Lemma 9.3-4]. Consequently, for $\phi \in \mathcal{A}$, we can state that $(G_{\alpha,r}\phi)(x) \to (G_\alpha\phi)(x)$ for all x, in contrast to Corollary 2.3 where convergence was for *almost* all $x \in I$.

4. Fractional Transforms of Generalised Functions

Our next task is to extend the equicontinuous (C_0)-group $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ on $\mathcal{A}$ to a group of generalised operators $\{\widetilde{G_\alpha}\}_{\alpha \in \mathbb{R}}$ on $\mathcal{A}'$. Each $f \in \mathcal{A}'$ assigns a number $< f, \phi >$ to each $\phi \in \mathcal{A}$. In the following it is convenient to use the notation

$$(f, \phi) :=< f, \overline{\phi} >, \quad f \in \mathcal{A}', \phi \in \mathcal{A}.$$

Note that each $\eta \in L^2(I)$ generates an element $\widetilde{\eta} \in \mathcal{A}'$ defined by

$$(\widetilde{\eta}, \phi) := (\eta, \phi)_2 = \int_I \eta(x)\overline{\phi(x)}\, dx, \ \phi \in \mathcal{A}. \tag{23}$$

It follows that for each $\eta \in L^2(I)$ and $\phi \in \mathcal{A}$,

$$(\widetilde{G_\alpha\eta}, \phi) = \sum_n e^{i\lambda_n\alpha}(\widetilde{\eta}, \psi_n)(\widetilde{\psi_n}, \phi)\,.$$

This indicates that the logical extension of G_α to $\mathcal{A}'$ is given by

$$\widetilde{G_\alpha}f = \sum_n e^{i\lambda_n\alpha}(f, \psi_n)\widetilde{\psi_n}, \ f \in \mathcal{A}' \tag{24}$$

where the series converges in $\mathcal{A}'$. We shall return to this series representation of $\widetilde{G_\alpha}$ later.

First we consider an alternative approach and proceed as follows. Let $\eta \in L^2(I)$ and $\phi \in \mathcal{A}$. Then by (23) and Theorem 2.2 we have

$$(\widetilde{G_\alpha\widetilde{\eta}}, \phi) := (\widetilde{G_\alpha\eta}, \phi) = (G_\alpha\eta, \phi)_2 = (\eta, G_{-\alpha}\phi)_2 = (\widetilde{\eta}, G_{-\alpha}\phi).$$

This suggests that we should define $\widetilde{G_\alpha}$ on $\mathcal{A}'$ as follows.

DEFINITION 4.1 *We define the generalised operator $\widetilde{G_\alpha}$ by*

$$(\widetilde{G_\alpha}f, \phi) := (f, G_{-\alpha}\phi) \quad f \in \mathcal{A}', \phi \in \mathcal{A}. \tag{25}$$

Then we have

$$\widetilde{G_\alpha} = (G_{-\alpha})' = (G_\alpha^*)'.$$

THEOREM 4.2 *Let $\{\widetilde{G_\alpha}\}_{\alpha \in \mathbb{R}}$ be the family of operators on $\mathcal{A}'$ defined by (25). Then for all $\alpha \in \mathbb{R}$, $\widetilde{G_\alpha}$ is a homeomorphism on $\mathcal{A}'$ with inverse $\widetilde{G_{-\alpha}}$.*

Proof From Theorem 3.4, $G_{-\alpha}$ is a homeomorphism on $\mathcal{A}$ with inverse G_α for each $\alpha \in \mathbb{R}$. Definition 4.1 and a standard result on adjoint operators (see [15, Theorem 1.10-2]) now establish that $\widetilde{G_\alpha}$ is a homeomorphism on $\mathcal{A}'$ with inverse $(G_{-\alpha}^{-1})' = G_\alpha' = \widetilde{G_{-\alpha}}$. ∎

Consider the operator T defined by (18)-(20) and let $\widetilde{T}$ denote the generalised version of T on $\mathcal{A}'$. For $\widetilde{T}$ to be an extension of T, we require

$$\widetilde{T}\widetilde{\varphi} = \widetilde{T\varphi}, \quad \forall \varphi \in D(T). \tag{26}$$

Let $\phi \in \mathcal{A}$. Then

$$(\widetilde{T}\widetilde{\varphi}, \phi) = (\widetilde{T\varphi}, \phi) = (T\varphi, \phi)_2 = (\varphi, T\phi)_2 = (\widetilde{\varphi}, T\phi).$$

This motivates the following definition.

