

## Strathprints Institutional Repository

Grimble, M.J. and Majecki, P.M. (2010) *Polynomial approach to nonlinear predictive generalized minimum variance control*. IET Control Theory and Applications, 4 (3). pp. 411-424. ISSN 1751-8644

Strathprints is designed to allow users to access the research output of the University of Strathclyde. Copyright © and Moral Rights for the papers on this site are retained by the individual authors and/or other copyright owners. You may not engage in further distribution of the material for any profitmaking activities or any commercial gain. You may freely distribute both the url (<http://strathprints.strath.ac.uk/>) and the content of this paper for research or study, educational, or not-for-profit purposes without prior permission or charge.

Any correspondence concerning this service should be sent to Strathprints administrator: <mailto:strathprints@strath.ac.uk>

# Polynomial Approach to Nonlinear Predictive Generalized Minimum Variance Control

Mike J Grimble and Pawel Majecki

Industrial Control Centre  
University of Strathclyde  
Graham Hills Building  
50 George Street  
GLASGOW G1 1QE  
United Kingdom

Telephone N<sup>o</sup>: +44 (0)141 548 2378

Facsimile N<sup>o</sup>: +44 (0)141 548 4203

Electronic Mail: [m.grimble@eee.strath.ac.uk](mailto:m.grimble@eee.strath.ac.uk)

## Abstract

A relatively simple approach to *Nonlinear Predictive Generalized Minimum Variance (NPGMV)* control is introduced for *nonlinear discrete-time multivariable systems*. The system is represented by a combination of a stable nonlinear subsystem where no structure is assumed and a linear subsystem that may be unstable and modelled in polynomial matrix form. The multi-step predictive control cost index to be minimised involves both weighted error and control signal costing terms. The *NPGMV* control law involves an assumption on the choice of cost-function weights to ensure the existence of a stable nonlinear closed-loop operator. A valuable feature of the control law is that in the asymptotic case, where the plant is linear, the

controller reduces to a polynomial matrix version of the well known *GPC* controller. In the limiting case when the plant is nonlinear and the cost-function is single step the controller becomes equal to the polynomial matrix version of the so called *Nonlinear Generalised Minimum Variance* controller. The controller can be implemented in a form related to a nonlinear version of the *Smith Predictor* but unlike this compensator a stabilizing control law can be obtained for open-loop unstable processes.

**Keywords:** polynomial systems, optimal, predictive, nonlinear, minimum variance, transport delays.

## 1 Introduction

The aim is to introduce a relatively simple controller for nonlinear systems and one that has some of the advantages of the popular polynomial based *Generalised Predictive Control (GPC)* algorithms. It is well known that nonlinear (*NL*) systems have more complex behaviour than linear systems, including limit cycle responses and chaotic behaviour. The proposed controller does not rely on local linearization and it provides a global optimal control solution.

The linear *model based predictive control (MBPC)* approach has been applied very successfully in the process industries, where it has improved the profitability and competitiveness of production plants. It has been used to improve performance in difficult systems which contain long dead times, time-varying system parameters and multivariable interactions. The approach is often thought to be easy to understand relative to other modern control design methods. The predictive control algorithms were first applied in slow processes for the chemical, petrochemical, food and cement industries, but are now used on faster applications, such as servo, hydraulic systems and gas turbine applications. A predictive controller uses future reference signal information and minimises a multi-step cost-function. The most popular algorithms are *Dynamic Matrix Control (DMC)* [1], *Generalized Predictive Control (GPC)* [2,3], and the algorithms of *Richalet* [4,5]. The relationship between *LQ* optimal and *predictive control* was explored in [6].

The *GPC* controller was originally obtained in a polynomial system form. The control strategy developed here also builds upon previous results on *Generalised Minimum Variance (GMV)* control. A *Nonlinear Generalized Minimum Variance (NGMV)* controller was derived recently for nonlinear *model*

based multivariable systems. The assumption was made that the plant model could be decomposed into a set of delay terms, a very general nonlinear subsystem that had to be stable and a linear subsystem that could be represented in polynomial matrix or state equation form and include unstable modes. This problem was analysed in [8-10]. The major development over the basic *NGMV* control law in [9] involves an extension of the *NGMV* cost-index to include future tracking error and control costing terms in a *GPC* type of problem where the linear sub-system of the plant model is represented in polynomial matrix equation form. When the system is linear the controller is equivalent to a *GPC* controller that is a practical solution for many applications.

There is of course a rich history of relatively recent research on nonlinear predictive control [11-24], but the proposed approach is somewhat different, since it is closer in spirit to that of fixed model based design than an online optimization algorithm. An advantage of the new predictive control approach is that the plant model can be in a very general nonlinear operator form, which might involve hard nonlinearities, a state-dependent state-space model, transfer operators or even nonlinear function look up tables. The plant can include both linear and nonlinear subsystems and no structure needs to be known for the nonlinear block but this must be assumed open-loop stable in an appropriate sense. To guarantee closed loop stability the assumption is made that a certain nonlinear operator has a stable inverse.

It is well known for linear systems that stability for this type of control law is ensured when the combination of a control weighting function and an error weighted plant model is strictly minimum phase [25]. For nonlinear systems a related operator is required to have a stable inverse. This implies that the cost-function weightings must be chosen to satisfy both performance and stability/robustness requirements.

The plan for this paper is as follows. The nonlinear plant and linear disturbance models in polynomial matrix form are described in § 2. It is shown in § 3 that the solution of the linear multi-step predictive (*GPC*) control problem can be found from the solution of an equivalent minimum variance control problem. The cost function and the solution of the *NPGMV* nonlinear optimal control problem are described in § 4, together with the main theorem. A design example is presented in § 5 and finally conclusions are summarised in § 6.



## 2 System Description

The assumed model for the plant can be severely nonlinear and dynamic and may have a very general form but the disturbance model is chosen to be linear so that relatively simple results are obtained. This is not restrictive, since in many applications the model for the disturbance signal is only an *LTI* approximation. The system shown in Fig. 1 includes the *nonlinear* (*NL*) plant model together with the linear reference, measurement noise and disturbance signals. The signals  $v(t)$  and  $\xi(t)$  are vector zero-mean, independent, white noise signals. The *measurement noise* signal  $v(t)$  is assumed to have a constant covariance matrix  $R_f = R_f^T \geq 0$ . There is no loss of generality in assuming that the *disturbance* white noise source  $\xi(t)$  has an identity covariance matrix. There is also no requirement to specify the distribution of the noise source, since the structure of the system leads to a prediction equation, which is only dependent upon the *linear* stochastic disturbance model. The plant model can have a very general nonlinear operator form, which might involve hard nonlinearities, a state-dependent state-space model, transfer operators or even nonlinear-function look up tables. Detailed knowledge of the *NL* system structure is not required.

$$\textbf{Nonlinear Plant:} \quad (\mathcal{W}_l u)(t) = z^{-k} (\mathcal{W}_{lk} u)(t) \quad (1)$$

where  $z^{-k}I$  denotes a diagonal matrix of the common delay elements in the output signal paths. The output of the non-linear subsystem  $\mathcal{W}_{lk}$ , that might represent actuators, will be denoted  $u_0(t) = (\mathcal{W}_{lk} u)(t)$ .

For simplicity the NL subsystem:  $\mathcal{W}_l$  is assumed to be *finite gain stable* but the linear subsystem  $W_0 = z^{-k}W_{0k}$ , introduced below, can contain any unstable modes. If the decomposition into a nonlinear and a linear sub-system is not relevant then let the linear sub-system  $W_{0k} = I$ . The generalisation to different delays in different paths is straightforward [26]. The vectors of signals in the system may be listed as follows:  $u_0(t) \in R^{m_0}$  (input to linear subsystem);  $u(t) \in R^m$  (control signal);  $y(t) \in R^r$  (plant output);  $z(t) \in R^r$  (observations);  $r(t) \in R^r$  (set-point / reference);  $y_p(t) \in R^m$  (weighted output);  $r_p(t) \in R^r$  (weighted set-point or reference).

## 2.1 Linear Subsystem Polynomial Matrix Models

The polynomial matrix system models, for the linear part of the  $(r \times m)$  multivariable system may now be introduced. The sub-systems to be defined are associated with any linear sub-system  $W_0$  in the plant model and the linear disturbance model. The *Controlled Auto-Regressive Moving Average (CARMA)* model, representing the linear subsystem of the plant is defined as:

$$A(z^{-1})y(t) = B_{0k}(z^{-1})u_0(t-k) + C_d(z^{-1})\xi(t) \quad (2)$$

where  $\xi(t)$  and the input signal channels in the plant model are assumed to include a  $k$ -steps ( $k \geq 0$ ) transport delay and  $B_0(z^{-1}) = B_{0k}(z^{-1})z^{-k}$ . The *delay free* plant transfer of the linear sub-system and the *disturbance model* may therefore be defined, in the left coprime form:

$$[W_{0k}(z^{-1}) \ W_d(z^{-1})] = A(z^{-1})^{-1} [B_{0k}(z^{-1}) \ C_d(z^{-1})] \quad (3)$$

Introduce a *stable cost-function weighting model* in left coprime form  $P_c(z^{-1}) = P_{cd}^{-1}(z^{-1})^{-1} P_{cn}(z^{-1})$ . The weighted output may be written as:

$$y_p(t) = P_c(z^{-1})y(t) = P_c(z^{-1})A(z^{-1})^{-1}(B_{0k}(z^{-1})u_0(t-k) + C_d(z^{-1})\xi(t)) \quad (4)$$

The *power spectrum* for the combined disturbance and noise signal  $f = d + v = W_d\xi + v$  can be computed, noting these are linear subsystems, using  $\Phi_{ff} = \Phi_{dd} + \Phi_{vv} = W_d W_d^* + R_f$ , where the notation for the adjoint of  $W_d$  implies  $W_d^*(z^{-1}) = W_d^T(z)$  and only in this case  $z$  represents the  $z$ -domain complex number. The *generalized spectral-factor*  $Y_f$  may be computed from this spectrum as  $Y_f Y_f^* = \Phi_{ff}$ , where  $Y_f = A^{-1}D_f$ . The system models are assumed to be such that  $D_f$  is a *strictly Schur* polynomial matrix [17,18]:

$$D_f D_f^* = C_d C_d^* + A R_f A^* \quad (5)$$

The model for the disturbance signal is linear, which is an assumption that does not affect stability properties but may cause a degree of sub-optimality in the disturbance rejection properties.

