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# The Geometry of Optimal Control Solutions on some Six Dimensional Lie Groups

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**Abstract**—This paper examines optimal solutions of control systems with drift defined on the orthonormal frame bundle of particular Riemannian manifolds of constant curvature. The manifolds considered here are the space forms Euclidean space  $\mathbb{E}^3$ , the spheres  $\mathbb{S}^3$  and the hyperboloids  $\mathbb{H}^3$  with the corresponding frame bundles equal to the Euclidean group of motions  $SE(3)$ , the rotation group  $SO(4)$  and the Lorentz group  $SO(1,3)$ . The optimal controls of these systems are solved explicitly in terms of elliptic functions. In this paper, a geometric interpretation of the extremal solutions is given with particular emphasis to a singularity in the explicit solutions. Using a reduced form of the Casimir functions the geometry of these solutions are illustrated.

## I. INTRODUCTION

This paper deals with affine control systems with drift defined on the frame bundles of simply connected manifolds of constant sectional curvature, denoted  $M$ . In particular the frame bundles of the space forms Euclidean space  $\mathbb{E}^3$ , the spheres  $\mathbb{S}^3$  and the hyperboloids  $\mathbb{H}^3$  with the corresponding frame bundles equal to the Euclidean group of motions  $SE(3)$ , the rotation group  $SO(4)$  and the Lorentz group  $SO(1,3)$  respectively. Indeed  $\mathbb{E}^3 = SE(3)/SO(3)$ ,  $\mathbb{S}^3 = SO(4)/SO(3)$  and  $\mathbb{H}^3 = SO(1,3)/SO(3)$ . Orthonormal frame bundles of space forms coincide with their isometry group and therefore the focus shifts to control systems defined on Lie groups. The cotangent bundle  $T^*G$  is then realized as the product of  $G$  with the dual of its Lie algebra  $\mathfrak{g}^*$  and leads to non-canonical coordinates, thus in this paper the use of the Maximum Principle of optimal control shifts the emphasis to 6-dimensional Hamiltonian systems defined on matrix Lie groups. Applications motivating this study are connected with controlling nonholonomic mechanical systems and in particular systems whose kinematics can be defined as affine control systems on Lie groups see, (see [1]). A simplified kinematic model of an airplane whose configuration can be described by  $SE(3)$  i.e it will always fly forward and the controls may yaw, pitch and roll the aircraft, has been used to find optimal landing trajectories for airplanes, (see [2]). The configuration  $SO(4)$  has a diverse range of applications from such fields as mathematical physics (see [3]), to modelling power conversion in electrical circuits and in particular, Wood [4] has shown that switched electrical networks such as those used in power conversion

can be modelled as bilinear systems with state transition matrices that evolve on the Simple Orthogonal group  $SO(n)$ , where  $n$  is dependent on the number of capacitors and/or inductors in the circuit. In addition the spherical space form  $\mathbb{S}^3$  can be used to represent spin systems in quantum control through the isomorphism  $\mathbb{S}^3 \rightarrow SU(2)$ . The Lorentz group has applications in physics and special relativity, indeed it is the isometry group of the 4-dimensional Minkowski metric. This paper explicitly solves for the optimal controls and the extremal solutions in terms of elliptic functions for the Euclidean, elliptic and hyperbolic case. In addition a geometric interpretation of these solutions is given using the invariant surfaces described by the Hamiltonian and Casimir functions. This illuminates the classical picture of elliptic curves as the intersection of quadric hypersurfaces in projective 3-space [5]. Finally, a geometric picture of the extremal solutions is given at a singularity of the system. At this singularity the solution is shown to be periodic and closed and the corresponding optimal controls trigonometric functions.

