On a Foundation for Cournot Equilibrium

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Abstract

We show in the context of a bilateral oligopoly where all agents are allowed to behave strategically the unexpected result that when the number of buyers becomes large the outcomes in a strategic market game do not converge to those at the Cournot equilibrium. However, convergence to Cournot outcomes is restored if the game is sequential: sellers move simultaneously as do buyers, but the former always move before the latter. This suggests that the ability to commit to supply decisions is an essential feature of Cournot equilibrium.

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1 Introduction

Cournot competition has become a widely used model of imperfect competition in modern economic theory. Its premise is that whilst firms behave strategically, buyers act as price takers. Just as in Walrasian models, this price-taking assumption requires a strategic foundation in order to be judged valid. A strategic market game with strategies as quantities is an appropriate framework within which to provide this foundation. We would want to show that when the number of buyers increases outcomes in the market game tend to those at the Cournot equilibrium. When the sellers move at the same time as the buyers in the market game this convergence does not generally occur. Conversely, in a sequential two-stage game in which sellers move simultaneously as do buyers, but the former move before the latter, convergence to Cournot outcomes is restored. This suggests that in order to provide the Cournot equilibrium concept with a strategic foundation, the sellers must have an opportunity to commit before the buyers make their choices, implying an essential feature of Cournot equilibrium is the ability of firms to commit to supply decisions.

The framework we use is that of bilateral oligopoly in which there are two commodities, the second thought of as money. Those agents endowed with the first commodity are called *sellers*, whilst those endowed with the second commodity are called *buyers*. In a strategic market game signals are quantity-based: each agent decides on a proportion of her endowment to send to the market to be exchanged for the other commodity. For sellers we call this an *offer*, whilst buyers make *bids*. In a traditional strategic market game of the type developed by Shapley and Shubik [5] the signals of all agents are placed simultaneously and the rate of exchange of the second commodity for the first (the price) is determined by the ratio of the aggregate bid to offer. In a Cournot market the buyers treat the price as uninfluenced by their actions and the sellers play a quantity-setting game with knowledge of the buyers' choices.

The price-taking hypothesis embedded in the Cournot model should be an accurate representation of behaviour for the buyers when their number is many, for then each should have a negligible influence on the price. As such, we would expect that as the number of buyers increases, the equilibrium outcomes in the strategic market game would converge to those at the Cournot equilibrium. In fact this is not generally the case, and we characterise when convergence does and does not occur. This extends the work of Codognato [2] who was the first to notice this phenomenon in the context of an example.

Busetto, Codognato and Ghosal [1] also analysed this issue, noting that it is the two-stage nature of Cournot competition that likely gives rise to the problems. Working in a continuum economy (with some atoms) they show equivalence between the (suitably refined) equilibria in a two-stage market game in which the atoms move first and the Cournot equilibria. Our analysis extends this work to the case of a finite economy where we not only gain intuition regarding the perverse results that emerge, we can also demonstrate asymptotic properties as the number of agents increase, not just the limiting results.

In market games where there are two distinct trading stages and all agents of the same type move at the same time, we are able to construct strategic versions of supply and demand and, using these, compare equilibrium outcomes with those at the Cournot equilibrium. We show that as the number of buyers increases without bound the equilibrium outcomes in the game in which the sellers are leaders converge to that at the Cournot equilibrium, whilst when the buyers are leaders outcomes remain distinct even in the limit.

Thus, in order to provide a strategic foundation for Cournot competition we require those agents that are permitted to behave strategically in the Cournot market to move first whilst those that are assumed to be price takers move second. If play is simultaneous or the timing order is reversed, the limit remains distinct from the Cournot equilibrium. This suggests that the essence of a strategically-behaving agent in Cournot competition is their ability to commit to their choices before those agents that behave as price takers. In any market regime where they are not able to make such a commitment, the Cournot equilibrium does not transpire as the natural limit.

The rest of the paper continues as follows. We briefly analyse a strategic market game where moves are simultaneous using the methodology from Dickson and Hartley [3] that exploits the aggregative properties of the game played. This allows us to construct strategic versions of supply and demand, and show that equilibria correspond to their intersection, a technique that is repeated in all the games we analyse. We then characterise the Cournot equilibrium and compare this with the market game equilibrium when the number of buyers increases, showing that generally the limit remains distinct. Next we turn to analyse two-stage strategic market games in which all the sellers move first and all the buyers move second, and the reverse timing structure. This analysis allows us to show that when the sellers are leaders convergence to the Cournot equilibrium always occurs, whilst if the buyers are leaders the equilibrium remains distinct from the Cournot equilibrium even in the many-buyer limit.

2 The Economic Framework

Throughout we consider the pure exchange economy ${}^{m,n}\mathcal{E} = \{(e_h, u_h, \mathbb{R}^2_+) : h \in {}^{m,n}H\}$. There are two commodities. The first is a standard consumption commodity, the second a commodity money. We partition the set of agents ${}^{m,n}H$ into $mH^{\mathrm{S}} \cup nH^{\mathrm{B}}$, $H^{\mathrm{S}} \cap H^{\mathrm{B}} = \emptyset$. Our ultimate intention is to have a fixed set of sellers and a variable set of buyers whose number may be increased through replication, in which case we fix m and increase n. Those in H^{S} are endowed only with the consumption commodity and so are termed sellers, those in H^{B} are endowed only with the commodity money and so we call them buyers.