DEFINITION 4.3 *We define the operator $\widetilde{T}$ on $\mathcal{A}'$ by*

$$(\widetilde{T}f, \phi) := (f, T\phi) \quad f \in \mathcal{A}', \phi \in \mathcal{A}.$$

THEOREM 4.4 *$\widetilde{T}$ is a continuous linear mapping from $\mathcal{A}'$ into $\mathcal{A}'$.*

Proof Since $\widetilde{T}$ is the adjoint of T and $T \in L(\mathcal{A})$, this follows from [15, Theorem 1.10-1]. ∎

We know that the operator $\mathcal{T}$ defined by (9) and (10) is self-adjoint. Moreover $\mathcal{A} \subset D(\mathcal{T})$ and $\mathcal{T}_{|\mathcal{A}} = T$. Therefore we have

$$(\widetilde{\mathcal{T}}f, \phi) := (f, T\phi) \qquad \forall f \in \mathcal{A}', \phi \in \mathcal{A}.$$

Hence $\widetilde{T} = \widetilde{\mathcal{T}}$ on $\mathcal{A}'$, that is $\widetilde{T}$ is also an extension of $\mathcal{T}$ to $\mathcal{A}'$.

THEOREM 4.5 *The family of operators $\{\widetilde{G_\alpha}\}_{\alpha \in \mathbb{R}}$, defined on $\mathcal{A}'$ by (25), is a weak*-continuous group of linear operators on $\mathcal{A}'$. Moreover the infinitesimal generator of $\{\widetilde{G_\alpha}\}_{\alpha \in \mathbb{R}}$ is $(-iT)' = i\widetilde{T}$.*

Proof Since $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ is an equicontinuous group of class (C_0) on $\mathcal{A}$ with infinitesimal generator iT, it follows that $\{G_{-\alpha}\}_{\alpha \in \mathbb{R}}$ is an equicontinuous group of class (C_0) on $\mathcal{A}$ with infinitesimal generator $-iT$. On applying [7, Proposition 2.1] (adapted for the case of equicontinuous groups) we deduce that $\{\widetilde{G_\alpha}\}_{\alpha \in \mathbb{R}} = \{G_{-\alpha}'\}_{\alpha \in \mathbb{R}}$ is a weak*-continuous group on $\mathcal{A}'$ with infinitesimal generator $(-iT)'$. Since

$$(\widetilde{iT}f, \psi) = i(\widetilde{T}f, \psi) = (\widetilde{T}f, -i\psi) = (f, -iT\psi)$$

for all $f \in \mathcal{A}'$, $\psi \in \mathcal{A}$, the operator $\widetilde{iT}$ is also the adjoint of $-iT$. ∎

To show that our definition of $\widetilde{G_\alpha}$ also agrees with (24), let $f \in \mathcal{A}'$. Then, by Definition 4.1,

$$(\widetilde{G_\alpha}f, \phi) = (f, G_{-\alpha}\phi) = (f, \sum_n e^{-i\lambda_n \alpha}(\phi, \psi_n)_2 \psi_n)$$

$$= \sum_n e^{i\lambda_n \alpha}(f, \psi_n)(\psi_n, \phi)_2 = \sum_n e^{i\lambda_n \alpha}(f, \psi_n)(\widetilde{\psi_n}, \phi).$$

Since the series on the right-hand side converges for each $\phi \in \mathcal{A}$, we obtain

$$(\widetilde{G_\alpha}f, \phi) = (\sum_n e^{i\lambda_n \alpha}(f, \psi_n)\widetilde{\psi_n}, \phi), \tag{27}$$

and so

$$\widetilde{G_\alpha}f = \sum_n e^{i\lambda_n \alpha}(f, \psi_n)\widetilde{\psi_n}, \ f \in \mathcal{A}'$$

where the series converges in $\mathcal{A}'$.