**Innovations signal:** Disturbance models are often approximated in real applications by linear systems driven by white noise. It is well known that the signal  $f = d + v$  may be modelled in innovations signal form as  $f(t) = Y_f \varepsilon(t)$ , where  $Y_f = A^{-1} D_f$  is defined via the spectral-factorisation (5) and  $\varepsilon(t)$  denotes a white noise signal of zero-mean and identity covariance matrix [8,21]. The system description may be assumed to be such that  $D_f$  is *strictly Schur*. The *observations signal* may therefore be written, using (2)

as:

$$\begin{aligned} z(t) = y(t) + v(t) &= A^{-1}(z^{-1})B_{0k}(z^{-1})u_0(t-k) + A^{-1}(z^{-1})C_d(z^{-1})\xi(t) + v(t) \\ &= A^{-1}(z^{-1})B_{0k}(z^{-1})u_0(t-k) + Y_f(z^{-1})\varepsilon(t) \end{aligned} \quad (6)$$

Define the right coprime model for the *weighted spectral factor*:

$$P_c(z^{-1})Y_f(z^{-1}) = D_{fp}(z^{-1})A_f^{-1}(z^{-1}) \quad (7)$$

Then the *weighted observations signal*  $z_p(t) = P_c(z^{-1})z(t)$  may be written as:

$$z_p(t) = P_c(z^{-1})W_{0k}(z^{-1})u_0(t-k) + D_{fp}(z^{-1})A_f^{-1}(z^{-1})\varepsilon(t) \quad (8)$$

## 2.2 Optimal Linear Prediction

The solution of the optimal control problem requires the introduction of a least squares predictor. This enables the inferred output  $y$  at times  $t+k+1, t+k+2, \dots$  to be calculated (assuming that the disturbance at future times is null). The cost-function to be minimised, which defines the least-squares predictor, is given as:

$$J = E\{\tilde{y}_p(t+j|t)^2\} \quad (9)$$

where the estimation error:

$$\tilde{y}_p(t+j|t) = y_p(t+j) - \hat{y}_p(t+j|t) \quad (10)$$

and  $\hat{y}_p(t+j|t)$  defines the predicted value of  $y_p(t)$  at a time  $j$  steps ahead. To generate the prediction algorithm the following *Diophantine equation* must be solved for the solution  $(E_j, H_j)$ , with  $E_j$  of smallest degree ( $\deg(E_j(z^{-1})) < j+k$ ):

**First Diophantine:**  $E_j(z^{-1})A_f(z^{-1}) + z^{-j-k}H_j(z^{-1}) = D_{fp}(z^{-1})$  (11)

This equation may be written as:

$$E_j(z^{-1}) + z^{-j-k}H_j(z^{-1})A_f^{-1}(z^{-1}) = D_{fp}(z^{-1})A_f^{-1}(z^{-1}) \quad (12)$$

**Prediction equation:** Substituting from (11) the expression for the weighted observations signal (8):

$$\begin{aligned} z_p(t) &= P_c(z^{-1})W_{0k}(z^{-1})u_0(t-k) + D_{fp}(z^{-1})A_f^{-1}(z^{-1})\varepsilon(t) \\ &= P_c(z^{-1})W_{0k}(z^{-1})u_0(t-k) + (E_j(z^{-1}) + z^{-j-k}H_j(z^{-1})A_f^{-1}(z^{-1}))\varepsilon(t) \end{aligned}$$

Substituting from the innovations (6)  $\varepsilon(t) = Y_f^{-1}z(t) - D_f^{-1}B_{0k}u_0(t-k)$  obtain:

$$\begin{aligned} z_p(t) &= P_c(z^{-1})W_{0k}(z^{-1})u_0(t-k) + E_j(z^{-1})\varepsilon(t) \\ &+ z^{-j-k}H_j(z^{-1})A_f^{-1}(z^{-1})(Y_f^{-1}z(t) - D_f^{-1}(z^{-1})B_{0k}(z^{-1})u_0(t-k)) \end{aligned}$$

Recall  $A_f^{-1}Y_f^{-1} = D_{fp}^{-1}P_c$  and substituting, the *weighted observations*:

$$\begin{aligned} z_p(t) &= E_j(z^{-1})\varepsilon(t) + z^{-j-k}H_j(z^{-1})D_{fp}^{-1}(z^{-1})P_c(z^{-1})z(t) \\ &+ (P_c(z^{-1})A^{-1}(z^{-1})B_{0k}(z^{-1}) - z^{-j-k}H_j(z^{-1})A_f^{-1}(z^{-1})D_f^{-1}(z^{-1})B_{0k}(z^{-1}))u_0(t-k) \end{aligned}$$

**Weighted Output:** To obtain the expression for the weighted output  $z_p(t) = P_c z(t) = y_p(t) + v_p(t)$ ,

where  $y_p(t) = P_c y(t)$  and  $v_p(t) = P_c v(t)$ :

$$\begin{aligned} z_p(t) &= E_j(z^{-1})\varepsilon(t) + z^{-j-k}H_j(z^{-1})D_{fp}^{-1}(z^{-1})z_p(t) \\ &+ (P_c(z^{-1})Y_f(z^{-1})A_f(z^{-1}) - z^{-j-k}H_j(z^{-1}))A_f^{-1}(z^{-1})D_f^{-1}(z^{-1})B_{0k}(z^{-1})u_0(t-k) \end{aligned} \quad (13)$$

but from (7)  $P_c Y_f A_f = D_{fp}$  and from (12) and (13):

$$\begin{aligned} y_p(t) &= E_j(z^{-1})\varepsilon(t) - v_p(t) + z^{-j-k}H_j(z^{-1})D_{fp}^{-1}(z^{-1})z_p(t) \\ &+ (D_{fp}(z^{-1}) - z^{-j-k}H_j(z^{-1}))A_f^{-1}(z^{-1})D_f^{-1}(z^{-1})B_{0k}(z^{-1})u_0(t-k) \end{aligned}$$

**Future Values of Weighted Output:** Using (11), the  $j+k$  steps ahead weighted output signal:

$$y_p(t+j+k) = E_j(z^{-1})\varepsilon(t+j+k) - v_p(t+j+k) + H_j(z^{-1})D_{fp}^{-1}(z^{-1})z_p(t)$$



$$+E_j(z^{-1})D_f^{-1}(z^{-1})B_{0k}(z^{-1})u_0(t+j) \quad (14)$$

To further simplify the equations (recalling  $D_f^{-1}$  is assumed to be stable), the right coprime model:

$$B_{1k}(z^{-1})D_{f1}^{-1}(z^{-1}) = D_f^{-1}(z^{-1})B_{0k}(z^{-1}) \quad (15)$$

Also let the signal  $u_f(t) = D_{f1}^{-1}(z^{-1})u_0(t)$ , then (14) may be written:

$$\begin{aligned} y_p(t+j+k) = & \left( E_j(z^{-1})\varepsilon(t+j+k) - v_p(t+j+k) \right) \\ & + \left[ H_j(z^{-1})D_{fp}^{-1}(z^{-1})z_p(t) + E_j(z^{-1})B_{1k}(z^{-1})u_f(t+j) \right] \end{aligned} \quad (16)$$

Note that the maximum degree of the polynomial matrix  $E_j$  is  $j+k-1$  and hence the noise components in  $E_j\varepsilon(t+j+k)$  includes  $\varepsilon(t+j+k), \dots, \varepsilon(t+1)$ , which are at future times.

### 2.3 The Prediction Equations

The optimal predictor at time  $t+j+k$ , given observations up to time  $t$ , can now be derived. Consider first the case where the noise  $\{v(t)\}$  is zero. The observations, up to time  $t$  are known and the future values of the control inputs  $\{u_0(t), \dots, u_0(t+j)\}$ , used in the predictor, are computed at time  $t$ , and hence the future control input is independent of the future disturbance and noise sequence. It follows that the expected value of the square  $[\cdot]$  and round  $(\cdot)$  bracketed terms in equation (16) must be zero. The predictor to minimise the cost (9), given that the cross terms in the cost are null, follows, from (16):

$$\hat{y}_p(t+j+k|t) = \left[ H_j(z^{-1})D_{fp}^{-1}(z^{-1})z_p(t) + E_j(z^{-1})B_{1k}(z^{-1})u_f(t+j) \right] \quad (17)$$

If the measurement noise signal is non-zero then the weighted noise term  $v_p(t+j+k) = P_{c0}(z^{-1})v(t+j+k)$ . If the weighting  $P_c(z^{-1})$  is a constant, which is usual in *GPC* control, or if it is assumed a polynomial matrix of degree  $j+k-1$ , then  $v_p(t+j+k)$  is only dependent on future white measurement noise and the expected value of such a term and the square bracketed terms in (16) must be zero. The optimal predictor is therefore again given by (17) and the *prediction error*:

$$\tilde{y}_p(t+j+k|t) = (E_j(z^{-1})\varepsilon(t+j+k) - v_p(t+j+k)) \quad (18)$$

A *second Diophantine equation* may now be introduced to break up the term  $E_j(z^{-1})B_{1k}(z^{-1})$  into a part with a  $j+1$  step delay and a part depending on  $D_{f1}(z^{-1})$  (recall  $u_f(t) = D_{f1}^{-1}(z^{-1})u_0(t)$ ). Thus, for  $j \geq 0$ , introduce the following Diophantine equation, with  $(G_j, S_j)$ , of *smallest degree* for  $G_j$ :

$$\textbf{Second Diophantine:} \quad G_j(z^{-1})D_{f1}(z^{-1}) + z^{-j-1}S_j(z^{-1}) = E_j(z^{-1})B_{1k}(z^{-1}) \quad (19)$$

where  $\deg(G_j(z^{-1})) = j$ . The prediction, from equation (17), may now be obtained (for  $j \geq 0$ ) as:

$$\hat{y}_p(t+j+k|t) = H_j(z^{-1})D_{fp}^{-1}(z^{-1})z_p(t) + G_j(z^{-1})u_0(t+j) + S_j(z^{-1})u_f(t-1) \quad (20)$$

The degree of  $G_j(z^{-1})$  is  $j$  and the second term in (20) therefore involves the inputs which are in the future.