In this bilateral oligopoly model we essentially have a partial equilibrium environment where the second commodity can be thought of as representing money used to acquire all other goods in the economy. As the consumption commodity in such circumstances is likely to have a low share in overall expenditure, the most natural assumption is that there are negligible income effects. Indeed, Marshall comments: "When a person buys anything for his own consumption, he generally spends on it a small part of his total resources;...[In such a] case there is no appreciable change in his willingness to part with money" (Marshall [4], pp 335). Zero income effects may be captured by assuming agents have preferences quasi-linear in the first commodity and we make the assumption that buyers are endowed with such preferences: $u_h(x_1, x_2) = v_h(x_1) + x_2 \, \forall h \in H^{\mathrm{B}}$. In addition, if the sellers are endowed with these preferences they may be thought of as profit-maximising firms in the standard sense, a natural interpretation in our environment, so we also make this assumption[1].

Assumption (Quasi-linearity)**.** *For all $h \in H^{\mathrm{S}} \cup H^{\mathrm{B}}$ preferences are representable by the quasi-linear utility function $v_h(x_1) + x_2$ where $v_h'(\cdot) \geq 0$ and $v_h''(\cdot) < 0$.*

In a strategic market game there is a trading post to which each seller may take a proportion of her endowment of the consumption commodity $q \in [0, e_h]$, which we call an *offer*, to be exchanged for money. Likewise, each buyer may take along a proportion of her endowment of the commodity money $b \in [0, e_h]$, called a *bid*, to be exchanged for the consumption commodity. The trading post then aggregates the offers and bids of each replica to Q and B, so the aggregate offer is $\mathcal{Q} = mQ$ and aggregate bid is $\mathcal{B} = nB$. Sellers are awarded their proportional share of the aggregate bid: $\frac{q}{\mathcal{Q}}\mathcal{B}$, whilst the number of units of the consumption commodity buyers are awarded equals their proportional share of the aggregate offer: $\frac{b}{\mathcal{B}}\mathcal{Q}$. If either $\mathcal{Q} = 0$ or $\mathcal{B} = 0$ no trade takes place. Payoffs take the form

$$v_h(e_h - q) + \frac{q}{\mathcal{Q}}\mathcal{B} \, \forall h \in H^{\mathrm{S}} \text{ and}$$
$$v_h\left(\frac{b}{\mathcal{B}}\mathcal{Q}\right) + e_h - b \, \forall h \in H^{\mathrm{B}}.$$

[1]Whilst we assume quasi-linearity of preferences throughout, many of our results hold for more general utility functions.

If play is simultaneous all agents approach the trading post at the same time and the equilibrium concept we use is (type-symmetric) Nash equilibrium in pure strategies. In a two-stage game those that move second observe the choices of those that move in the first stage. In this case we consider subgame-perfect Nash equilibria.

We could alternatively define the rate of exchange of the commodity money for the consumption commodity (i.e. the price of the consumption commodity) in the strategic market game as the ratio of aggregate bid to aggregate offer: $p = \frac{\mathcal{B}}{\mathcal{Q}}$, and state that allocations take the form

$$v_h(e_h - q) + qp \, \forall h \in H^{\mathrm{S}} \text{ and}$$
$$v_h\left(\frac{b}{p}\right) + e_h - b \, \forall h \in H^{\mathrm{B}}.$$

In a Cournot (oligopoly) market the buyers are assumed to behave as price-takers whilst the sellers are permitted to behave strategically in the knowledge of the buyers' behaviour. Each buyer can be thought of as choosing a level of b given that her allocation will be $(x_{h1}, x_{h2}) = (b/p, e_h - b)$ and the fact that she treats the price as a parameter. This corresponds to a standard competitive maximisation problem resulting from which will be a competitive demand schedule. Then the sellers know that if their aggregate supply is $\mathcal{Q}$ the price will be such that this supply is matched with demand. We denote this price $\tilde{p}(\mathcal{Q})$. Then the payoff to seller $h \in H^{\mathrm{S}}$ from the ensuing non-cooperatively played quantity-setting game is

$$v_h(e_h - q) + q\tilde{p}(q + \mathcal{Q}_{-h}).$$

Again we consider (type-symmetric) pure strategy Nash equilibria.

3 The Strategic Market Game

In this section we undertake an analysis of the simultaneous-move strategic market game. Recently Dickson and Hartley [3] have performed such an analysis that exploits the aggregative structure of the game in order to characterise behaviour at the aggregate level consistent with Nash equilibrium, and to provide a characterisation of equilibrium. We refer the reader to the original paper for the details of this process.