5. Fractional Fourier Transforms

We know that $\{G_\alpha\}_{\alpha \in \mathbb{R}} = \{e^{i\alpha T}\}_{\alpha \in \mathbb{R}}$ is an equicontinuous group on the space $\mathcal{A}$ for any differential operator T of the form (18)-(20). We now consider the specific case when the symmetric operator T is the differential expression E given in (5). Let $I = \mathbb{R}$ and let $\{\psi_n\}_{n=0}^\infty$ be the complete orthonormal set of eigenfunctions of T given by (2). The corresponding eigenvalues are $\lambda_n = n$, for all $n = 0, 1, \dots$. and therefore we have

$$G_\alpha \phi = \sum_{n=0}^\infty e^{in\alpha}(\phi, \psi_n)_2 \psi_n.$$

Comparison with (4) shows that this particular choice of differential operator leads to the FrFT. Consequently, we shall denote the corresponding fractional transforms by F_α rather than G_α in this case, that is

$$F_\alpha \phi = \sum_{n=0}^\infty e^{in\alpha}(\phi, \psi_n)_2 \psi_n, \tag{28}$$

where ψ_n is defined by (2). As indicated earlier, $F_{\pi/2} = F$, the classical Fourier transformation.

Our general theory leads immediately to the following.

THEOREM 5.1 *Let F_α be defined by (28). Then for each $\alpha \in \mathbb{R}$, F_α is a homeomorphism on $L^2(\mathbb{R})$ with inverse $F_{-\alpha}$. Furthermore, the family of operators $\{F_\alpha\}_{\alpha \in \mathbb{R}}$ is a strongly continuous group of unitary operators on $L^2(\mathbb{R})$, with infinitesimal generator given by $i\mathcal{T}$ where*

$$D(\mathcal{T}) := \{\phi \in L^2(\mathbb{R}) : \sum_{n=0}^\infty n^2 |(\phi, \psi_n)_2|^2 < \infty\} \tag{29}$$

and

$$\mathcal{T}\phi := \sum_{n=0}^{\infty} n(\phi, \psi_n)_2 \psi_n. \tag{30}$$

Proof This follows from Theorem 2.2. ∎

We turn now to the FrFT on the space S of test functions of rapid descent. For this we require the following result which will enable us to use our theory of $\{G_\alpha\}_{\alpha \in \mathbb{R}}$ on the space $\mathcal{A}$.

LEMMA 5.2 *If T is the differential operator E given in (5), with domain defined via (20), then the corresponding space $\mathcal{A}$ is the space S of test functions of rapid descent and $\mathcal{A}'$ is the space S' of distributions of slow growth.*

Proof See [11] and [15, p.267]. ∎

The main properties of F_α on S follow immediately.

THEOREM 5.3 *For each $\alpha \in \mathbb{R}$, F_α, defined by (28), is a homeomorphism on S with inverse $F_{-\alpha}$. Moreover, $\{F_\alpha\}_{\alpha \in \mathbb{R}}$ is an equicontinuous group of class (C_0) on S with infinitesimal generator iT, where $D(T) = \mathcal{A} = S$.*

Proof These results are a direct consequence of Theorems 3.4 and 3.5, and Lemma 5.2.

∎

Note that a number of mapping properties of F_α on S were previously established by McBride and Kerr [4] via different, but lengthier, arguments that relied on the integral formula (6). The advantage of our approach is that we obtain, not only the results in [4], but also additional information on the associated group of fractional Fourier transforms on S as a particular case of a more general theory. We now give a rigorous justification that the eigenfunction-eigenvalue definition that we have used leads directly to the integral formula (6).

To obtain an integral representation of $F_\alpha \phi$ in the form

$$(F_\alpha \phi)(x) = \int_{-\infty}^{\infty} k_\alpha(x, y)\phi(y)dy, \quad \phi \in L^2(\mathbb{R}), \tag{31}$$

we define

$$F_{\alpha,r}\phi := \sum_{n=0}^{\infty} r^n e^{in\alpha}(\phi, \psi_n)_2 \psi_n, \tag{32}$$

where $0 < r \leq 1$. Then from (14),

$$F_{\alpha,r}\phi \to F_{\alpha,1}\phi := F_\alpha \phi \text{ in } L^2(\mathbb{R}) \text{ as } r \to 1^-,$$

for each $\phi \in L^2(\mathbb{R})$. We now verify that assumption (15) is valid for the specific case when ψ_n is defined by (2).