## II. EXPLICIT SOLUTIONS ON $\mathfrak{se}(3)^*$ , $\mathfrak{so}(4)^*$ AND $\mathfrak{so}(1,3)^*$

The group  $G$  is used to represent the frame bundle of the space forms corresponding to the matrix Lie groups  $SE(3)$ ,  $SO(4)$  and  $SO(1,3)$  respectively. We identify  $TG$  with  $G \times \mathfrak{g}$  where  $\mathfrak{g}$  is the Lie algebra of  $G$  and consider only the left translation. The elastic problem concern the solutions  $g(t)$  of the left-invariant differential system

$$\frac{dg}{dt}(t) = g(t)(A_0 + \sum_{i=1}^3 u_i A_i) \quad (1)$$

where  $A_0, \dots, A_3$  are given matrices in the Lie algebra  $\mathfrak{g}$  of  $G$  and for our particular case (see [7] for a derivation):

$$\frac{dg}{dt}(t) = g(t) \begin{pmatrix} 0 & -\varepsilon & 0 & 0 \\ 1 & 0 & -u_3 & u_2 \\ 0 & u_3 & 0 & -u_1 \\ 0 & -u_2 & u_1 & 0 \end{pmatrix} \quad (2)$$

where  $\varepsilon=0$  for the Euclidean case  $\mathbb{E}^3$ ,  $\varepsilon=1$  for the elliptic case  $\mathbb{S}^3$  and  $\varepsilon=-1$  for the hyperbolic case  $\mathbb{H}^3$ . Equation (2) describes the deformations on the frame bundle  $G$  of  $M$  subject to the assumption that for any curve  $x(t) \in M$ ,

$\left\| \frac{dx}{dt} \right\| = 1$ , and  $x$  is parameterized by its length from the initial point on the curve. In this optimal control problem we wish to minimize the expression  $\frac{1}{2} \int_0^T (u(t)Qu(t))dt$ , subject to the given boundary condition  $g(0) = g_0$ ,  $g(T) = g_1$ ,  $Q$  is diagonal and positive definite, and the diagonal entries will be denoted by  $c_i$ 's. In the mechanics literature the  $c_i$ 's are analogous to its principle moments of inertia and in the analogy to the elastic rod reflect the physical characteristics of the bar related to the geometric shape of its cross section [6]. Here  $u_i$  play the role of the controls. It is interesting to note that  $\varepsilon$  coincides with the constant sectional curvature of the corresponding space form. In this sense  $\varepsilon$  can be viewed as a continuous parameter representing the curvature of an arbitrary Riemannian manifold with constant curvature  $\varepsilon$ . However, in this paper  $\varepsilon$  will be discrete and its value will distinguish between the three cases. The maximum principle of optimal control then identifies the appropriate left-invariant Hamiltonian  $H$  on the dual of the Lie algebra  $\mathfrak{g}^*$ , specifically  $\mathfrak{se}(3)^*$ ,  $\mathfrak{so}(4)^*$  and  $\mathfrak{so}(1,3)^*$ . The maximum principle considers the lift of the optimization problem to the cotangent manifold  $T^*G$ . The control Hamiltonian is written as:

$$H(p, g, u) = p(gA_0) + \sum_{i=1}^3 u_i p(gA_i) - p_0 \frac{1}{2} \sum_{i=1}^3 c_i u_i^2 \quad (3)$$

where  $p \in T_g^*G$  and  $p_0 \in \mathbb{R}$  is a constant of motion. The two cases where  $p_0$  is either 1 or 0, correspond to normal extremals and abnormal extremals respectively. That is there are two Hamiltonian functions to consider. Indeed, all these cases admit abnormal extremals. However, because of the regularity of these variational problems each optimal trajectory is a projection of a regular extremal curve (see [7]), therefore, we assume  $p_0 = 1$ . The Hamiltonian is defined on the cotangent manifold  $T^*G$  which can be pulled back to  $G \times \mathfrak{g}^*$ . The Hamiltonian function can be pulled back by the left or right action of an element  $g \in G$ . Explicitly the pullback mapping by the left translated value can be defined as  $\hat{p}(\cdot) = p(g(\cdot))$  hence  $p(\cdot) = \hat{p}(g^{-1}(\cdot))$ . i.e  $p \in T^*G$  is pulled back to give a function  $\hat{p} \in \mathfrak{g}^*$ . Specifically,  $\hat{p}(\cdot)$  is defined via the non-degenerate Killing form on  $\mathfrak{so}(4)$  and  $\mathfrak{so}(1,3)$ . In the case of  $\mathfrak{se}(3)$  the trace form is degenerate but the function  $\hat{p}(\cdot)$  is derived using a combination of the Euclidean inner product and the Killing form (see [7] for details). The control Hamiltonian on  $\mathfrak{g}^*$  can be written as:

$$H(\hat{p}, u) = \hat{p}(A_0) + \sum_{i=1}^3 u_i \hat{p}(A_i) - \frac{1}{2} \sum_{i=1}^3 c_i u_i^2 \quad (4)$$