By considering the 'partial game' played by each side of the market when the strategies of the other side remain fixed, we can characterise the behaviour of each replica of sellers using what we call strategic supply, denoted ${}^m\mathcal{X}_1^{\mathrm{S}}(p)$, and the behaviour of a replica of buyers by strategic demand, denoted ${}^n\mathcal{X}_1^{\mathrm{B}}(p)$. For a given price strategic supply gives the sum of the offers of a replica of sellers consistent with a Nash equilibrium at that price. Likewise, strategic demand gives the level of demand (equal to the ratio of aggregate bid to price) consistent with a Nash equilibrium at a given price. Aggregate supply and demand are simply $m\,{}^{m}\mathcal{X}_1^{\mathrm{S}}(p)$ and $n\,{}^{n}\mathcal{X}_1^{\mathrm{B}}(p)$ respectively.

Strategic supply and demand have several desirable properties, summarised in the following two lemmata.

Lemma 1. *Strategic supply ${}^{m}\mathcal{X}_1^{\mathrm{S}}(p)$ is defined for all prices exceeding some lower cutoff ${}^{m}P^{\mathrm{S}}$ where it is a function that is positive, continuous and non-decreasing in p. ${}^{m}P^{\mathrm{S}}$ is the price below which no agent would make a positive offer with this price, and is such that*

$$m \sum_{H^{\mathrm{S}}} \max\left\{0, 1 - \frac{v_h'(e_h)}{{}^{m}P^{\mathrm{S}}}\right\} = 1.$$

Lemma 2. *Strategic demand ${}^{n}\mathcal{X}_1^{\mathrm{B}}(p)$ is defined for all $0 < p < {}^{n}P^{\mathrm{B}}$ where it is a function that is positive, continuous and strictly decreasing in p. ${}^{n}P^{\mathrm{B}}$ is the price above which all buyers have zero demand in an equilibrium with this price, and is such that*

$$n \sum_{H^{\mathrm{B}}} \max\left\{0, 1 - \frac{{}^{n}P^{\mathrm{B}}}{v_h'(0)}\right\} = 1.$$

The purpose of constructing these functions is the following fundamental insight: non-autarkic Nash equilibria in the strategic market game are in one-to-one correspondence with intersections of strategic supply and demand.

Proposition 1. *There is a non-autarkic Nash equilibrium in the economy ${}^{m,n}\mathcal{E}$ in which the price is p if and only if*

$$m\,{}^{m}\mathcal{X}_1^{\mathrm{S}}(p) = n\,{}^{n}\mathcal{X}_1^{\mathrm{B}}(p).$$

This equivalence then gives us a handle on when a non-autarkic Nash equilibrium will exist, and whether it will be unique[2].

Theorem 1. *In any economy ${}^{m,n}\mathcal{E}$, if ${}^{m}P^{\mathrm{S}} \geq {}^{n}P^{\mathrm{B}}$ there is no non-autarkic Nash equilibrium. Conversely, if ${}^{m}P^{\mathrm{S}} < {}^{n}P^{\mathrm{B}}$ there is a single non-autarkic Nash equilibrium in which the price is ${}^{m,n}\hat{p}$ such that $m\,{}^{m}\mathcal{X}_1^{\mathrm{S}}({}^{m,n}\hat{p}) = n\,{}^{n}\mathcal{X}_1^{\mathrm{B}}({}^{m,n}\hat{p})$.*

[2]In addition, there is always an autarkic Nash equilibrium in the simultaneously-played strategic market game. To see this, consider whether the strategies $(\mathbf{0}, \mathbf{0})$ are an equilibrium. When everyone makes a zero offer/bid payoffs are $v_h(e_h)\,\forall h \in H^{\mathrm{S}}$ and $e_h\,\forall h \in H^{\mathrm{B}}$. If any seller considers a unilateral deviation to $q > 0$ then her payoff will be $v_h(e_h - q)$ as there is no bid in the market. Clearly this is worse for her. Likewise, if any buyer considers a unilateral deviation to $b > 0$ her payoff will be $e_h - b < e_h$.

4 Cournot Oligopoly

In this section we intend to provide a characterisation of equilibrium in Cournot oligopoly. In such a market the buyers are assumed to be price takers and, knowing this (and so the resulting demand at any given price) the sellers play a game amongst themselves where their strategic variables are the quantities they supply.

Formally, each buyer wishes to maximise her utility from consumption at each price taking the price as given and uninfluenced by her choice of action. In this bilateral oligopoly setting she may be seen as choosing the amount of money she is willing to forego at each price in exchange for b/p units of the consumption commodity. As such, competitive demand for each replica of buyers will take the form

$$\tilde{\mathcal{X}}_1^{\mathrm{B}}(p) = \sum_{H^{\mathrm{B}}} \frac{\tilde{\mathfrak{b}}_h(p)}{p}.$$

where $\tilde{\mathfrak{b}}_h(p)$ is the solution in b to $\max_{b \in [0,e_h]} v_h(b/p) + e_h - b$. Aggregate demand is simply $n\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$.

It transpires that competitive demand has several desirable properties.

Lemma 3. *Competitive demand $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$ is zero for all $p \geq \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ whilst for $0 < p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ it is a function that is positive, continuous and strictly decreasing in the price.*

In equilibrium supply must equate to demand at the aggregate level. If the supply from the sellers as a result of their quantity-setting game is $\mathcal{Q}$ then this requires the price to be such that the level of demand at that price is the same as this supply. Thus, the price will be of the form ${}^n\tilde{p}(\mathcal{Q})$ which is defined such that

$${}^n\tilde{p}(\mathcal{Q}) = \{p : \mathcal{Q} = n\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\}.$$

This price functional has several desirable properties, easily discerned from the preceding lemma.