Define the signal  $f_j(t)$ , in terms of past outputs and inputs, as:

$$f_j(t) = H_j(z^{-1})D_{fp}^{-1}(z^{-1})z_p(t) + S_j(z^{-1})u_f(t-1) \quad (21)$$

Thus, the *predicted weighted output* (20) may be written, for  $j \geq 0$ , as:

$$\hat{y}_p(t+j+k|t) = G_j(z^{-1})u_0(t+j) + f_j(t) \quad (22)$$

**Coefficients of the Polynomial matrix  $G_j(z^{-1})$ .** From equations (11) and (19):

$$G_j(z^{-1})D_{f1}(z^{-1}) + z^{-j-1}S_j(z^{-1}) = E_j(z^{-1})B_{1k}(z^{-1})$$

$$E_j(z^{-1})B_{1k}(z^{-1}) + z^{-j-k}H_j(z^{-1})A_f(z^{-1})^{-1}B_{1k}(z^{-1}) = D_{fp}(z^{-1})A_f(z^{-1})^{-1}B_{1k}(z^{-1})$$

From these and from (7)  $P_c Y_f = D_{fp} A_f^{-1}$  and from (15)  $D_f B_{1k} = B_{0k} D_{f1}$ :

$$G_j D_{f1} = P_c Y_f B_{1k} - z^{-j-k} H_j A_f^{-1} B_{1k} - z^{-j-1} S_j$$

$$G_j = P_c A^{-1} B_{0k} - (z^{-j-k} H_j A_f^{-1} B_{1k} + z^{-j-1} S_j) D_{f1}^{-1}$$

The  $G_j(z^{-1})$  therefore includes the first  $j+1$  Markov parameters  $g_j$  of the weighted plant  $G_j = P_c W_{0k}$ .

Thus,  $\deg(G_j(z^{-1})) = j$  and  $G_j(z^{-1}) = g_0 + g_1 z^{-1} + \dots + g_j z^{-j}$ , where  $G_0(z^{-1}) = g_0$ .

## 2.4 Vector/Matrix Prediction Equations

The future weighted outputs are to be predicted for inputs computed in the interval  $\tau \in [t, t + N]$

where  $N \geq 0$ . Equation (22) may therefore be used to obtain:

$$\begin{bmatrix} \hat{y}_p(t+k|t) \\ \hat{y}_p(t+1+k|t) \\ \vdots \\ \hat{y}_p(t+N+k|t) \end{bmatrix} = \begin{bmatrix} g_0 & 0 & \dots & 0 & 0 \\ g_1 & g_0 & 0 & \dots & 0 \\ \vdots & g_1 & g_0 & \ddots & \vdots \\ g_{N-1} & \vdots & \ddots & \ddots & \\ g_N & g_{N-1} & \dots & g_1 & g_0 \end{bmatrix} \begin{bmatrix} u_0(t) \\ u_0(t+1) \\ \vdots \\ u_0(t+N) \end{bmatrix} + \begin{bmatrix} f_0(t) \\ f_1(t) \\ \vdots \\ f_N(t) \end{bmatrix} \quad (23)$$

The vector form of the predicted weighted outputs:

$$\hat{Y}_{t+k,N} = G_N U_{t,N}^0 + F_{t,N} \quad (24)$$

Using (21), the free response predictions  $F_{t,N}$ :

$$\begin{aligned} F_{t,N} &= \begin{bmatrix} f_0(t) \\ f_1(t) \\ \vdots \\ f_N(t) \end{bmatrix} = \begin{bmatrix} H_0(z^{-1}) \\ H_1(z^{-1}) \\ \vdots \\ H_N(z^{-1}) \end{bmatrix} D_{\hat{p}}^{-1}(z^{-1}) z_p(t) + \begin{bmatrix} S_0(z^{-1}) \\ S_1(z^{-1}) \\ \vdots \\ S_N(z^{-1}) \end{bmatrix} u_f(t-1) \\ &= H_{NZ}(z^{-1}) z_p(t) + S_{NZ}(z^{-1}) u_f(t-1) \end{aligned} \quad (25)$$

The functions  $H_{NZ}(z^{-1})$  and  $S_{NZ}(z^{-1})$  are defined in an obvious way from (25). The prediction error

$E_j(z^{-1})\epsilon(t+j+k) - v_p(t+j+k)$  may be written, recalling  $\deg(E_j(z^{-1})) < j+k$ , as:

$$\tilde{Y}_{t+k,N} = \begin{bmatrix} e_0 \epsilon(t+k) + \dots + e_{k-1} \epsilon(t+1) - v_p(t+k) \\ e_0 \epsilon(t+1+k) + \dots + e_k \epsilon(t+1) - v_p(t+1+k) \\ \vdots \\ e_0 \epsilon(t+N+k) + \dots + e_{N+k-1} \epsilon(t+1) - v_p(t+N+k) \end{bmatrix} \quad (26)$$

**Future set point knowledge:** The future variations of the reference signal  $r(t)$  are assumed known over  $N$  steps and the weighted reference  $r_p(t) = P_c(z^{-1})r(t)$ . The vectors of *future weighted* signals:

$$R_{t,N} = \begin{bmatrix} r_p(t) \\ r_p(t+1) \\ \vdots \\ r_p(t+N) \end{bmatrix}, \quad Y_{t,N} = \begin{bmatrix} y_p(t) \\ y_p(t+1) \\ \vdots \\ y_p(t+N) \end{bmatrix}, \quad U_{t,N}^0 = \begin{bmatrix} u_0(t) \\ u_0(t+1) \\ \vdots \\ u_0(t+N) \end{bmatrix} \quad (27)$$

The  $k$  steps-ahead future weighted outputs can be written in vector terms  $Y_{t+k,N} = \hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N}$  and the *future tracking error*, that includes a dynamic error weighting, may therefore be written as:

$$E_{t+k,N} = R_{t+k,N} - Y_{t+k,N} = R_{t+k,N} - (\hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N}) \quad (28)$$

The vector of predicted signals  $\hat{Y}_{t+k,N}$  in (28) and the prediction error  $\tilde{Y}_{t,N}$  are orthogonal.

### 3 Main Features of Generalised Predictive Control

A review of the derivation of the *GPC* controller is provided below where the input will be taken to be that for the linear sub-system ( $u_0$ ), since it provides results that are needed for the definition of the *NL* problem of interest. The *GPC performance index*, to be minimised:

$$J = E \left\{ \sum_{j=0}^N e_p(t+j+k)^T e_p(t+j+k) + \lambda_j^2 u_0(t+j)^T u_0(t+j) \right\} \Big| t \quad (29)$$

where  $E\{.\mid t\}$  denotes the conditional expectation, conditioned on measurements up to time  $t$ ;  $\lambda_j$  denotes a scalar control signal weighting and the vector of *future weighted* error signal values  $e_p(t+j+k) = P_c(z^{-1})(r(t+j+k) - y(t+j+k))$ . The future optimal control is to be calculated for the interval  $\tau \in [t, t+N]$  and the *GPC cost-function*:

$$J = E\{J_t\} = E\{(R_{t+k,N} - Y_{t+k,N})^T (R_{t+k,N} - Y_{t+k,N}) + U_{t,N}^{0T} \Lambda_N^2 U_{t,N}^0 \mid t\} \quad (30)$$

Introducing the optimal predictor, using (28) and (30), obtain,



$$J = E\{(R_{t+k,N} - (\hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N}))^T (R_{t+k,N} - (\hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N})) + U_{t,N}^{0T} \Lambda_N^2 U_{t,N}^0 \mid t\} \quad (31)$$

where the cost weightings on the future inputs  $u_0$  are written  $\Lambda_N^2 = \text{diag}\{\lambda_0^2, \lambda_1^2, \dots, \lambda_N^2\}$ .

### 3.1 GPC Optimal Control Solution

The terms in the performance criterion can be simplified by noting the prediction errors in  $\tilde{Y}_{t+k,N}$  depends on future values of the signal  $\varepsilon(t)$ , which are assumed to be independent of future controls. The estimate  $\hat{Y}_{t+k,N}$  is therefore orthogonal to the estimation error  $\tilde{Y}_{t+k,N}$  and  $R_{t+k,N}$  is assumed to be known over the  $(N+1)$  steps. The cost may therefore be obtained as:

$$J = (R_{t+k,N} - \hat{Y}_{t+k,N})^T (R_{t+k,N} - \hat{Y}_{t+k,N}) + U_{t,N}^{0T} \Lambda_N^2 U_{t,N}^0 + J_0 \quad (32)$$

where  $J_0 = E\{\tilde{Y}_{t+k,N}^T \tilde{Y}_{t+k,N} \mid t\}$  is independent of the control action. Substituting (24) into (32) obtain:

$$J = (R_{t+k,N} - (G_N U_{t,N}^0 + F_{t,N}))^T (R_{t+k,N} - (G_N U_{t,N}^0 + F_{t,N})) + U_{t,N}^{0T} \Lambda_N^2 U_{t,N}^0 + J_0$$

Thus, define:

$$\tilde{R}_{t+k,N} = R_{t+k,N} - F_{t,N} \quad (33)$$

and

$$\begin{aligned} J &= (\tilde{R}_{t+k,N} - G_N U_{t,N}^0)^T (\tilde{R}_{t+k,N} - G_N U_{t,N}^0) + U_{t,N}^{0T} \Lambda_N^2 U_{t,N}^0 + J_0 \\ &= \tilde{R}_{t+k,N}^T \tilde{R}_{t+k,N} - U_{t,N}^{0T} G_N^T \tilde{R}_{t+k,N} - \tilde{R}_{t+k,N}^T G_N U_{t,N}^0 + U_{t,N}^{0T} (G_N^T G_N + \Lambda_N^2) U_{t,N}^0 + J_0 \end{aligned} \quad (34)$$

To minimise this conditional cost term the gradient of the cost must be set to zero to obtain the vector of future controls. Note the  $J_0$  term is independent of the control action and a perturbation and gradient calculation may be applied [27] to obtain the vector of *GPC future optimal controls* as:

$$U_{t,N}^0 = (G_N^T G_N + \Lambda_N^2)^{-1} G_N^T (\tilde{R}_{t+k,N} - F_{t,N}) \quad (35)$$

The *GPC* optimal control signal at time  $t$  is based on the *receding horizon* principle [27] and the optimal control is taken as the first element in the vector of future controls  $U_{t,N}^0$ .

### 3.2 Equivalent GPC Cost Minimisation Problem

The above is equivalent to a special cost minimisation problem which is needed to motivate the *NPGMV* problem introduced later. Let the constant matrix  $X_N = G_N^T G_N + \Lambda_N^2$  be factorised as:

$$Y^T Y = X_N = G_N^T G_N + \Lambda_N^2 \quad (36)$$

Completing the squares in (34) the cost:

$$\begin{aligned} J &= \tilde{R}_{t+k,N}^T \tilde{R}_{t+k,N} - U_{t,N}^{0T} G_N^T \tilde{R}_{t+k,N} - \tilde{R}_{t+k,N}^T G_N U_{t,N}^0 + U_{t,N}^{0T} Y^T Y U_{t,N}^0 + J_0 \\ &= \left( \tilde{R}_{t+k,N}^T G_N Y^{-1} - U_{t+k,N}^{0T} Y^T \right) \left( Y^{-T} G_N^T \tilde{R}_{t+k,N} - Y U_{t,N}^0 \right) + \tilde{R}_{t+k,N}^T (I - G_N Y^{-1} Y^{-T} G_N^T) \tilde{R}_{t+k,N} + J_0 \end{aligned}$$

The cost-function:  $J = \hat{\Phi}_{t+k,N}^T \hat{\Phi}_{t+k,N} + J_{10}(t) \quad (37)$

where  $\hat{\Phi}_{t+k,N} = Y^{-T} G_N^T (R_{t+k,N} - F_{t,N}) - Y U_{t,N}^0 \quad (38)$

The terms that are independent of the control action may be written as  $J_{10}(t) = J_0 + J_1(t)$  where

$$J_1(t) = \tilde{R}_{t+k,N}^T (I - G_N Y^{-1} Y^{-T} G_N^T) \tilde{R}_{t+k,N} \quad (39)$$

The last term  $J_{10}(t)$  in equation (37) does not depend upon control action and the optimal control is found by setting the first term to zero, giving the same control as in (35). Thence, the *GPC* controller for the above linear system is the same as the controller to minimise the norm of the signal  $\hat{\Phi}_{t+k,N}$  in (38).