The maximum principle states that the optimal controls  $u^*$  will maximize the control Hamiltonian at every point of  $T^*G$ . The control Hamiltonian is a quadratic function of the scalar  $u_i$  and  $\frac{d^2 H}{du_i^2} < 0$  implies that there exists exactly one global maximum at each point of the Hamiltonian function. Differentiating (4) with respect to  $u_i$  gives:

$$\frac{dH}{du_i} = \hat{p}(A_i) - c_i u_i \quad (5)$$

$i = 1, 2, 3$

Therefore, the optimal controls are defined in terms of the momentum function  $\hat{p}(\cdot)$ .

$$u_i^* = \frac{1}{c_i} \hat{p}(A_i) \quad (6)$$

where  $i = 1, 2, 3$ . Also let  $p_1 = \hat{p}(A_0)$  and  $M_i = \hat{p}(A_i)$ , then substituting these back into (4) gives the optimal Hamiltonian

$$H^* = p_1 + \frac{1}{2} \left( \frac{M_1^2}{c_1} + \frac{M_2^2}{c_2} + \frac{M_3^2}{c_3} \right) \quad (7)$$

the optimal controls  $u_i^*$  can be substituted into (2) and rearranged to give

$$g^{-1} \frac{dg}{dt} = \begin{pmatrix} 0 & -\varepsilon & 0 & 0 \\ 1 & 0 & -M_3/c_3 & M_2/c_2 \\ 0 & M_3/c_3 & 0 & -M_1/c_1 \\ 0 & -M_2/c_2 & M_1/c_1 & 0 \end{pmatrix} \quad (8)$$

In addition to the Hamiltonian defined on these groups it is also essential to recognize some geometric facts about these Lie algebras. Any element of these Lie algebras can be naturally split into two spaces  $\mathfrak{p}$  and  $\mathfrak{k}$ , following the Cartan decomposition;  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ , which satisfy the classic relations.  $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$ ,  $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$  and  $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$  where  $\mathfrak{k}$  consists of all matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a_3 & a_2 \\ 0 & a_3 & 0 & -a_1 \\ 0 & -a_2 & a_1 & 0 \end{pmatrix} \quad (9)$$

and  $\mathfrak{p}$  consists of the matrices

$$\begin{pmatrix} 0 & -\varepsilon b_1 & -\varepsilon b_2 & -\varepsilon b_3 \\ b_1 & 0 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 \end{pmatrix} \quad (10)$$

For notational convenience write the element  $A_0$  as  $B_1$ . The corresponding adjoint representation for  $\mathfrak{k}$  and  $\mathfrak{p}$  where  $A_i \in \mathfrak{k}$  and  $B_i \in \mathfrak{p}$  are:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & -\varepsilon & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & -\varepsilon & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 & 0 & -\varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (11)$$

Define the Lie bracket to be  $[X, Y] = XY - YX$  for any element  $X$  and  $Y$  in  $\mathfrak{g}$ . The corresponding Lie bracket table is:

$[\cdot, \cdot]$	$A_1$	$A_2$	$A_3$	$B_1$	$B_2$	$B_3$
$A_1$	0	$A_3$	$-A_2$	0	$-B_3$	$-B_2$
$A_2$	$-A_3$	0	$A_1$	$-B_3$	0	$B_1$
$A_3$	$A_2$	$-A_1$	0	$B_2$	$-B_1$	0
$B_1$	0	$B_3$	$-B_2$	0	$\varepsilon A_3$	$-\varepsilon A_2$
$B_2$	$-B_3$	0	$B_1$	$-\varepsilon A_3$	0	$\varepsilon A_1$
$B_3$	$B_2$	$-B_1$	0	$\varepsilon A_2$	$-\varepsilon A_1$	0