Lemma 4. *${}^n\tilde{p}(\mathcal{Q})$ is a function that is strictly decreasing in $\mathcal{Q}$ with the property that $\lim_{\mathcal{Q}\to 0} {}^n\tilde{p}(\mathcal{Q}) = \max_{H^{\mathrm{B}}}\{v_h'(0)\}$.*

Now, as the sellers are assumed to know the buyers' behaviour before they play their quantity-setting game, they know that if the aggregate offer is $\mathcal{Q}$ then the price will be ${}^n\tilde{p}(\mathcal{Q})$. As such if seller $h \in H^{\mathrm{S}}$ uses the offer q whilst other sellers' offers total $\mathcal{Q}_{-h}$ her payoff takes the form

$$v_h(e_h - q) + q\,{}^n\tilde{p}(q + \mathcal{Q}_{-h}).$$

In order that this program is concave we require that $2^n\tilde{p}'(\mathcal{Q}) + \mathcal{Q}^n\tilde{p}''(\mathcal{Q}) \leq 0$, a restriction that requires total revenue to be concave, which is standard in the literature on the uniqueness of Cournot equilibrium.

Again we can characterise the behaviour of each replica of sellers by constructing strategic supply, in this case denoted ${}^{m,n}\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ which, for each price, gives the supply consistent with a Nash equilibrium in which the price is p (given the behaviour of the buyers). We summarise the properties of strategic supply in the following lemma.

Lemma 5. *Suppose the price functional is such that* ${}^n\tilde{p}'(\mathcal{Q}) + \mathcal{Q}^n\tilde{p}''(\mathcal{Q}) \leq 0$*. Then strategic supply* ${}^{m,n}\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ *is defined for all* $p > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ *where it is a function that is positive, continuous and non-decreasing in* p*.*

The condition ${}^n\tilde{p}'(\mathcal{Q}) + \mathcal{Q}^n\tilde{p}''(\mathcal{Q}) \leq 0$ is slightly stronger than $2^n\tilde{p}'(\mathcal{Q}) + \mathcal{Q}^n\tilde{p}''(\mathcal{Q}) \leq 0$ (as ${}^n\tilde{p}'(\mathcal{Q}) < 0$), but again is a standard condition seen in the literature on Cournot competition.

Having characterised the (competitive) behaviour of the buyers and the strategic behaviour of the sellers at the aggregate level, we are now in a position to discuss the identification of Nash equilibria in the Cournot game. It transpires that there will be a non-autarkic Nash equilibrium if and only if the aggregate strategic supply of the sellers intersects with the aggregate competitive demand of the buyers.

Proposition 2. *Suppose the price functional is such that* ${}^n\tilde{p}'(\mathcal{Q}) + \mathcal{Q}^n\tilde{p}''(\mathcal{Q}) \leq 0$*. Then there is a non-autarkic Nash equilibrium in the economy* ${}^{m,n}\mathcal{E}$ *with price* p *if and only if*

$$m^{m,n}\tilde{\mathcal{X}}_1^{\mathrm{S}}(p) = n\tilde{\mathcal{X}}_1^{\mathrm{B}}(p).$$

This equivalence between intersections of strategic supply in the Cournot market and competitive demand and non-autarkic Nash equilibria then allows us to determine exactly when a non-autarkic Nash equilibrium will exist, and if so whether it is unique.

Theorem 2. *Suppose the price functional is such that* ${}^n\tilde{p}'(\mathcal{Q}) + \mathcal{Q}^n\tilde{p}''(\mathcal{Q}) \leq 0$*. Then in any economy* ${}^{m,n}\mathcal{E}$*, if* $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} \geq \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ *there is no non-autarkic Nash equilibrium; the only equilibrium is autarky. Conversely, if* $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ *there is a single non-autarkic Nash equilibrium (and no autarkic equilibrium) in which the price is* ${}^{m,n}\hat{p}^{\mathrm{C}}$ *such that* $m^{m,n}\tilde{\mathcal{X}}_1^{\mathrm{S}}({}^{m,n}\hat{p}^{\mathrm{C}}) = n\tilde{\mathcal{X}}_1^{\mathrm{B}}({}^{m,n}\hat{p}^{\mathrm{C}})$*.*

5 Non-Convergence

The aim of this paper is to provide a strategic foundation for Cournot competition. In a Cournot oligopoly the buyers are assumed to behave as price takers, and we expect this model to be valid when the number of buyers is many and each has no market power with which she can manipulate the market price. Such an equilibrium concept uses price-taking assumptions and implicit in the market clearing requirement is the notion of a Walrasian Auctioneer. In order to provide the equilibrium concept with a strategic foundation we would like to show that the price-taking assumption is justified; that is, price-taking behaviour is the natural limit of strategic behaviour in the market game as the number of buyers increases without bound.