### 3.3 Modified Cost-Index

The above discussion motivates the definition of a new *multi-step minimum variance cost* problem that has the same solution for the optimal controller. Consider a new signal to be minimised of the form:

$$\phi = P_c(z^{-1})(r(t) - y(t)) + F_{c0} u_0 \quad (40)$$

The vector of future values of this signal,

$$\Phi_{t,N} = P_{cN} E_{t,N} + F_{cN}^0 U_{t,N}^0 \quad (41)$$

Also introduce cost weightings, using the original *GMV* weightings, to have the constant matrix form:

$$P_{CN} = G_N^T \quad \text{and} \quad F_{CN}^0 = -\Lambda_N^2. \quad (42)$$

The reason for this choice of cost terms becomes apparent below. Define a *MV multi-step* cost-function as:

$$\tilde{J} = E\{\tilde{J}_t\} = E\{\Phi_{t+k,N}^T \Phi_{t+k,N} \mid t\} \quad (43)$$

Predicting forward  $k$ -steps:

$$\Phi_{t+k,N} = P_{CN}(R_{t+k,N} - Y_{t+k,N}) + F_{CN}^0 U_{t,N}^0 \quad (44)$$

Now consider the signal  $\Phi_{t+k,N}$  and substitute for  $Y_{t+k,N} = \hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N}$ . Then from (44) obtain:

$$\Phi_{t+k,N} = P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) + F_{CN}^0 U_{t,N}^0 - P_{CN} \tilde{Y}_{t+k,N} \quad (45)$$

This expression may be written in terms of an *estimate* and *estimation error* vector as:

$$\Phi_{t+k,N} = \hat{\Phi}_{t+k,N} + \tilde{\Phi}_{t+k,N} \quad (46)$$

The estimated prediction  $\hat{\Phi}_{t+k,N} = P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) + F_{CN}^0 U_{t,N}^0$  and prediction error:

$$\tilde{\Phi}_{t+k,N} = -P_{CN} \tilde{Y}_{t+k,N} \quad (47)$$

**Multi-Step Cost Index:** The performance index (43) may therefore be simplified as:

$$\tilde{J} = E\{\tilde{J}_t\} = E\{\Phi_{t+k,N}^T \Phi_{t+k,N} \mid t\} = E\{(\hat{\Phi}_{t+k,N} + \tilde{\Phi}_{t+k,N})^T (\hat{\Phi}_{t+k,N} + \tilde{\Phi}_{t+k,N}) \mid t\}$$

The terms in (43) can be simplified, recalling the optimal estimate  $\hat{Y}_{t+k,N}$  and the estimation error  $\tilde{Y}_{t+k,N}$  are orthogonal and the future reference  $R_{t+k,N}$  is a known signal. Expanding:

$$\tilde{J} = \hat{\Phi}_{t+k,N}^T \hat{\Phi}_{t+k,N} + E\{\tilde{\Phi}_{t+k,N}^T \tilde{\Phi}_{t+k,N} \mid t\} \quad (48)$$

$$\text{Thence, the cost-function:} \quad \tilde{J}(t) = \hat{\Phi}_{t+k,N}^T \hat{\Phi}_{t+k,N} + \tilde{J}_1(t) \quad (49)$$

The part of the cost term *independent of control action* may be written as:

$$\tilde{J}_1(t) = E\{\tilde{\Phi}_{t+k,N}^T \tilde{\Phi}_{t+k,N} \mid t\} = E\{\tilde{Y}_{t+k,N}^T P_{CN}^T P_{CN} \tilde{Y}_{t+k,N} \mid t\} \quad (50)$$

Now simplify the vector  $\hat{\Phi}_{t+k,N}$  by substituting for  $\hat{Y}_{t+k,N}$  from (24) and using (42) and (36) obtain:

$$\hat{\Phi}_{t+k,N} = P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) + F_{CN}^0 U_{t,N}^0 = P_{CN} R_{t+k,N} - P_{CN}(G_N U_{t,N}^0 + F_{t,N}) + F_{CN}^0 U_{t,N}^0$$

$$= P_{CN}(R_{t+k,N} - F_{t,N}) - (\Lambda_N^2 + G_N^T G_N) U_{t,N}^0 = P_{CN}(R_{t+k,N} - F_{t,N}) - X_N U_{t,N}^0 \quad (51)$$

From a similar argument the *multi-step predictive control* sets the *squared term* in (49) to zero  $\hat{\Phi}_{t+k,N} = 0$ .

Clearly the resulting optimal control  $U_{t,N}^0 = X_N^{-1} P_{CN}(R_{t+k,N} - F_{t,N})$  is the same as the vector of future *GPC* controls in (35).

**Theorem 3.1: Equivalent Minimum Variance Predictive Control Problem**

Consider the minimisation of the *GPC* cost index (29) for the system and assumptions introduced in §2, where the nonlinear subsystem  $\mathcal{W}_{1k} = I$  and the vector of optimal *GPC* controls is given by (35).

Redefine the cost-index to have a multi-step variance form (43)  $\tilde{J}(t) = E\{\Phi_{t+k,N}^T \Phi_{t+k,N} | t\}$ , where

$\Phi_{t+k,N} = P_{CN}(R_{t+k,N} - Y_{t+k,N}) + F_{CN}^0 U_{t,N}^0$  and the cost weightings  $P_{CN} = G_N^T$  and  $F_{CN}^0 = -\Lambda_N^2$ . Then the vector of future optimal controls is identical to the *GPC* controls defined in (35). ■

**Proof:** Follows by collecting together the above results. ■

#### 4 NPGMV Optimal Control Problem

The *Nonlinear Predictive Generalised Minimum Variance (NPGMV)* control problem of interest is now considered. The actual input to the system is of course the control signal  $u(t)$ , shown in Fig. 1, rather than the input to the linear sub-system  $u_0$ . The cost-function for the nonlinear control problem must include a control signal costing term, although the costing on the intermediate signal  $u_0(t)$  can be retained to examine limiting cases. This signal may also represent an *actuator output* that may be costed in some problems. If the smallest delay in each output channel of the plant is of magnitude  $k$ -steps this implies that the control signal  $u$  affects the output at least  $k$ -steps later. For this reason the control signal costing should be defined to have the form:

$$(\mathcal{F}_c u)(t) = z^{-k} (\mathcal{F}_c u)(t) \quad (52)$$



Typically this weighting on the nonlinear sub-system input will be a *linear dynamic operator* but it may also be chosen to be nonlinear to introduce an anti-windup capability [10]. This operator  $\mathcal{F}_{ck}$  can be assumed to be full rank and invertible. Thus, consider a new signal whose variance is to be minimised, involving the weighted sum of error, subsystem input and control signals:

$$\phi_0(t) = P_c e(t) + F_{c0} u_0(t) + (\mathcal{F}_c u)(t) \quad (53)$$

In analogy with the *GPC* problem a *multi-step cost index* may be defined that is an extension of (43):

$$\textbf{Extended Multi-Step Performance Index: } J_p = E\{\Phi_{t+k,N}^{0T} \Phi_{t+k,N}^0 \mid t\} \quad (54)$$

The signal  $\Phi_{t+k,N}^0$  is therefore extended to include the additional future control signal costing term:

$$\Phi_{t+k,N}^0 = P_{CN} E_{t+k,N} + F_{CN}^0 U_{t,N}^0 + (\mathcal{F}_{c k,N} U_{t,N}) = P_{CN} (R_{t+k,N} - Y_{t+k,N}) + F_{CN}^0 U_{t,N}^0 + (\mathcal{F}_{c k,N} U_{t,N}) \quad (55)$$

The non-linear function  $\mathcal{F}_{c k,N} U_{t,N}$  will normally be defined to have the simple diagonal operator form:

$$(\mathcal{F}_{c k,N} U_{t,N}) = \text{diag}\{(\mathcal{F}_{ck} u)(t), (\mathcal{F}_{ck} u)(t+1), \dots, (\mathcal{F}_{ck} u)(t+N)\} \quad (56)$$

where  $U_{t,N}^0 = (\mathcal{W}_{1k,N} U_{t,N})$  and  $\mathcal{W}_{1k,N}$  also has a block diagonal matrix form:

$$(\mathcal{W}_{1k,N} U_{t,N}) = \text{diag}\{\mathcal{W}_{1k}, \mathcal{W}_{1k}, \dots, \mathcal{W}_{1k}\} U_{t,N} = [(\mathcal{W}_{1k} u)(t)^T, \dots, (\mathcal{W}_{1k} u)(t+N)^T]^T \quad (57)$$

**Remarks:** The problem simplifies when  $N = 0$  to the single-step *non predictive* control problem, which is the same as the so-called *NGMV* control problem [9].