Using the optimal Hamiltonian (7), it is possible to construct the Hamiltonian vector fields using the Poisson bracket defined on the symplectic manifold. The Poisson bracket is associated with the Lie bracket by  $\{M_i, M_j\} = -\hat{p}([A_i, A_j])$ , where  $p_i = \hat{p}(B_i)$  and  $M_i = \hat{p}(A_i)$  for  $i = 1, 2, 3$ . Therefore, deriving:

$$\begin{aligned} \frac{dM_1}{dt} &= \{M_1, H^*\} \\ &= \{M_1, p_1\} + \frac{M_1}{c_1} \{M_1, M_1\} + \frac{M_2}{c_2} \{M_1, M_2\} + \\ &\frac{M_3}{c_3} \{M_1, M_3\} = 0 + 0 - \frac{1}{c_2} M_2 M_3 + \frac{1}{c_3} M_3 M_2 \\ &= \frac{c_2 - c_3}{c_2 c_3} M_2 M_3 \end{aligned} \quad (12)$$

the remaining derivations of the Hamiltonian vector fields are left to the reader and yield:

$$\left\{ \begin{aligned} \frac{dM_1}{dt} &= \{M_1, H^*\} = \frac{-M_2 M_3}{c_2} + \frac{M_2 M_3}{c_3} \\ \frac{dM_2}{dt} &= \{M_2, H^*\} = \frac{M_1 M_3}{c_1} - \frac{M_1 M_3}{c_3} + p_3 \\ \frac{dM_3}{dt} &= \{M_3, H^*\} = \frac{-M_1 M_2}{c_1} + \frac{M_1 M_2}{c_2} - p_2 \\ \frac{dp_1}{dt} &= \{p_1, H^*\} = \frac{-M_2 p_3}{c_2} + \frac{p_2 M_3}{c_3} \\ \frac{dp_2}{dt} &= \{p_2, H^*\} = \frac{M_1 p_3}{c_1} - \frac{p_1 M_3}{c_3} + \varepsilon M_3 \\ \frac{dp_3}{dt} &= \{p_3, H^*\} = -\frac{M_1 p_2}{c_1} + \frac{p_1 M_2}{c_2} - \varepsilon M_2 \end{aligned} \right. \quad (13)$$

where  $p_i = \hat{p}(B_i)$ . The Casimir functions are constant on co-adjoint orbits of  $G$ , they are integrals of motion for any left-invariant Hamiltonian  $H$ . In addition the intersection of the surface described by the Casimir functions and the energy surface described by the Hamiltonian gives a geometric interpretation of the extremal solutions. A treatment of the three dimensional case is given in [8]. However, in the study of 6-dimensional Hamiltonian systems, there is an additional Casimir function. These are derived explicitly in [9] using the property that the Cartan-Killing form  $\langle L, \cdot \rangle = -\varepsilon \frac{1}{2} \text{Trace}(L, \cdot)$  is invariant, where  $L$  is the projection of extremal solutions on the Lie algebra, specifically

$$L = \begin{pmatrix} 0 & -\varepsilon p_1 & -\varepsilon p_2 & -\varepsilon p_3 \\ p_1 & 0 & -M_3 & M_2 \\ p_2 & M_3 & 0 & -M_1 \\ p_3 & -M_2 & M_1 & 0 \end{pmatrix} \quad (14)$$

then calculating  $\langle L, L \rangle$  and  $\langle L^2, L^2 \rangle$  gives the Casimir functions  $I_2$  and  $I_3$  respectively. In the Euclidean case the Cartan-Killing form is non-degenerate, however, it is shown

in [10] that the Casimir functions can be derived using a combination of the Euclidean inner product on  $\mathfrak{p}$  and the Cartan-Killing form on  $\mathfrak{k}$ . The Hamiltonian and these Casimir functions are:

$$H = p_1 + \frac{1}{2} \left( \frac{M_1^2}{c_1} + \frac{M_2^2}{c_2} + \frac{M_3^2}{c_3} \right) \quad (15)$$

$$I_2 = p_1^2 + p_2^2 + p_3^2 + \varepsilon(M_1^2 + M_2^2 + M_3^2) \quad (16)$$

$$I_3 = p_1 M_1 + p_2 M_2 + p_3 M_3 \quad (17)$$

$H$ ,  $I_2$  and  $I_3$  are all constants of motion. Thus these functions are constant along the Hamiltonian flow and geometrically interpret hypersurfaces. The extremal solutions must exist on each of these surfaces and thus are defined at their intersection. For simplicity let  $c_1 = c_2 = c_3 = 1$  in (13), which is analogous to a particular case of the integrable Lagrange top in the Euclidean case, and immediately notice that