Thus, our next task resides in checking whether the outcomes in the market game converge to those at the Cournot equilibrium in the many-buyer limit. In order to achieve this we fix the set of sellers (setting $m = 1$ and dropping the ^{m} notation, and denoting by Q (rather than $\mathcal{Q}$) the aggregate offer) and consider a sequence of economies $\{^n\mathcal{E}\}_{n=1}^{\infty}$. Our task is made easier by the results we have gained so far. In particular, in any economy both the Cournot equilibrium and the strategic market game equilibrium are unique so the sequences of these will be single-valued. As such, in order to show convergence to the Cournot equilibrium, *and* that any limit Cournot equilibrium can be supported as the limit of the sequence of strategic market game equilibria, it will suffice to demonstrate the first result[3]. We focus on the case in which the Cournot equilibrium is non-autarkic, i.e. $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$.

The allocation structure in both the strategic market game and the Cournot market take the same form; namely

$$(x_{h1}, x_{h2}) = \begin{cases} (e_h - q, qp) & \text{if } h \in H^{\mathrm{S}} \text{ or} \\ \left(\frac{b}{p}, e_h - b\right) & \text{if } h \in H^{\mathrm{B}}. \end{cases}$$

Then to demonstrate convergence in equilibrium outcomes it will suffice to demonstrate convergence in the equilibrium price and in individual bids and offers, for then convergence in allocations will follow.

We first show that as the number of buyers increases their strategic demand in the strategic market game converges to their competitive demand for all prices.

Lemma 6. *Strategic demand in the simultaneously-played strategic market game converges to competitive demand as the number of buyers increases without bound, i.e.*

$$^n\mathcal{X}_1^{\mathrm{B}}(p) \to_{n\to\infty} \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\,\forall 0 < p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}.$$

Thus, at any given price the behaviour of the buyers in the market game converges to that were they to behave as price takers as their number increases.

[3]Our analysis is, however, limited to pure strategy equilibria; we do not consider mixed strategies.

The equilibrium in the strategic market game is identified by the intersection of strategic supply $\mathcal{X}_1^{\mathrm{S}}(p)$ and strategic demand $n^n\mathcal{X}_1^{\mathrm{B}}(p)$. According to the above lemma, in the many-buyer limit this intersection corresponds to the intersection of strategic supply and competitive demand. The Cournot equilibrium on the other hand is identified by the intersection of strategic supply in the Cournot market ${}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ and competitive demand ${}^n\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$.

As such, the intersection point (and therefore the equilibrium price and quantity of the consumption commodity traded) in the strategic market game will converge to that at the Cournot equilibrium if and only if the strategic supply in the strategic market game is the same as that in the Cournot market in the limit, at least in a neighborhood of the Cournot equilibrium. If it is larger then the limit equilibrium price will be lower and quantity traded higher. If it is smaller then the limit equilibrium price will be higher and quantity traded smaller.

Thus, the key to our convergence (indeed non-convergence) argument lies in determining exactly when the two strategic supplies (in the strategic market game and in the Cournot oligopoly) are equal, and the following lemma addresses this point.

Lemma 7. *Suppose the price functional is such that* ${}^n\tilde{p}'(Q) + Q{}^n\tilde{p}''(Q) \leq 0$. *Then for any* n

$$\mathcal{X}_1^{\mathrm{S}}(p) \gtreqless {}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p) \Leftrightarrow {}^n\eta({}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p), p) \lesseqgtr 1$$

where ${}^n\eta(Q,p) = \left|\frac{p}{Q}\frac{1}{{}^n\tilde{p}'(Q)}\right|$ *is the elasticity of competitive demand.*

Thus, in any economy ${}^n\mathcal{E}$, even the limit economy, strategic supply in the strategic market game will exceed that in the Cournot market at a given price if competitive demand evaluated at the Cournot supply is inelastic at that price. If it is elastic the strategic market game strategic supply will be less than that in the Cournot market. Only when the elasticity of demand is unity will the two be equal. In particular, at the Cournot equilibrium in which the price is ${}^n\hat{p}^{\mathrm{C}}$ we have that

$$\mathcal{X}_1^{\mathrm{S}}({}^n\hat{p}^{\mathrm{C}}) \gtreqless \tilde{\mathcal{X}}_1^{\mathrm{S}}({}^n\hat{p}^{\mathrm{C}}) \Leftrightarrow {}^n\eta({}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}({}^n\hat{p}^{\mathrm{C}}), {}^n\hat{p}^{\mathrm{C}}) \lesseqgtr 1.$$

In a neighborhood of the intersection between strategic supply in the Cournot oligopoly and competitive demand, the strategic supply in the strategic market game will be equal to the Cournot strategic supply if and only if the elasticity of competitive demand at the Cournot equilibrium is unity.

As such, the price and aggregate quantity of the consumption commodity traded at the limit strategic market game equilibrium will coincide with those at the limit Cournot equilibrium if and only if the elasticity of competitive demand at the Cournot equilibrium is unity.

Proposition 3. *Suppose the price functional is such that* ${}^n\tilde{p}'(Q) + Q^n\tilde{p}''(Q) \leq 0$. *Then the price and aggregate amount of the consumption commodity traded in the strategic market game equilibrium converge to those at the Cournot equilibrium as the number of buyers increases without bound if and only if the elasticity of demand at the limit Cournot equilibrium is unity.*

If competitive demand is inelastic, the price in the strategic market game in the limit will be lower and the quantity traded higher than those at the Cournot equilibrium. Conversely, if it is elastic, the price in the limit strategic market game will be higher and the quantity traded lower.