#### 4.1 The NPGMV Control Solution

The solution follows from very similar steps to those in §3.3 and will therefore be summarised only briefly

below. Observe from (44) that  $\Phi_{t,N}^0 = \Phi_{t,N} + z^{-k}(\mathcal{F}_{c k,N} U_{t,N})$  and  $\Phi_{t+k,N}^0 = \hat{\Phi}_{t+k,N}^0 + \tilde{\Phi}_{t+k,N}^0$  where

$$\hat{\Phi}_{t+k,N}^0 = \hat{\Phi}_{t+k,N} + (\mathcal{F}_{c k,N} U_{t,N}) = P_{CN} (R_{t+k,N} - \hat{Y}_{t+k,N}) + F_{CN}^0 U_{t,N}^0 + (\mathcal{F}_{c k,N} U_{t,N}) \quad (58)$$

and the estimation error:

$$\tilde{\Phi}_{t+k,N}^0 = \tilde{\Phi}_{t+k,N} = -Y^{-T} G_N^T \tilde{Y}_{t+k,N} \quad (59)$$

The future predicted values in the signal  $\hat{\Phi}_{t+k,N}^0$  involves the estimated vector of weighted outputs  $\hat{Y}_{t+k,N}$  and these are orthogonal to  $\tilde{Y}_{t+k,N}$ . Also note that the estimation error is zero mean and hence the expected value of the product with any known signal is null. The cost-function may therefore be written as:

$$\tilde{J}(t) = \hat{\Phi}_{t+k,N}^{0T} \hat{\Phi}_{t+k,N}^0 + \tilde{J}_1(t) \quad (60)$$

where the optimal control sets  $\hat{\Phi}_{t+k,N}^0 = 0$ . The condition for optimality therefore has the form.

$$P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) + (\mathcal{F}_{Ck,N} + F_{CN}^0 \mathcal{W}'_{1k,N})U_{t,N} = 0 \quad (61)$$

#### 4.2 The Nonlinear Predictive GMV Control Signal

The future optimal control, to minimize (60), follows from the condition for optimality in (61):

$$U_{t,N} = -(\mathcal{F}_{Ck,N} - \Lambda_N^2 \mathcal{W}'_{1k,N})^{-1} P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) \quad (62)$$

An alternative solution of (61), in an easier form for implementation, gives:

$$U_{t,N} = -\mathcal{F}_{Ck,N}^{-1} \left( P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) - \Lambda_N^2 \mathcal{W}'_{1k,N} U_{t,N} \right) \quad (63)$$

The optimal predictive control law is nonlinear, since it involves the nonlinear control signal costing term:

$\mathcal{F}_{Ck,N}$  and the nonlinear model for the plant  $\mathcal{W}'_{1k,N}$ . Further simplification is possible by substituting from

(24) for the estimate  $\hat{Y}_{t+k,N}$ , so that the condition for optimality in (61) may be written as:

$$P_{CN}(R_{t+k,N} - F_{t,N}) + (\mathcal{F}_{Ck,N} - (P_{CN}G_N - F_{CN}^0)\mathcal{W}'_{1k,N})U_{t,N} = 0$$

Substituting from (42) the condition for optimality becomes:

$$P_{CN}(R_{t+k,N} - F_{t,N}) + (\mathcal{F}_{Ck,N} - Y^T Y \mathcal{W}'_{1k,N})U_{t,N} = 0 \quad (64)$$

The two alternative solutions for the *vector of future optimal controls*, noting (36), therefore becomes:

$$U_{t,N} = -(\mathcal{F}_{Ck,N} - X_N \mathcal{W}'_{1k,N})^{-1} P_{CN}(R_{t+k,N} - F_{t,N}) \quad (65)$$

or

$$U_{t,N} = -\mathcal{F}_{ck,N}^{-1} \left( P_{CN} (R_{t+k,N} - F_{t,N}) - X_N \mathcal{W}_{1k,N} U_{t,N} \right) \quad (66)$$

**Remarks:** The *NPGMV* control law in equation (65) is model based and includes an internal model for the nonlinear process. The control law is to be implemented using a *receding horizon philosophy* and from the preceding discussion it becomes identical to the *GPC* controller (35) in the limiting linear case when the control costing tends to zero ( $\mathcal{F}_{ck,N} \rightarrow 0, \mathcal{W}_{1k,N} = I$ ). The problem construction enables an important property to be predicted and confirmed from (65). That is, if the control weighting  $\mathcal{F}_{ck,N} \rightarrow 0$  then  $U_{t,N}$  should introduce the inverse of the plant model  $\mathcal{W}_{1k,N}$  (if one exists) and the resulting vector of future controls  $U_{t,N}^0$  will then be the same as the *GPC* controls for the resulting linear system.

**Theorem 4.1: Nonlinear Predictive GMV Optimal Control Law**

Consider the system described in §2 and the predictive control problem for the cost index (54) ( $N > 0$ ). The nonlinear plant operator  $\mathcal{W}_k$  is assumed to be *finite gain stable*. For closed-loop stability the operator  $(I - \mathcal{F}_{ck,N}^{-1} (X_N + P_{CN} W_{Nk} C_{10}) \mathcal{W}_{1k,N})^{-1}$  is assumed to be *finite gain stable*, due to the choice of control  $(\mathcal{F}_c u)(t) = (\mathcal{F}_{ck} u)(t-k)$ , dynamic error  $P_c(z^{-1})$  and input  $\{\lambda_0, \dots, \lambda_N\}$  *cost weightings*. The *multi-step predictive control* cost-function to be minimised, involves a sum of future cost terms  $J_p = E\{\Phi_{t+k,N}^{0T} \Phi_{t+k,N}^0 \mid t\}$ , where  $\Phi_{t+k,N}^0$  includes the vector of future error, input and control costing terms:

$$\Phi_{t+k,N}^0 = P_{CN} E_{t+k,N} + F_{CN}^0 U_{t,N}^0 + (\mathcal{F}_{ck,N} U_{t,N}) \quad (67)$$

and in terms of the weightings  $P_{CN} = G_N^T$  and  $F_{CN}^0 = -\Lambda_N^2$ . The *NPGMV* optimal control law to minimize the variance of signal (67) is given as:

$$U_{t,N} = -(\mathcal{F}_{ck,N} - X_N \mathcal{W}_{1k,N})^{-1} P_{CN} (R_{t+k,N} - F_{t,N}) \quad (68)$$

where  $X_N = G_N^T G_N + \Lambda_N^2$ . For implementation of the *vector of future optimal control signals*:

$$U_{t,N} = -\mathcal{F}_{ck,N}^{-1} \left( P_{cN} (R_{t+k,N} - F_{t,N}) - X_N \mathcal{W}_{1k,N} U_{t,N} \right) \quad (69)$$

and the current control may be found from the first element of the vector (invoking the receding horizon philosophy). The signals  $F_{t,N} = H_{NZ}(z^{-1})z(t) + S_{NZ}(z^{-1})u_f(t-1)$  and  $u_f(t) = D_{f1}^{-1}(z^{-1})u_0(t)$ . ■

**Solution:** The proof of the *NPGMV* optimal control follows by collecting the results in the above section. The necessary condition for stability can be established using the same argument as after the main theorem in [9]. This requires the introduction of some linear or *NL* plant sub-system models. Write the linear plant in the right coprime form  $W_0 = B_0 A_0^{-1}$  and the corresponding block structure as  $W_{0,N} = B_{0,N} A_{0,N}^{-1}$ . Also write the plant model in a polynomial *NL* operator form:  $A_{0,N}^{-1} \mathcal{W}_{1k,N} = \mathcal{B}_{1k,N} \mathcal{A}_{1k,N}^{-1}$  so that  $\mathcal{W}_{k,N} = B_{0,N} A_{0,N}^{-1} \mathcal{W}_{1k,N} = B_{0,N} \mathcal{B}_{1k,N} \mathcal{A}_{1k,N}^{-1}$ . Also note that  $P_{cN} W_{Nk}$  may be written  $P_{cN} W_{Nk} = P_w A_0^{-1}$ . Then the following relationship may be established:

$$(I - \mathcal{F}_{ck,N}^{-1} (X_N + P_{cN} W_{Nk} C_{10}) \mathcal{W}_{1k,N})^{-1} = \left( I - \mathcal{F}_{ck,N}^{-1} (X_N A_{0,N} + P_w C_{10}) \mathcal{B}_{1k,N} \mathcal{A}_{1k,N}^{-1} \right)^{-1}$$

which may be used to show that the model for the predicted outputs involves only stable operators. ■

**Remarks:** The two expressions for the *NPGMV* control signal (68) and (69) lead to the two alternative structures, shown in Figs. 2 and 3, respectively. The second, shown in Fig. 3 shows how the current and future controls may be separated from the full vector of future controls, as explained below. If the error and input cost-function weightings are defined in the *GPC* motivated form  $P_{cN} = G_N^T$  and  $F_{cN}^0 = -\Lambda_N^2$  then for a linear system ( $\mathcal{W}_{1k} = I$ ) the optimal control, when  $\mathcal{F}_c \rightarrow 0$ , is identical to a *GPC* control law.

### 4.3 Implementation of the Predictive Optimal Control

A useful partition may be introduced which later enables the algorithm to be simplified. The control at time  $t$  is computed for  $N > 0$  from the vector of current and future controls by introducing the matrix:



$$C_{I0} = [I, 0, \dots, 0] \quad (70)$$

This enables the control at time  $t$  to be found from the vector of current and future controls as:

**Current control:** 
$$u(t) = [I, 0, \dots, 0] U_{t,N} \quad (71)$$

To compute the vector of future controls for  $t > 0$  also introduce: 
$$C_{0I} = \begin{bmatrix} 0 & I_N \end{bmatrix} \quad (72)$$

**Future controls:** 
$$U_{t,N}^f = C_{0I} U_{t,N} = \begin{bmatrix} 0 & I_N \end{bmatrix} \begin{bmatrix} u(t) \\ \vdots \\ u(t+N) \end{bmatrix} = \begin{bmatrix} u(t+1) \\ \vdots \\ u(t+N) \end{bmatrix} \quad (73)$$

Note from (70), because of the block diagonal structure of the control signal costing  $\mathcal{F}_{ck,N}$ , then

$$C_{I0} \mathcal{F}_{ck,N}^{-1} = [\mathcal{F}_{ck}^{-1}, 0, \dots, 0] = \mathcal{F}_{ck}^{-1} C_{I0} \quad (74)$$

The *optimal control at time  $t$*  can then be computed, using (69) as:

$$u(t) = -\mathcal{F}_{ck}^{-1} C_{I0} \left( P_{cn} (R_{t+k,N} - F_{t,N}) - X_N \mathcal{W}_{1k,N} U_{t,N} \right) \quad (75)$$

The *vector of future controls*, computed at time  $t$ , may also be found as:

$$U_{t,N}^f = -C_{0I} \mathcal{F}_{ck,N}^{-1} \left( P_{cn} (R_{t+k,N} - F_{t,N}) - X_N \mathcal{W}_{1k,N} U_{t,N} \right) \quad (76)$$

where from (72) write  $C_{0I} \mathcal{F}_{ck,N}^{-1} = \begin{bmatrix} 0_{(N+1) \times m, m} & I_N \end{bmatrix} \mathcal{F}_{ck,N}^{-1} = \begin{bmatrix} 0_{(N+1) \times m, m} & \mathcal{F}_{ck,N-1}^{-1} \end{bmatrix}$ .