LEMMA 5.4 *For each $x \in \mathbb{R}$,*

$$\sum_{n=0}^{\infty} r^{2n}|\psi_n(x)|^2 < \infty \quad \forall\, r \in (0,1), \tag{33}$$

where $\psi_n(x)$ is defined by (2).

Proof The asymptotic representation of the Hermite polynomial [8, p.67] shows that, for each fixed x and n sufficiently large, there exists a constant M such that

$$|H_n(x)| \leq M2^{(n+1)/2}n^{n/2}e^{-n/2}e^{x^2/2}.$$

Therefore it is sufficient to establish the convergence of the infinite series

$$\sum_{n=0}^{\infty} u_n, \quad \text{where } u_n = \frac{r^{2n}n^n e^{-n}}{n!}, \quad n = 0, 1, 2, \ldots.$$

Since $u_{n+1}/u_n \to r^2 < 1$ as $n \to \infty$, it follows that this series does indeed converge. ∎

Consequently, we have the following lemma.

LEMMA 5.5 *For $\phi \in L^2(\mathbb{R})$ and $0 < r < 1$,*

$$(F_{\alpha,r}\phi)(x) = \int_{-\infty}^{\infty} k_{\alpha,r}(x,y)\phi(y)dy, \tag{34}$$

where

$$k_{\alpha,r}(x,y) = \sum_{n=0}^{\infty} r^n e^{in\alpha}\psi_n(x)\overline{\psi_n(y)}. \tag{35}$$

Proof We know from Lemma 5.4 that $\sum_{n=0}^{\infty} r^{2n}|\psi_n(x)|^2 < \infty$, $\forall \, r \in (0,1)$ and for each fixed $x \in \mathbb{R}$. Hence the result follows from Lemma 2.4. ∎

If we examine the function $k_{\alpha,r}$ given by (35), then we obtain

$$k_{\alpha,r}(x,y) = \sum_{n=0}^{\infty} r^n e^{in\alpha}\psi_n(x)\overline{\psi_n(y)}$$

$$= \frac{e^{-(\frac{x^2+y^2}{2})}}{\sqrt{\pi}} \sum_{n=0}^{\infty} (re^{i\alpha})^n \frac{H_n(x)H_n(y)}{2^n n!}.$$

Now using Mehler's formula [8, p.61], and setting $z = re^{i\alpha}$ with $|z| = r < 1$, we have

$$k_{\alpha,r}(x,y) = \frac{1}{\sqrt{\pi}}(1 - (re^{i\alpha})^2)^{-1/2}\exp\left(\frac{2xyre^{i\alpha}}{1 - (re^{i\alpha})^2} - (\frac{x^2+y^2}{2})\frac{1 + (re^{i\alpha})^2}{1 - (re^{i\alpha})^2}\right). \tag{36}$$

We shall examine each term in (36) in turn for $0 < |\alpha| < \pi$. Firstly

$$\frac{1 + (re^{i\alpha})^2}{1 - (re^{i\alpha})^2} = \frac{(1 - r^4) + 2ir^2 \sin 2\alpha}{(1 + r^4) - 2r^2 \cos 2\alpha},$$

and it is straightforward to show that

$$Re\left(\frac{1 + (re^{i\alpha})^2}{1 - (re^{i\alpha})^2}\right) > 0 \tag{37}$$

and

$$\lim_{r \to 1^-} \frac{1 + (re^{i\alpha})^2}{1 - (re^{i\alpha})^2} = i \cot \alpha. \tag{38}$$

For the term $\dfrac{re^{i\alpha}}{1 - (re^{i\alpha})^2}$ in (36), we obtain

$$\frac{re^{i\alpha}}{1 - (re^{i\alpha})^2} = \frac{(r - r^3)\cos\alpha + i(r + r^3)\sin\alpha}{1 + r^4 - 2r^2\cos 2\alpha}, \tag{39}$$

and so

$$\lim_{r \to 1^-} \frac{re^{i\alpha}}{1 - (re^{i\alpha})^2} = \frac{2i\sin\alpha}{2(1 - \cos 2\alpha)} = \frac{i}{2\sin\alpha}. \tag{40}$$