It only remains to show that in the limit the individual bids and offers are the same in the strategic market game as in the Cournot market if we achieve convergence in the equilibrium price and quantity of the consumption commodity traded (i.e. if the elasticity is one). For the sellers this follows immediately from the realisation that the elasticity must be one if we achieve convergence, which implies shares and, therefore, offers must be the same in the limit (see the proof of Lemma 7). For the buyers, we revealed in the proof of Lemma 6 that for each buyer $h \in H^{\mathrm{B}}$ we have that $nBS_h(nB, p) \to_{n\to\infty} \tilde{\mathfrak{b}}_h(p)\,\forall B > 0$, $\forall p$, so convergence in the price implies convergence in individual bids.

Thus, when the demand elasticity is unity we find that the sequence of equilibrium outcomes in the strategic market game converge to the Cournot equilibrium outcome as the number of buyers increases without bound. Conversely, when competitive demand is not unit elastic at the Cournot oligopoly equilibrium convergence will not (generically at least) occur.

Notice that even though the Cournot equilibrium is non-autarkic (by presumption) it may be the case that the only strategic market game equilibrium is autarky even in the limit. Indeed, since[4] $P^{\mathrm{S}} > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$, even if $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ it is not ruled out that $P^{\mathrm{S}} \geq \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ (so the only limit strategic market game equilibrium is autarky). Moreover, there is also always an autarkic equilibrium in the strategic market game even if a non-autarkic equilibrium exists. Of course, this autarkic equilibrium never converges to the (non-autarkic) Cournot equilibrium.

Our results here generalise and extend the work of Codognato [2], who showed that convergence occurs in the case of Cobb-Douglas preferences (in which case demand is everywhere unit elastic) whereas when the buyers have preferences quasi-linear in the second commodity the equilibrium remains distinct even in the limit.

[4] To see this, note that $1 = \sum_{H^{\mathrm{S}}} \max\left\{0, 1 - \frac{v_h'(e_h)}{P^{\mathrm{S}}}\right\} \leq |H^{\mathrm{S}}|\left(1 - \frac{\min_{H^{\mathrm{S}}}\{v_h'(e_h)\}}{P^{\mathrm{S}}}\right)$ and this implies $P^{\mathrm{S}} \geq \frac{|H^{\mathrm{S}}|}{|H^{\mathrm{S}}|-1}\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$.

6 Two-Stage Strategic Market Games

The result we have presented seems rather paradoxical: one would expect that as the market power of one side of the market diminishes the equilibrium would tend to that which assumes they are price takers. But this only occurs in the special case where the elasticity of competitive demand at the Cournot equilibrium is unity. In any other case the sequence of equilibria remains distinct from the Cournot equilibrium even in the limit. This suggests that the simultaneous-move strategic market game is not an appropriate framework in which to provide a strategic foundation for Cournot equilibrium.

The case in which the competitive demand is unit elastic and convergence is achieved is intriguing. This singular case of convergence presents itself for the following reason: the elasticity of 'demand' in the strategic market game is always unity as well. To see this note that the price is $p = \mathcal{B}/\mathcal{Q}$, so that $\mathrm{d}p/\mathrm{d}\mathcal{Q} = -\mathcal{B}/\mathcal{Q}^2$ and then we find that the elasticity is

$$\left| \frac{p}{\mathcal{Q}} \frac{1}{\frac{\mathrm{d}p}{\mathrm{d}\mathcal{Q}}} \right| = \frac{\mathcal{B}}{\mathcal{Q}^2} \frac{\mathcal{Q}^2}{\mathcal{B}} = 1.$$

Analytically, non-convergence emerges because in the Cournot market the sellers have information about the buyers' choices before they make their quantity decisions whereas in the strategic market game no such *ex ante* inference is possible: sellers make conjectures about demand but these are made simultaneous to their quantity decisions. This is a fundamental difference in the information that the sellers have. In this case, even as the number of buyers increases without bound the sellers' behaviour remains inherently different to that were they to infer demand prior to making their decisions, even though the buyers are behaving as if they are price takers in the strategic market game. But in the case where the elasticity of competitive demand is unity the detail about the buyers' demand available to the sellers in the Cournot market gives no more information than is available (by inference) in the strategic market game (as the demand has the same 'structure').

This suggests that in order to align the strategic market game with the Cournot equilibrium in the limit, we need for the sellers to infer, before they play their quantity setting game, the decisions of the buyers. Another way of putting this is that we need the sellers to commit to their quantity choices before the buyers make their choices. One way of doing this is via a market game that has two distinct trading stages in which all the sellers move at the first stage whilst all the buyers move at the second stage.

We now turn to an analysis of two-stage strategic market games in which the order of moves is exogenously specified, and we focus on the cases where all agents of the same 'type' move at the same time. We begin with the game in which the sellers move first and the buyers move second, followed by an analysis of the reverse timing structure.