The vector  $\mathcal{W}_{1k,N} U_{t,N}$  may be written, from equation (57) (partitioning current and future terms) as:

$$(\mathcal{W}_{1k,N} U_{t,N}) = [(\mathcal{W}_{1k} u)(t)^T, \dots, (\mathcal{W}_{1k} u)(t+N)^T]^T = [(\mathcal{W}_{1k} u)(t)^T, (\mathcal{W}_{1k,N-1} U_{t,N}^f)^T]^T \quad (77)$$

Using a related partition, write the matrix  $X_N$  in the form  $X_N = Y^T Y = [Y_1 \ Y_2]$ , where  $Y_1$  has  $m_0$  columns, so that  $X_N \mathcal{W}_{1k,N} U_{t,N} = [Y_1 \ Y_2] \mathcal{W}_{1k,N} U_{t,N} = Y_1 (\mathcal{W}_{1k} u)(t) + Y_2 (\mathcal{W}_{1k,N-1} U_{t,N}^f)$ . Thus the second equation for implementing the optimal control (69), may be split into the current and future controls as shown in Fig. 3.

#### 4.4 Marine Predictive Control Design Example

Consider the problem of the simultaneous control of the roll and yaw motions of a ship. A supply vessel with the conventional angle notation is shown in Fig. 5. The ship heading (yaw angle) is controlled by the rudder, and it is assumed that the heading trajectory to follow is known. The rolling motion caused by the force of the sea wave disturbances can be counteracted by the use of fin roll stabilizers. However, this undesirable movement may also be reduced by active use of the rudder, and a number of commercial rudder roll stabilization systems have been developed (see [28] and references therein). This strategy requires high-performance rudder machinery but can provide improved performance or enable smaller fins to be used. The basic dynamics of the ship roll and yaw motion with respect to the fin and rudder, for particular ship speed and encounter angle, are shown in Figure 6.

**Roll model:** 
$$G_{\phi}(s) = \frac{(0.8)^2}{s^2 + 2 \cdot 0.2 \cdot 0.8s + (0.8)^2}$$

**Yaw model:** 
$$G_{\psi}(s) = \frac{0.2}{s(10s + 1)}$$

**Rudder to roll interaction:** 
$$G_{\delta\phi}(s) = \frac{0.1(1 - 4s)}{(6s + 1)}$$

The model includes non-minimum phase interaction from the rudder to roll motion and there is an integrator in the yaw model. The roll characteristics of the ship are modelled using a resonant second-order system, with a natural frequency of 0.8 rad/sec and a low damping factor. The frequency responses of the models are shown in Fig. 7. The fin and rudder actuators  $G_{\alpha}$  and  $G_{\delta}$  have hard constraints on the achievable angle and rate. The actuator limits are set as 25 deg and 10 deg/sec for the fins, and 30 deg and 7 deg/sec for the rudder servo, respectively.

**Disturbances:** The effect of the wave disturbance on the roll and yaw motion is represented in Fig. 13 by

the signals  $d_{\phi}$  and  $d_{\psi}$ , where  $d_{\phi} = \frac{5s}{s^2 + 2 \cdot 0.1 \cdot 0.7s + (0.7)^2} \xi$ ,  $d_{\psi} = \frac{0.5}{s} \zeta$  and  $\xi$  and  $\zeta$  are white noise

sequences. The model for the roll wave disturbance provides a second order linear approximation to the

*Pierson-Moskowitz* spectrum, and the yaw disturbance is assumed to be of low-frequency nature and is modelled by an integrator driven by white noise.

**Control objectives:** The main control objectives are the reduction of roll motion and tracking of the heading set-point. The former can be characterized by the *Roll Reduction Ratio* (*RRR*) defined as:

$$RRR = \left( 1 - \frac{\text{closed-loop roll variance}}{\text{open-loop roll variance}} \right) \quad (78)$$

This ratio represents the improvement in roll reduction achieved by using feedback control, with 100% corresponding to the ideal null roll motion. The yaw tracking performance can be measured using conventional measures such as rise time / settling time, or, alternatively, by integral square error (*ISE*). In a classical control scheme, rolling motion is regulated using fin stabilizers, and the heading is controlled with the rudder, involving two SISO systems. A multivariable control scheme will take the system interactions into account, allowing the rudder to actively attenuate the roll and to control yaw, which is possible due to the separation in the roll and yaw motion frequency content.

**Results:** For the purpose of controller design, the continuous-time models of the system were discretized using the sample time of 0.5 seconds. In the simulations, the ship yaw angle was required to follow a known trajectory consisting of two step changes, while minimizing the roll motion, according to the specified criterion. In the limiting case when  $\mathcal{W}_{lk} = I$  (i.e. no constraints in the ship model) and  $\mathcal{F}_{ck} \rightarrow 0$ , the *NPGMV* controller collapses to a version of the standard *GPC* controller but with weighted output and reference signals in the cost criterion. The results for the nominal settings of  $N=0$ ,  $\Lambda_0 = 3 \times \text{diag}\{10^{-3}, 5 \times 10^{-3}\}$  and the linear case are shown in Fig. 8. The  $P_c$  weighting was chosen based on a multi-loop classical controller (see [9]), the performance of which is also shown.

The *GPC* results for the roll attenuation in this unconstrained case are somewhat unrealistic and detuning the controller (increasing  $\lambda$  weighting) is normally needed in the presence of fin and rudder constraints (in particular, fin rate limits are exceeded). The predictive action can also be utilized when the future yaw trajectory is known, and this is illustrated in Fig. 9 (the stochastic noise has been removed and

the time scale magnified to show the predictive action more clearly). Increasing values of  $N$  are indicated by the arrow. A long prediction horizon often leads to a faster response and also improves the robustness of the solution (as measured by the response overshoot), which is illustrated by the yaw angle response.

When the constraints are present, the *GPC* controller needs to be detuned to maintain stability. The nonlinearities can be accounted for more effectively by introducing the nonlinear control weighting  $\mathcal{F}_{ck}$  into the *NPGMV* control structure. For example, defining this weighting appropriately leads to an anti-windup structure for controllers that include integral action [10]. After tuning, the results are shown in Fig. 10 (for  $N = 5$ ), where the *NPGMV* satisfies the rudder angle limits that are much exceeded by the *GPC* design. The roll reduction is also more effective with the nonlinear control (about 40% improvement in Roll Reduction Ratio) since the servo nonlinearities are explicitly accounted for in the controller structure.

## 5 Concluding Remarks

There are many nonlinear predictive control strategies based on *state dependent* models, linearization around a trajectory and others. The aim was to try to produce a control law which is simple to implement and the result is an algorithm which closer to traditional model based designs than to current nonlinear predictive control strategies. The *NL Predictive Generalised Minimum Variance (NPGMV)* control problem involves a multi-step predictive control cost-function and the introduction of future set-point information. The predictive controls strategy is a development of the *NGMV* design method which is easy to design and implement.

It has the nice property that if the system is linear the control reverts to the *GPC* design method which is well known in industry. That is, the *NPGPC* control design method reduces to that of *GPC* control design when the weight  $\mathcal{F}_{ck}$  tends to zero and the system is linear ( $\mathcal{M}'_k = I$ ). This suggests a 2 stage design process might be used where the first stage is for a free choice of *GPC* weightings based upon the linear sub-systems. The engineer only need consider the selection of desirable weightings, which satisfy suitable performance requirements for the multivariable system. The *NL* system characteristics can then be



considered in the second stage of the design where the control signal costing ( $\mathcal{F}_{ck}$  possibly nonlinear) is selected and stability issues are considered.

If the cost horizon becomes only a single step then the control law reverts to the so called *NGMV* solution. A method is available for generating cost weightings that will provide a starting point for design [9] and guarantee a stabilising initial solution. This may be a useful starting point and the number of steps in the predictive control horizon can then be increased which normally improves robustness at the expense of additional computations. Clearly if the responses are not improving by using further steps there is no need to increase the computations. The control law includes an internal model but many of the computations, as in traditional polynomial equation based predictive control, simply involve the solution of Diophantine equations and matrix multiplications.

**Acknowledgements:** We are grateful for the support of the EPSRC on the Platform Grant EP/C526422/1.

## References

- [1]. Cutler C.R. and Ramaker B.L., 1979, *Dynamic matrix control - A computer control algorithm*, A.I.C.H.E, 86th National Meeting, April 1979.
- [2]. Clarke, D.W., C. Mohtadi and P.S. Tuffs, 1987, *Generalized predictive control - Part 1, The basic algorithm, Part 2, Extensions and interpretations*, Automatica, 23, 2, pp.137-148.
- [3]. Clarke, D.W., and C. Montadi, 1989, *Properties of generalised predictive control*, Automatica, Vol. 25, No. 6, pp. 859-875.
- [4]. Richalet J., A. Rault, J.L. Testud and J. Papon 1978, *Model predictive heuristic control applications to industrial processes*, Automatica, 14, pp. 413-428.
- [5]. Richalet, J., 1993, *Industrial applications of model based predictive control*, Automatica, Vol. 29, No. 8, pp. 1251-1274.
- [6]. Bitmead R., M. Gevers, and V. Wertz, 1990, *Adaptive Optimal Control: The Thinking Man's GPC*. Prentice Hall.

- [7]. Kwon W.H. and Pearson, A.E., 1977, *A modified quadratic cost problem and feedback stabilization of a linear system*, IEEE Transactions on Automatic Control, Vol. AC-22, No. 5, pp. 838-842.
- [8]. Grimble, M J, 2004, *GMV control of nonlinear multivariable systems*, UKACC Conference Control 2004, University of Bath, 6-9 September.
- [9]. Grimble, M J, 2005, *Non-linear generalised minimum variance feedback, feedforward and tracking control*, Automatica, Vol. 41, pp 957-969.
- [10]. Grimble, M J, and P Majecki, 2005, *Nonlinear Generalised Minimum Variance Control Under Actuator Saturation*, IFAC World Congress, Prague, Friday 8 July, 2005.
- [11]. Cannon, M. and B. Kouvaritakis, 2001, Open-loop and closed-loop optimality in interpolation MPC, in *Nonlinear Predictive Control: Theory and Practice*, pp. 131-149. IEE, London.
- [12]. Cannon, M. and B. Kouvaritakis, 2002, Efficient constrained model predictive control with asymptotic optimality, *Siam J. Control Optimisation*, 41(1), pp. 60-82.
- [13]. Cannon, M and B. Kouvaritakis, Y.I. Lee and A.C. Brooms, 2001, Efficient nonlinear predictive control, *International J. Control*, 74(4), pp.361-372.
- [14]. Cannon, M and V. Deshmukh and B. Kouvaritakis, 2003, Nonlinear model predictive control with polytopic invariant sets, *Automatica* 39(8), pp. 1487-1494.
- [15]. Kothare, M. V, V. Balakrishnan and M. Morari, 1996, Robust constrained model predictive control using linear matrix inequalities, *Automatica* 32(10), pp. 1361-1379.
- [16]. Michalska, H. and D.Q. Mayne, 1993, Robust receding horizon control of constrained non-linear systems, *IEEE Transactions on Automatic Control*, 38, pp.1623-1633.
- [17]. Shamma, J.S. and M. Athans, 1990, Analysis of gain scheduled control for non-linear plants, *IEEE Transactions on Automatic Control*, 35, pp. 898-907.
- [18]. Kouvaritakis, B., M. Cannon and J.A. Rossiter, 1999, Nonlinear model based predictive control, *Int. J. Control*, 72(10), pp. 919-928.
- [19]. Lee, Y.I., B. Kouvaritakis and M. Cannon, 2003, Constrained receding horizon predictive control for nonlinear systems, *Automatica*, 38(12), pp. 2093-2102.