$$\frac{dM_1}{dt} = 0 \quad (18)$$

and therefore  $M_1$  is a constant of motion denoted as  $k$ , the Hamiltonian vector fields (13) become

$$\left\{ \begin{aligned} \frac{dM_2}{dt} &= p_3 \\ \frac{dM_3}{dt} &= -p_2 \\ \frac{dp_1}{dt} &= p_2 M_3 - M_2 p_3 \\ \frac{dp_2}{dt} &= p_3 k - M_3 p_1 + \varepsilon M_3 \\ \frac{dp_3}{dt} &= p_1 M_2 - k p_2 - \varepsilon M_2 \end{aligned} \right. \quad (19)$$

Therefore, the equations of motion in the Hamiltonian can be written in a reduced form. Proceeding to solve for the optimal controls:

$$\frac{dp_1}{dt} = p_2 M_3 - M_2 p_3 \quad (20)$$

$$\therefore \left( \frac{dp_1}{dt} \right)^2 = p_2^2 M_3^2 + p_3^2 M_2^2 - 2p_2 p_3 M_2 M_3$$

Multiplying equation (15) by equation (16) write;

$$\begin{aligned} I_2 - p_1^2 - \varepsilon(k^2) &= p_2^2 + p_3^2 + \varepsilon(M_2^2 + M_3^2) \\ 2(H - p_1) - k^2 &= M_2^2 + M_3^2 \\ \therefore (I_2 - p_1^2 - \varepsilon(k^2))(2(H - p_1) - k^2) \\ &= p_2^2 M_2^2 + p_2^2 M_3^2 + p_3^2 M_2^2 + p_3^2 M_3^2 + \varepsilon((M_2^2 + M_3^2)^2) \end{aligned} \quad (21)$$

Another useful relation comes from the Casimir function (17), writing this in a reduced form and squaring gives:

$$\begin{aligned} I_3 - p_1 k &= p_2 M_2 + p_3 M_3 \\ \therefore (I_3 - p_1 k)^2 &= p_2^2 M_2^2 + p_3^2 M_3^2 + 2p_2 M_2 p_3 M_3 \end{aligned} \quad (22)$$

Therefore, substituting (22) and (21) into (20) yields:

$$\begin{aligned} f(p_1) &= \left( \frac{dp_1}{dt} \right)^2 = (I_2 - p_1^2 - \varepsilon(k^2))(2(H - p_1) - k^2) \\ &\quad - (I_3 - p_1 k)^2 - \varepsilon((2H - k^2 - 2p_1)^2) \end{aligned} \quad (23)$$

The function  $f(p_1)$  is then a cubic function of  $p_1$  and the qualitative behavior of the system will depend on where the roots of this cubic lie. Explicit solutions of (23) can be solved in terms of elliptic functions, see [11] and [2]. Proceeding to solve for  $M_2$  and  $M_3$  using the Casimir function (15) where  $M_1$  is a constant  $k$ , the reduced Casimir is:

$$M_2^2 + M_3^2 = 2(H - p_1) - k^2 \quad (24)$$

This suggests using polar coordinates for  $M_2$  and  $M_3$  ( $M_3 \neq 0$ );

$$\theta = \arctan\left(\frac{M_2}{M_3}\right) \quad (25)$$

$$\dot{\theta} = \frac{M_3 \dot{M}_2 - M_2 \dot{M}_3}{M_2^2 + M_3^2} \quad (26)$$

substituting in the values for  $M_2$  and  $M_3$  from (19) gives

$$\begin{aligned} \dot{\theta} &= \frac{M_3 p_3 - M_2 p_2}{M_2^2 + M_3^2} \\ &= \frac{I_3 - k p_1}{2(H - p_1) - k^2} \end{aligned} \quad (27)$$

Solving explicitly for the radius in (24) gives:

$$r = \sqrt{2(H - p_1) - k^2} \quad (28)$$

so, the optimal controls are

$$\begin{aligned} M_1(t) &= k \\ M_2(t) &= r(t) \cos(\theta(t)) \\ M_3(t) &= r(t) \sin(\theta(t)) \end{aligned} \quad (29)$$