Examining now the remaining term in (36) that involves r, we express $1 - (re^{i\alpha})^2$ in polar coordinates as $\rho_r e^{i\theta_r}$, where

$$\rho_r = |1 - r^2 e^{2i\alpha}| \to 2|\sin\alpha| \text{ as } r \to 1^-,$$

and

$$\lim_{r \to 1^-} \theta_r = \tan^{-1}\left(\frac{-\sin 2\alpha}{1 - \cos 2\alpha}\right) = \tan^{-1}(-\cot\alpha)$$

$$= \begin{cases} \alpha - \pi/2, & 0 < \alpha < \pi \\ \alpha + \pi/2, & -\pi < \alpha < 0. \end{cases}$$

Therefore,

$$(\rho_r e^{i\theta_r})^{-1/2} \to \frac{1}{\sqrt{2|\sin\alpha|}} e^{i(\frac{\pi}{4}\hat{\alpha} - \alpha/2)} \text{ as } r \to 1^-, \text{ where } \hat{\alpha} = \mathrm{sgn}(\sin\alpha). \tag{41}$$

Hence, for $0 < |\alpha| < \pi$,

$$\lim_{r \to 1^-} k_{\alpha,r}(x, y) = k_\alpha(x, y),$$

where

$$k_\alpha(x, y) = \frac{1}{\sqrt{2\pi|\sin\alpha|}} \exp\left(i\frac{\pi}{4}\hat{\alpha} - \alpha/2\right) \exp\left(\frac{-i}{2}(x^2 + y^2)\cot\alpha + \frac{ixy}{\sin\alpha}\right). \tag{42}$$

The function k_α is the kernel of the integral operator $\mathcal{F}_\alpha$ defined by (6).

Our next task is to show that the operators $\mathcal{F}_\alpha$ and F_α agree on the space $L^2(\mathbb{R})$ and hence also on S.

THEOREM 5.6 *Let $\mathcal{F}_\alpha$ be defined by (6) and (7), and let F_α be defined by (28). Then as operators on $L^2(\mathbb{R})$, $\mathcal{F}_\alpha = F_\alpha$ for all $\alpha \in \mathbb{R}$.*

Proof Clearly $F_0 = \mathcal{F}_0$, $\mathcal{F}_{\pi/2} = F_{\pi/2} = F$ and $\mathcal{F}_{-\pi/2} = F_{-\pi/2} = F^{-1}$. Also,

$$(F_{\pm\pi}\phi)(x) = \sum_{n=0}^{\infty}(-1)^n(\phi, \psi_n)_2\psi_n(x)$$

and, since

$$\int_{-\infty}^{\infty} \psi_n(x)\phi(-x) \ dx = \begin{cases} (\phi, \psi_n)_2 & \text{if } n \text{ is even} \\ -(\phi, \psi_n)_2 & \text{if } n \text{ is odd} \end{cases}$$
$$= (-1)^n(\phi, \psi_n)_2,$$

we obtain

$$(F_{\pm\pi}\phi)(x) = \phi(-x) = (\mathcal{F}_{\pm\pi}\phi)(x).$$

Now we proceed to the case when $0 < |\alpha| < \pi$, with $\alpha \neq \pm\pi/2$. Let $\phi \in C_0^\infty(\mathbb{R})$. Then

$$(F_{\alpha,r}\phi)(x)$$
$$= \frac{1}{\sqrt{\pi}} \frac{e^{-i\theta_r/2}}{\sqrt{\rho_r}} \int_{-\infty}^{\infty} \exp\left(\frac{2xyre^{i\alpha}}{1 - (re^{i\alpha})^2} - \left(\frac{x^2 + y^2}{2}\right)\frac{1 + (re^{i\alpha})^2}{1 - (re^{i\alpha})^2} \right) \phi(y)dy, \quad (43)$$

where $\theta_r = \arg(1 - r^2 e^{2i\alpha})$ and $\rho_r = |1 - r^2 e^{2i\alpha}|$. From (41),

$$\frac{1}{\sqrt{\pi}} \frac{e^{-i\theta_r/2}}{\sqrt{\rho_r}} \to \frac{1}{\sqrt{2\pi|\sin\alpha|}} e^{i(\frac{\pi}{4}\hat{\alpha} - \alpha/2)}, \text{ as } r \to 1^-. \quad (44)$$