- [20]. Mayne, D.Q., J.B. Rawlings, C.V. Rao and P.O.M. Scokaert, 2000, Constrained model predictive control: stability and optimality, *Automatica* 36(6), pp. 789-814.
- [21]. Scokaert, P.O.M., D.Q. Mayne and J.B. Rawlings, 1999, Suboptimal model predictive control (feasibility implies stability), *IEEE Transactions on Automatic Control*, 44(3), pp. 648-654.
- [22]. Brooms, A.C. and B. Kouvaritakis, 2000, Successive constrained optimisation and interpolation in non-linear model based predictive control, *Int. J. Control*, 73(4), pp. 312-316.
- [23]. Allgower, F., and R. Findeisen, 1998, Non-linear predictive control of a distillation column, *International Symposium on Non-linear Model Predictive Control*, Ascona, Switzerland.
- [24]. Camacho, E.F., 1993, Constrained generalized predictive control, *IEEE Transactions on Automatic Control*, 38, pp. 327-332.
- [25]. Grimble, M J, 2006, *Robust industrial control*, John Wiley, Chichester.
- [26]. Grimble, M J, 2001, *Industrial control systems design*. John Wiley, Chichester.
- [27]. Grimble M.J. and Johnson, M.A, 1988, Optimal control and stochastic estimation, Vols. I and II, John Wiley, Chichester.
- [28]. Perez T., 2005, *Ship Motion Control*, Springer-Verlag, London.

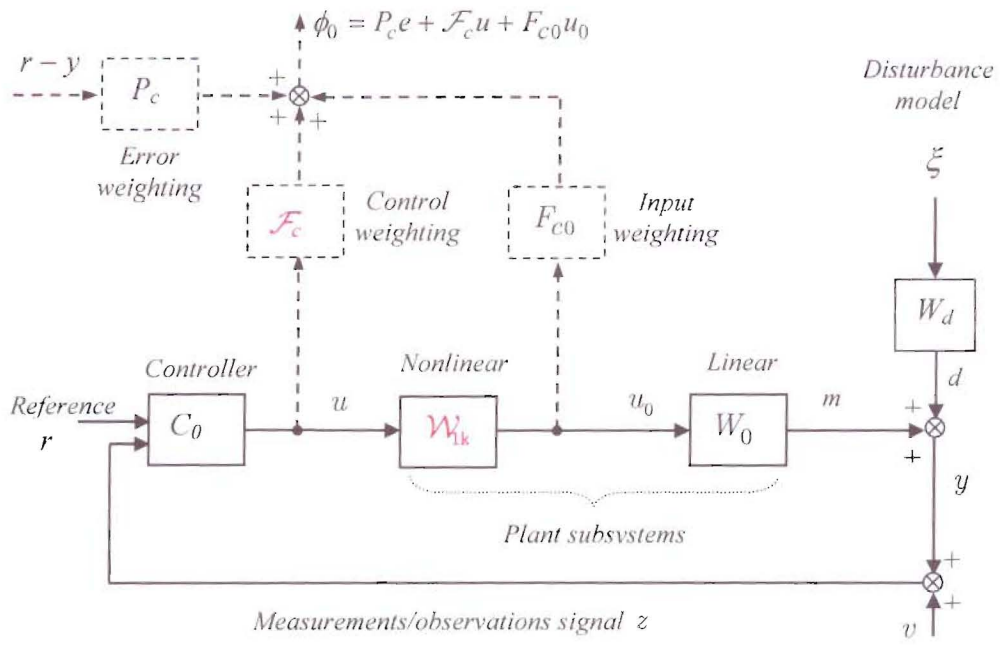


Fig.1: NPGMV 2-Degrees of Freedom Feedback Control System for Nonlinear Plant

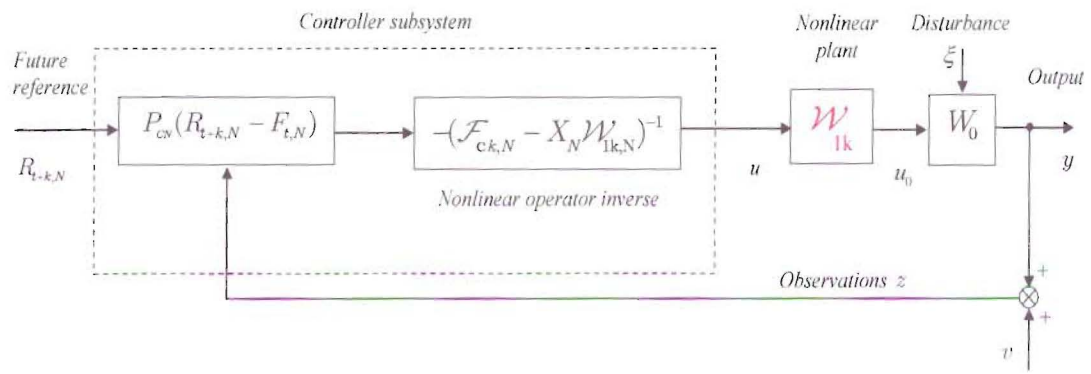


Fig. 2: First Form of the NPGMV Polynomial Controller Structure



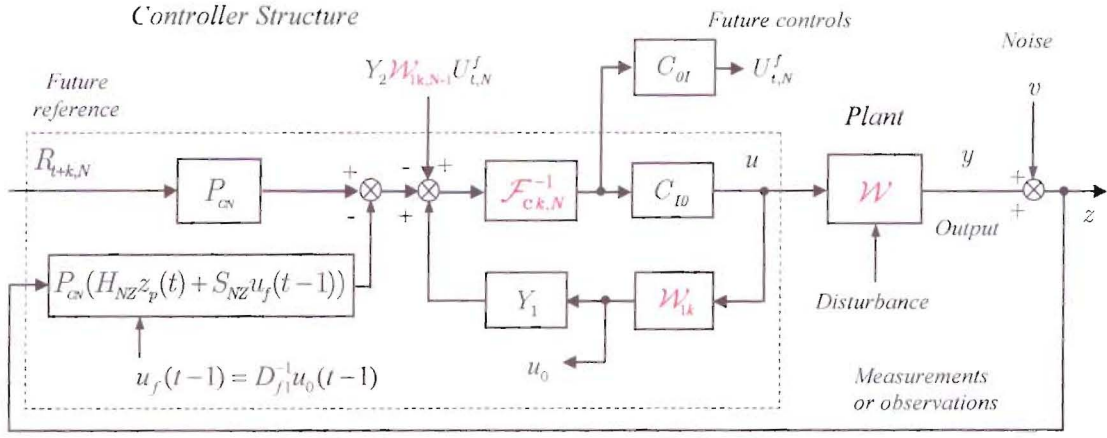


Fig. 3: **Second Form of NPGMV Controller** (showing future control signal generation)

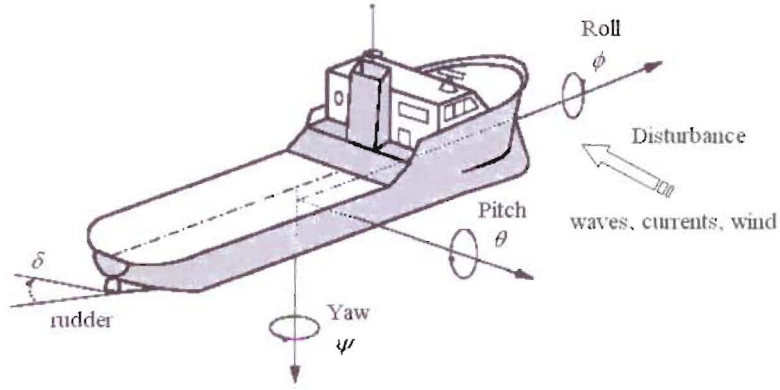


Fig. 5: **Standard Ship Motion Description** (from Perez [28])