6.1 Sellers as Leaders

There are two trading stages. At the first stage the sellers approach the trading post; each seller may offer a proportion of her endowment $q \in [0, e_h]$ to be exchanged for money. At the second stage, having observed the sellers' actions, each buyer then approaches the trading post and offers a proportion of her endowment of money $b \in [0, e_h]$ to be exchanged for the consumption commodity. At the end of the second stage the offers are aggregated to $\mathcal{Q}$ and the bids are aggregated to $\mathcal{B}$. The rate of exchange of the commodity money for the consumption commodity is determined as $p = \mathcal{B}/\mathcal{Q}$ and allocations are determined in the usual way so that the payoff to each seller $h \in H^{\mathrm{S}}$ is $v_h(e_h - q) + qp$ whilst that to each buyer $h \in H^{\mathrm{B}}$ is $v_h(b/p) + e_h - b$. This dynamic game of complete but imperfect information is well defined, as the set of players, their available strategies, their payoffs and the order of moves are all specified.

The equilibrium concept we use is subgame-perfect Nash equilibrium (henceforth SPNE). To identify such an equilibrium we need to fix the set of offers in the first stage (so we specify the subgame) and then compute the optimal actions of the buyers when the offers take said values, and repeat for all possible offer combinations. We then use the optimal (re)actions of the buyers in the sellers' payoff functions (as they infer these actions) and determine a set of mutually consistent best responses from the sellers given the reactions of the buyers.

Let us fix the offers from the sellers and consider the second-stage game played by the buyers. If the offers of the sellers total $\mathcal{Q}$ then each buyer will seek to solve the problem

$$\max_{b \in [0, e_h]} v_h \left(\frac{b}{b + \mathcal{B}_{-h}} \mathcal{Q} \right) + e_h - b.$$

This problem corresponds exactly to that in the simultaneous-move game. Our analysis there suggested that in any 'partial game' in which the aggregate offer was fixed there would be a unique set of strategies consistent with equilibrium (share correspondences, and strategic supply correspondences, were functions). As this is a fundamental result in this dynamic game we formalise it in the following proposition.

Proposition 4. *In any subgame in which the offers of the sellers are $\{q_h\}_{h \in H^{\mathrm{S}}}$ there is a unique Nash equilibrium among the buyers moving at the second stage. Moreover, in any subgame in which the aggregate offer takes some specified value*

$\mathcal{Q}$, the equilibrium among the buyers is the same (i.e. it is independent of the microstructure of $\mathcal{Q}$).

This fact allows us to focus on SPNE rather than considering Markov perfect equilibria, as Busetto, Codognato and Ghosal [1] do. The reason is because in any subgame we have a unique equilibrium and it is the same in subgames that have the same aggregate offer. Thus, regardless of how that aggregate offer is constructed, buyers will play the same strategies, so no payoff irrelevant considerations are necessary.

Now, at the per-replica level, the buyers' behaviour can be characterised by their consistent bids in response to the aggregate offer, or, indeed, by their strategic demand function ${}^{n}\mathcal{X}_1^{\mathrm{B}}(p)$.

The sellers, moving at the first stage infer that, if their collective actions result in an aggregate offer of $\mathcal{Q}$ then the price will be such that this supply equates to the demand forthcoming at such a price. Thus, the price will take the form ${}^{n}\dot{p}(Q)$ which is such that

$$ {}^{n}\dot{p}(\mathcal{Q}) = \{p : \mathcal{Q} = n^{n}\mathcal{X}_1^{\mathrm{B}}(p)\}. $$

From Lemma 2 we know that strategic demand is a function that is continuous and strictly decreasing in the price. This implies that for any $\mathcal{Q}$ there will be a single such price, i.e. ${}^{n}\dot{p}(\mathcal{Q})$ will be a function. This, along with other properties, are summarised in the following lemma. The proof parallels that of Lemma 4 and so is omitted.

Lemma 8. *${}^{n}\dot{p}(\mathcal{Q})$ is a function that is continuous and strictly decreasing in $\mathcal{Q}$ with the property that ${}^{n}\dot{p}(\mathcal{Q}) \to_{\mathcal{Q}\to 0} {}^{n}P^{\mathrm{B}}$.*

Sellers infer that this is the price they face and so the payoff to each seller moving at the first stage is

$$ v_h(e_h - q) + q^{n}\dot{p}(q + \mathcal{Q}_{-h}), $$

and each will choose her level of q to maximise this payoff given the offers of the other sellers totalling $\mathcal{Q}_{-h}$. In order to ensure this program is concave we require some conditions on the price functional. In particular, sufficient to ensure concavity of the objective is to have $2^{n}\dot{p}'(\mathcal{Q}) + \mathcal{Q}^{n}\dot{p}''(\mathcal{Q}) \leq 0$. A more useful condition, and one we use in the sequel, is ${}^{n}\dot{p}'(\mathcal{Q}) + \mathcal{Q}^{n}\dot{p}''(\mathcal{Q}) \leq 0$ which, since $\dot{p}'(\mathcal{Q}) < 0$ implies $2^{n}\dot{p}'(\mathcal{Q}) + \mathcal{Q}^{n}\dot{p}''(\mathcal{Q}) \leq 0$. This condition places restrictions on the buyers' preferences in very much the same way that the analogous condition used in a Cournot oligopoly does. For brevity we omit the derivation of these conditions.