Examining the integrand in (43) with $x \in \mathbb{R}$ fixed, we have, from (39),

$$\left| \exp\left(\frac{2xyre^{i\alpha}}{1 - (re^{i\alpha})^2} \right) \right| = \exp\left(\frac{2xy(r - r^3)\cos\alpha}{1 + r^4 - 2r^2\cos^2 2\alpha} \right).$$

Clearly,

$$2xy(r - r^3)\cos\alpha \leq 2|x||y||r - r^3||\cos\alpha| \leq 2|x||y|, \forall r \in (0, 1].$$

Moreover,

$$1 + r^4 - 2r^2\cos^2 2\alpha = (r^2 - \cos 2\alpha)^2 + \sin^2 2\alpha \geq \sin^2 2\alpha, \quad \forall r.$$

Hence,

$$\left| \exp\left(\frac{2xyre^{i\alpha}}{1 - (re^{i\alpha})^2} \right) \right| \leq \exp\left(\frac{2|x||y|}{\sin^2 2\alpha} \right), \quad \forall r \in (0, 1].$$

Also, from (37), we obtain

$$\left| \exp\left(-\left(\frac{x^2 + y^2}{2}\right)\frac{1 + (re^{i\alpha})^2}{1 - (re^{i\alpha})^2} \right) \right| = \exp\left(-\left(\frac{x^2 + y^2}{2}\right) Re \frac{1 + (re^{i\alpha})^2}{1 - (re^{i\alpha})^2} \right) \leq 1.$$

Let supp $\phi \subseteq [-R, R]$. Then

$$\int_{-\infty}^{\infty} \left| \exp\left(\frac{2xyre^{i\alpha}}{1 - (re^{i\alpha})^2} \right) \exp\left(-\left(\frac{x^2 + y^2}{2}\right)\frac{1 + (re^{i\alpha})^2}{1 - (re^{i\alpha})^2} \right) \right| |\phi(y)|dy$$
$$\leq \exp\left(\frac{2|x|R}{\sin^2 2\alpha} \right) \int_{-R}^{R} |\phi(y)|dy \ \forall \ r \in (0, 1].$$

Consequently, we can take the limit inside the integral to obtain

$$(F_\alpha \phi)(x) = \lim_{r \to 1^-} (F_{\alpha,r}\phi)(x)$$

$$= \frac{1}{\sqrt{2\pi|\sin\alpha|}} e^{i(\frac{\pi}{4}\hat{\alpha}-\alpha/2)} \int_{-\infty}^{\infty} e^{\frac{ixy}{\sin\alpha}} e^{-i\frac{x^2+y^2}{2}\cot\alpha} \phi(y) \; dy. \qquad (45)$$

It follows from Lemma 3.6 that for each $\phi \in S = \mathcal{A}$, we have $F_{\alpha,r}\phi \to F_{\alpha,1}\phi = F_\alpha\phi$ in S as $r \to 1^-$. Hence $(F_{\alpha,r}\phi)(x) \to (F_\alpha\phi)(x)$ uniformly with respect to x on compact subsets of $\mathbb{R}$. But from (45) we have, for each $\phi \in C_0^\infty(\mathbb{R})$, $(F_{\alpha,r}\phi)(x) \to (\mathcal{F}_\alpha\phi)(x)$ as $r \to 1^-$. We know from Theorem 5.1 that F_α is continuous on $L^2(\mathbb{R})$. Also it has been proved in [5] that $\mathcal{F}_\alpha$ is continuous on $L^2(\mathbb{R})$. Therefore, since $C_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$,

$$F_\alpha\phi = \mathcal{F}_\alpha\phi,$$

for all $\phi \in L^2(\mathbb{R})$. ∎

Now we turn to the theory of FrFTs on the space of generalised functions $\mathcal{A}'$, which, in view of Lemma 5.2, can be identified in this particular case with the space S' of tempered distributions. We recall that the generalised Fourier transformation $\widetilde{F}$ is defined as a homeomorphism on S' by

$$< \widetilde{F}f, \phi > := < f, F\phi > \quad \text{for } f \in S', \phi \in S. \qquad (46)$$

From (25), we obtain the following definition for the generalised FrFT $\widetilde{F}_\alpha$.