Note that if  $p_1$  is constant in (27) and (28) then the optimal controls  $M_2$  and  $M_3$  in (29) degenerate to trigonometric functions (a geometric interpretation of the extremal solutions at these singularities is given in Section (III)). With the  $M_1$  constant denoted  $k$ , the Casimir functions reduce to:

$$H - \frac{1}{2}k^2 = p_1 + \frac{1}{2}(M_2^2 + M_3^2) \quad (30)$$

$$I_2 - \varepsilon k^2 = p_1^2 + p_2^2 + p_3^2 + \varepsilon(M_2^2 + M_3^2) \quad (31)$$

$$I_3 - k = \frac{p_2 M_2 + p_3 M_3}{p_1} \quad (32)$$

The left hand side of the equations (30), (31) and (32) are constants and thus a reduced form of the Casimir functions. Each of these equations implicitly describe an invariant surface. (30) describes an elliptic paraboloid, (31) describes the sphere for  $\varepsilon=0$ , the 6-dimensional sphere for  $\varepsilon=1$  and finally a non-generic 5-dimensional surface for  $\varepsilon=-1$ . The Casimir function (32) also implicitly defines a non-generic 5-dimensional surface. It is possible to write the two Casimir functions as a single surface. Rearranging (32) in terms of  $p_2$ , squaring and substituting in (31) gives:

$$\begin{aligned} \frac{1}{M_2^2} (\varepsilon M_2^2 (k^2 + M_2^2 + M_3^2) + I_3^2 p_1^2 + k^2 p_1^2 + M_2^2 p_2^2 \\ + 2k M_3 p_1 p_3 + M_2^2 p_3^2 + M_3^2 p_3^2 - 2I_3 p_1 (k p_1 + M_3 p_3)) = I_2 \end{aligned} \quad (33)$$

This is a 4-dimensional hypersurface and the intersection of this with the Hamiltonian function (30) gives us the extremal solutions. It is interesting to note that a classical picture of an elliptic curve is the smooth intersection of two quadric hypersurfaces in projective three space [5]. In the Euclidean case where  $\varepsilon = 0$ , the Hamiltonian and the function (33) implicitly define two quadric surfaces and the extremal solutions can be expressed in terms of elliptic curves, reinforcing the classical picture. In the non-Euclidean case the hypersurface (33) is not a quadric, although the explicit solutions are in terms of elliptic functions.

### III. DEGENERATE SOLUTIONS ON $\mathfrak{se}(3)^*$ , $\mathfrak{so}(4)^*$ , $\mathfrak{so}(1,3)^*$ AND THEIR GEOMETRY

Recall that  $p_1$  is a solution of the cubic equation (23). There is a qualitative difference in the solutions depending on where the three roots of the cubic lie. At any one of these roots the solutions for the optimal controls degenerate from elliptic functions to trigonometric functions as  $p_1$  is constant. A plot of the real roots (singularities) are given in Fig.(1) for  $\varepsilon = 0$ . The figure illustrates the real roots  $p_1$  as a function of  $k$ .

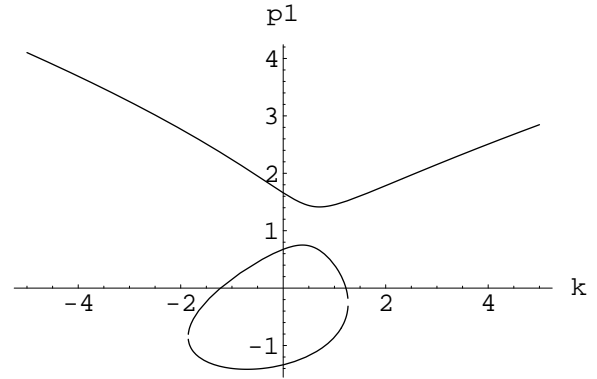


Fig. 1. The singularities of the system  $\varepsilon=0$

For  $\varepsilon=1$  and  $\varepsilon=-1$ , the analysis is also restricted to the real roots of the cubic (23). Continuing to study the system when  $p_1$  is constant and denoting this constant as  $c$ , the Casimir functions (15), (16) and (17) reduce further to:

$$2(H - c) - k^2 = M_2^2 + M_3^2 \quad (34)$$

$$I_2 - c^2 - \varepsilon(k^2) = p_2^2 + p_3^2 + \varepsilon(M_2^2 + M_3^2) \quad (35)$$