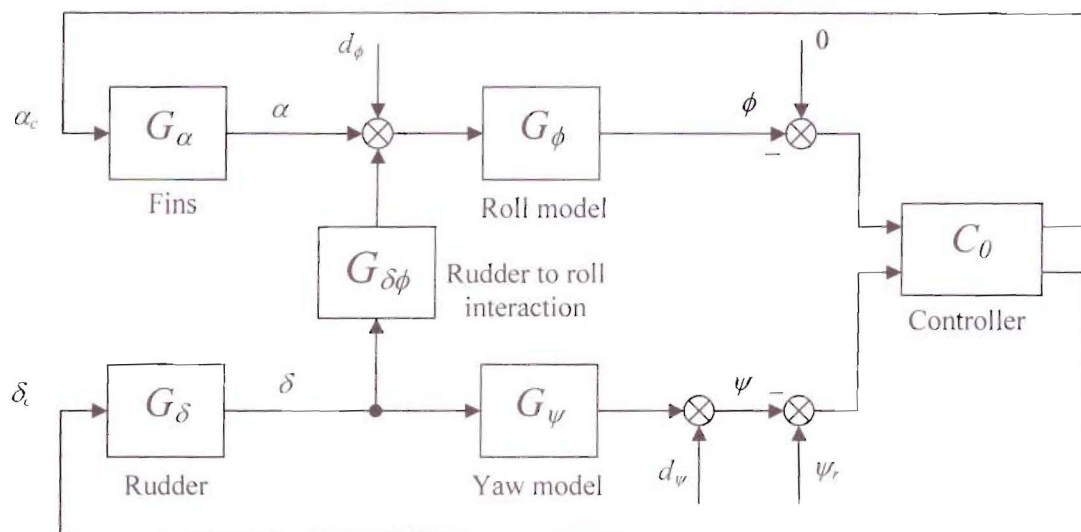


Fig. 6: Block Diagram of the Ship Model

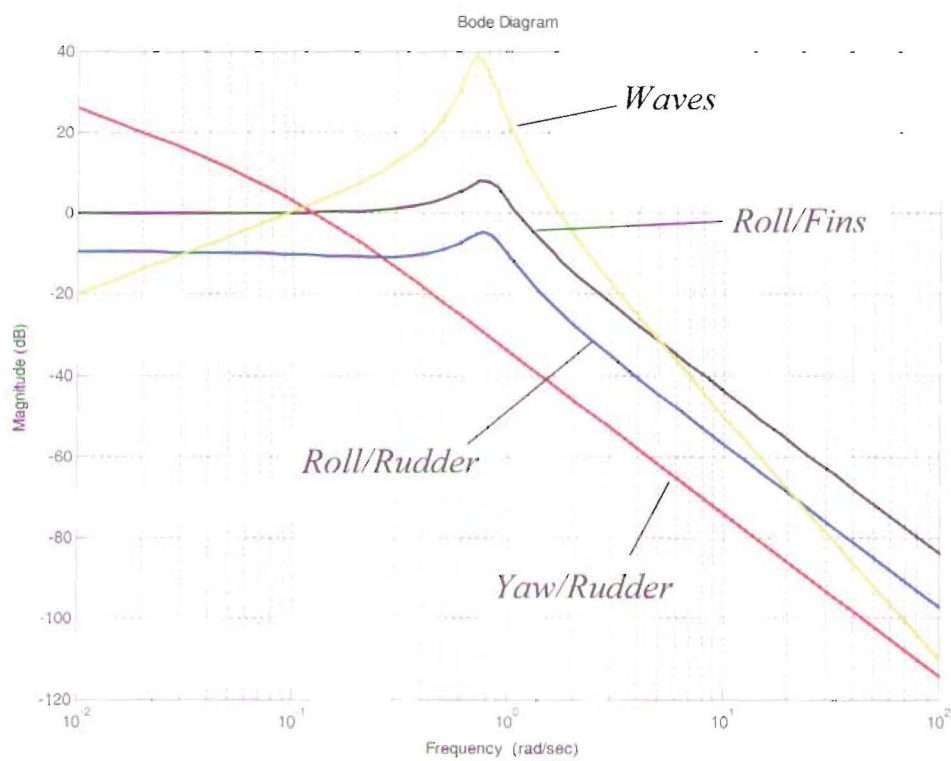


Fig. 7: Frequency Responses of the System Models and Wave Spectra

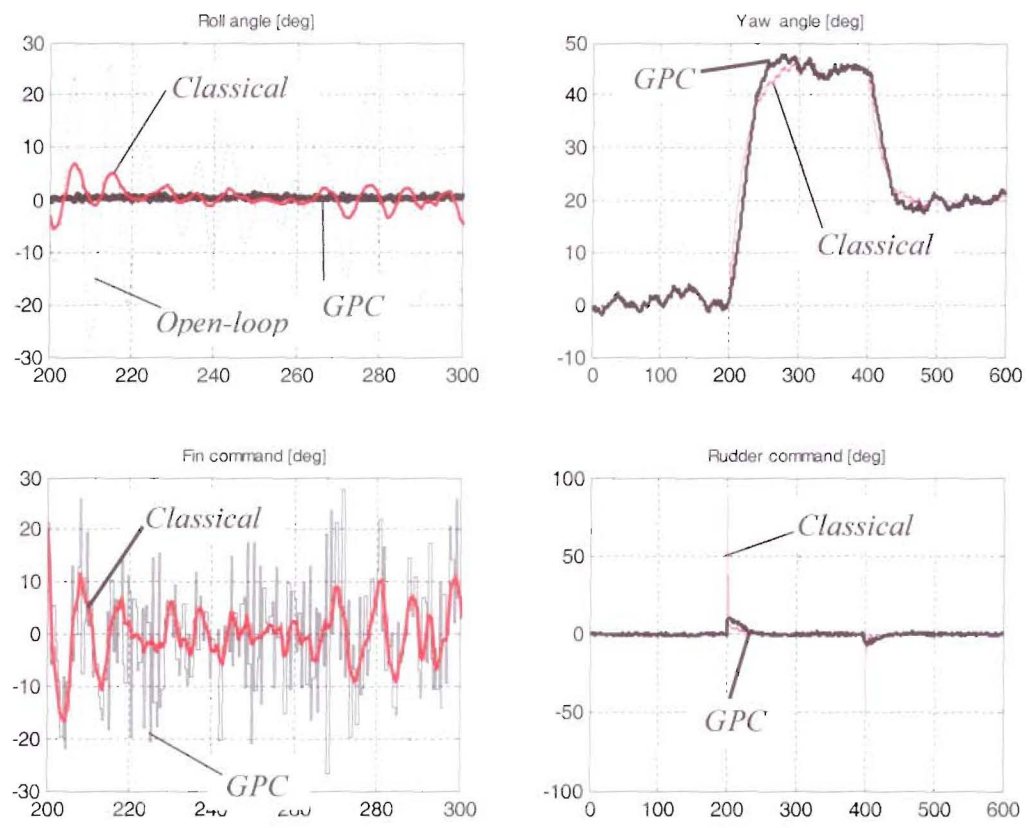


Fig. 8: Comparison Nominal GPC and Classical Control Results ( $N = 0$ )

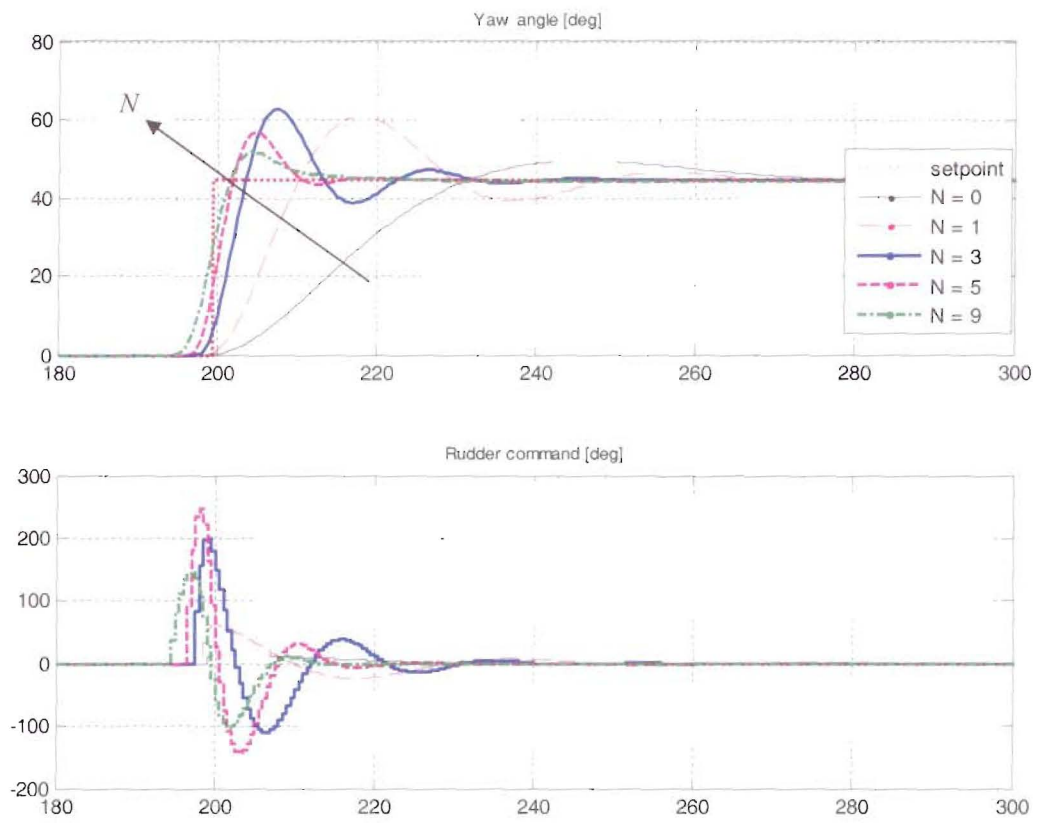


Fig. 9: NPGMV Results – Effect of Varying the Prediction Horizon ( $N = 0, 1, 3, 5, 9$ )



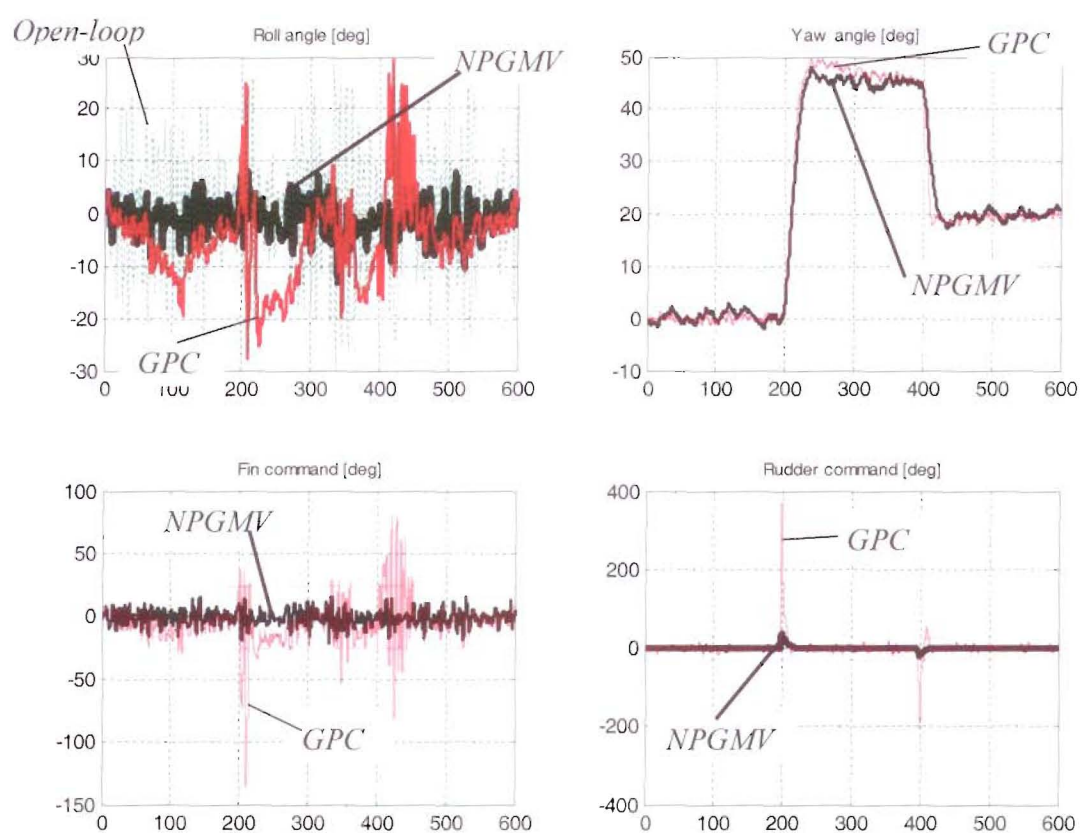


Fig. 10: NPGMV and linear GPC responses for the constrained system ( $N = 5$ )