DEFINITION 5.7 *The generalised fractional Fourier transform of order α of $f \in S'$ is given by*

$$(\widetilde{F}_\alpha f, \phi) := (f, F_{-\alpha}\phi), \quad \phi \in S. \qquad (47)$$

Since

$$< \widetilde{F}_\alpha f, \phi > = (\widetilde{F}_\alpha f, \overline{\phi})$$

$$= (f, \overline{F_{-\alpha}\overline{\phi}}) = < f, F_\alpha\phi >,$$

this definition is consistent with (46), which corresponds to the case $\alpha = \pi/2$. Note also that

$$\widetilde{F_\alpha}\widetilde{\eta} = \widetilde{F_\alpha\eta} \quad \forall\eta \in L^2(I). \qquad (48)$$

THEOREM 5.8 *Let $\{\widetilde{F_\alpha}\}_{\alpha\in\mathbb{R}}$ be the family of operators defined by (47). Then for each $\alpha \in \mathbb{R}$, $\widetilde{F_\alpha}$ is a homeomorphism on S' with inverse $\widetilde{F_{-\alpha}}$. Moreover,*

$$\widetilde{F_\alpha}f = \sum_n e^{in\alpha}(f, \psi_n)\widetilde{\psi_n},$$

where the series converges in S'.

Proof See Theorem 4.2 and (24). ∎

THEOREM 5.9 *For each $f \in S'$ and $\alpha, \beta \in \mathbb{R}$,*

(i) $\widetilde{F}_\alpha \widetilde{F}_\beta f = \widetilde{F}_{\alpha+\beta} f$

(ii) $\widetilde{F}_\alpha f \to \widetilde{F}_\beta f$ *in* S' *as* $\alpha \to \beta$.

Proof This follows from Definition 5.7 and Theorem 5.3. ■

THEOREM 5.10 *The family of operators $\{\widetilde{F}_\alpha\}_{\alpha \in \mathbb{R}}$, defined on S' by (47), is a weak*-continuous group of linear operators on S'. Moreover the infinitesimal generator of $\{\widetilde{F}_\alpha\}_{\alpha \in \mathbb{R}}$ is $(-iT)' = \widetilde{iT}$, where $\widetilde{T}$ is defined via (5), (20) and Definition 4.3.*

Proof See Theorem 4.5. ■

6. Conclusion

We have used a systematic procedure for defining a unitary group of fractional transformations on $L^2(I)$ with infinitesimal generator $i\mathcal{T}$ and a corresponding equicontinuous group on the space of test functions $\mathcal{A}$ that was constructed around a symmetric restriction T of $\mathcal{T}$. This led to a weak*-continuous group of fractional transforms defined on the space $\mathcal{A}'$ of generalised functions. The generator of this group is the extended operator $\widetilde{iT} = \widetilde{i\mathcal{T}} \in L(\mathcal{A}')$. As a special case of our theory, we obtained the distributional theory of the fractional Fourier transform that was developed, via a different approach, in [6]. Note that the abstract Cauchy problem associated with $\widetilde{iT}$ is

$$\frac{du(t)}{dt} = \widetilde{iT}u(t), \quad u(0) = f \in \mathcal{A}'. \tag{49}$$

The theory presented above shows that the unique solution to (49) is

$$u(t) = \widetilde{G_t}f \in \mathcal{A}' \quad \forall f \in \mathcal{A}',$$

where we work with the weak*-topology in $\mathcal{A}'$. Full details will be given in a future paper.

The constructive and general nature of our approach immediately suggests that it could be used to produce distributional theories for other fractional transformations. For example, we have obtained similar results for the Hankel transformation which will be discussed elsewhere.

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