$$I_3 - ck = p_2 M_2 + p_3 M_3 \quad (36)$$

where the left hand side of these equations are all constants along the Hamiltonian flow. Expressing equation (36) in terms of  $p_2$  and squaring gives;

$$p_2^2 = \frac{(I_3^2 - 2I_3 ck + c^2 k^2) - 2I_3 p_3 M_3 + 2ck p_3 M_3 + p_3^2 M_3^2}{M_2^2} \quad (37)$$

defining new constants  $\alpha = (I_3^2 - 2I_3 ck + c^2 k^2)$  and  $\beta = I_2 - c^2 - \varepsilon(k^2)$  for simplicity, then by substituting  $p_2^2$  into (35)

the reduced Casimir function can be written as a non-generic quadric surface in  $p_3, M_2$  and  $M_3$ , giving:

$$\begin{aligned}
 2(H - c) - k^2 &= M_2^2 + M_3^2 \\
 \beta M_2^2 &= \alpha + p_3^2 M_2^2 + 2ckp_3 M_3 - 2I_3 p_3 M_3 \\
 &+ p_3^2 M_2^2 + \varepsilon(M_2^4 + M_2^2 M_3^2)
 \end{aligned} \quad (38)$$

Proceeding more geometrically we analyze the solutions in terms of the intersection of these two invariant surfaces. It is necessary for illustration purposes to consider only the critical values of  $p_1$  that are real and give a positive reduced Hamiltonian (the left hand side of the first equation in (38) is positive). In these cases the Hamiltonian invariant surface (cylinder), intersects the reduced Casimir surface (non-generic) see Fig.(2) for  $\varepsilon = 0$ , Fig.(3) for  $\varepsilon = 1$  and Fig.(4) for  $\varepsilon = -1$ . In all of these cases the vertical axis is the  $p_3$  variable, and the horizontal axis are  $M_2$  and  $M_3$  respectively:

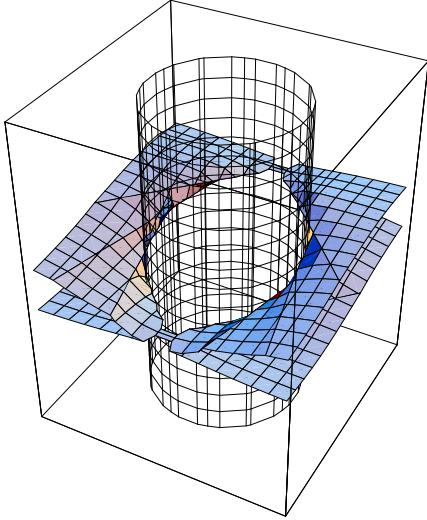


Fig. 2. The Euclidean case: intersection of invariant surfaces

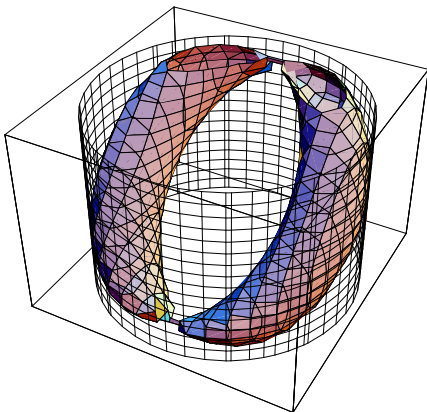


Fig. 3. The elliptic case: intersection of invariant surfaces

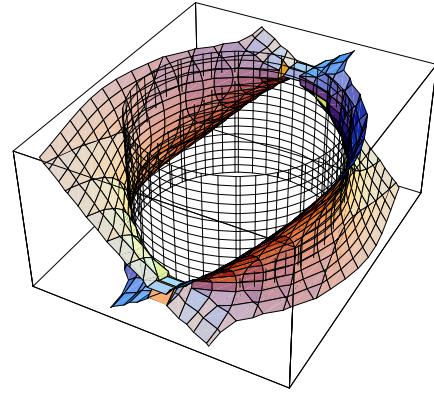


Fig. 4. The hyperbolic case: intersection of invariant surfaces

These surfaces were drawn using ImplicitPlot3D code written by Steven Wilkinson for Mathematica see <http://library.wolfram.com/infocenter/MathSource/4189/> and as the name suggests enables one to plot surfaces that are implicitly defined in 3 dimensions. The surfaces make contact and in each case the intersection is a closed periodic orbit. Fig.(5) shows the points of intersection for  $\varepsilon = 0$ , this remains qualitatively unchanged in each case  $\varepsilon = 1$  and  $\varepsilon = -1$ .

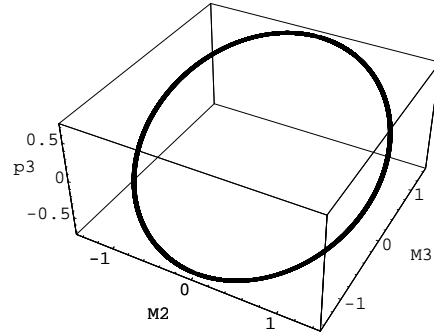


Fig. 5. Closed periodic orbit; intersection of invariant surfaces

At this singularity ( $\frac{dp_1}{dt} = 0$ ) the Hamiltonian vector fields (19) reduce to;

$$\begin{aligned}
 \frac{dM_2}{dt} &= p_3 \\
 \frac{dM_3}{dt} &= -p_2 \\
 \frac{dp_2}{dt} &= kp_3 - cM_3 + \varepsilon M_3 \\
 \frac{dp_3}{dt} &= cM_2 - kp_2 - \varepsilon M_2 \\
 \frac{dp_1}{dt} &= 0 \Rightarrow p_2 = \frac{M_2 p_3}{M_3}
 \end{aligned} \quad (39)$$

in the frame  $(M_2, M_3, p_3)$  these Hamiltonian vector fields

can be expressed independently of  $p_2$  as

$$\begin{aligned}\frac{dM_2}{dt} &= p_3 \\ \frac{dM_3}{dt} &= -\frac{M_2 p_3}{M_3} \\ \frac{dp_3}{dt} &= M_2\left(c - k\frac{p_3}{M_3} - \varepsilon\right)\end{aligned}\quad (40)$$

Changing the initial conditions to perturb away from the periodic orbit will mean that the constant  $p_1 = c$  will have to change if it is to remain a constant, i.e. the Casimir functions (34), (35) and (36) constrain the flow. The radius (28) will then change accordingly and the extremal solution will again be a closed periodic orbit with a different radius. From (40) it is deduced that the flow of these closed periodic orbits are in a clockwise direction. From (40) it can be seen that there is a fixed point at  $p_3 = M_2 = 0$ , this corresponds to the explicit solutions when  $r = 0$  in (28) i.e.  $2(H - p_1) - k^2 = 0$ , in other words when the reduced Hamiltonian is zero the periodic orbit degenerates to a fixed point. Perturbing away from the periodic orbit such that  $p_1$  does not change accordingly to remain a constant, then  $p_1$  will be an elliptic function. Consequently the dynamics will then be described by the Hamiltonian vector fields (19).

#### IV. CONCLUSION

The optimal controls and the extremal solutions for these systems are explicitly solved in terms of elliptic functions using the Hamiltonian formalism of Pontryagin's maximum principle. In addition it is shown that at a real singularity of the system with positive Hamiltonian function the optimal controls degenerate from elliptic to trigonometric functions. This is an interesting case as it shows that trigonometric controls are in some sense optimal. Indeed trigonometric functions have been used to control systems for the Euclidean group of motion  $SE(3)$ , (see [1]). The elliptic solutions are shown to correspond to the intersection of hypersurfaces and in particular, for the Euclidean case the intersection of quadric hypersurfaces in projective 3 space illustrates the classical picture of elliptic curves. Finally, a geometric picture of the extremal solutions at a singularity is given and is shown to be the intersection of the reduced Hamiltonian function (cylinder) and a non-generic quadric dependent on  $\varepsilon$ . At this singularity the extremal solutions are shown to be periodic and closed. The differential equations describing these periodic orbits can be interpreted as a constrained dynamical system embedded in a higher dimension Hamiltonian system. Current and future research concerns the corresponding elastic curves in the base spaces  $\mathbb{E}^3$ ,  $\mathbb{S}^3$  and  $\mathbb{H}^3$  when the optimal controls are trigonometric functions.

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