When ${}^{n}\dot{p}'(\mathcal{Q}) + \mathcal{Q}^{n}\dot{p}''(\mathcal{Q}) \leq 0$ we can consider the best response of each agent as the solution to the first order condition of the above problem. Simple calculations show that the best response takes the form

$$ {}^{n}\dot{\mathrm{BR}}_{h}^{\mathrm{S}}(\mathcal{Q}_{-h}) = \begin{cases} 0 & \text{if } {}^{n}\dot{p}(\mathcal{Q}_{-h}) \le v_h'(e_h) \text{ or} \\ \min\{{}^{n}\dot{\mathrm{br}}_{h}^{\mathrm{S}}(\mathcal{Q}_{-h}), e_h\} & \text{if } {}^{n}\dot{p}(\mathcal{Q}_{-h}) > v_h'(e_h) \end{cases} $$

where

$$ {}^{n}\dot{\mathrm{br}}_{h}^{\mathrm{S}}(\mathcal{Q}_{-h}) = \{q : v_h'(e_h - q) = {}^{n}\dot{p}(q + \mathcal{Q}_{-h}) + q\,{}^{n}\dot{p}'(q + \mathcal{Q}_{-h})\}\,. $$

We consider offers consistent with a SPNE in which the aggregate offer is $\mathcal{Q}$ and the price is p. By replacing $\mathcal{Q}_{-h}$ with $\mathcal{Q} - q$ and ${}^{n}\dot{p}(q + \mathcal{Q}_{-h})$ with p we find the replacement correspondence, and by dividing elements in the replacement correspondence by $\mathcal{Q}$ we find the share correspondence of each seller in this two-stage market game. This takes the form

$$ {}^{n}\dot{S}_{h}^{\mathrm{S}}(\mathcal{Q}, p) = \begin{cases} 0 & \text{if } p \le v_h'(e_h) \text{ or} \\ \min\left\{{}^{n}\dot{s}_{h}^{\mathrm{S}}(\mathcal{Q}, p), \frac{e_h}{\mathcal{Q}}\right\} & \text{if } p > v_h'(e_h) \end{cases} $$

where

$$ {}^{n}\dot{s}_{h}^{\mathrm{S}}(\mathcal{Q}, p) = \{s : v_h'(e_h - s\mathcal{Q}) = p + s\mathcal{Q}\,{}^{n}\dot{p}'(\mathcal{Q})\}\,. $$

When multiplied by $\mathcal{Q}$, elements in the share correspondence give the offers of seller $h \in H^{\mathrm{S}}$ consistent with a Nash equilibrium in which the aggregate offer is $\mathcal{Q}$ and the price is p.

At any given price we then look for the consistent aggregate offers; those that generate individual offers that sum to the aggregate offer, or where the aggregate share function is equal to one. At a type-symmetric equilibrium these consistent offers will be equal to mQ where Q is the consistent offer of one replica of sellers which we call strategic supply, and denote by ${}^{m,n}\dot{\mathcal{X}}_{1}^{\mathrm{S}}(p)$ the correspondence that contains such offers:

$$ {}^{m,n}\dot{\mathcal{X}}_{1}^{\mathrm{S}}(p) = \{Q : m \sum_{H^{\mathrm{S}}} {}^{n}\dot{S}_{h}^{\mathrm{S}}(mQ, p) = 1\}. $$

We summarise the properties of strategic supply in the following lemma.

Lemma 9. *Suppose the price functional is such that* ${}^{n}\dot{p}'(\mathcal{Q}) + \mathcal{Q}\,{}^{n}\dot{p}''(\mathcal{Q}) \le 0$*. Then strategic supply* ${}^{m,n}\dot{\mathcal{X}}_{1}^{\mathrm{S}}(p)$ *is defined for all* $p > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ *where it is a function that is positive, continuous and non-decreasing in* p*.*

We can then use strategic supply at the first stage, and strategic demand at the second stage, to identify SPNE in this two-stage game in which the sellers are leaders.

Proposition 5. *Suppose the price functional is such that* ${}^n\dot{p}'(\mathcal{Q}) + \mathcal{Q}^n\dot{p}''(\mathcal{Q}) \leq 0$. *Then there is a non-autarkic SPNE in the two-stage game where the sellers move first and the buyers move second in which the price is* p *if and only if*

$$m^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(p) = n^n\mathcal{X}_1^{\mathrm{B}}(p).$$

This equivalence between strategic supply and demand then gives us a handle on when a non-autarkic SPNE will exist, and whether it will be unique.

Theorem 3. *Suppose the price functional is such that* ${}^n\dot{p}'(\mathcal{Q}) + \mathcal{Q}^n\dot{p}''(\mathcal{Q}) \leq 0$. *Then in the economy* ${}^{m,n}\mathcal{E}$, *if* $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} \geq {}^nP^{\mathrm{B}}$ *there is no non-autarkic SPNE; only autarky is an equilibrium. Conversely, if* $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < {}^nP^{\mathrm{B}}$ *there is a unique non-autarkic SPNE (and no autarkic equilibrium) in which the price is* ${}^{m,n}\hat{p}^{\mathrm{SB}}$ *such that* $m^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}({}^{m,n}\hat{p}^{\mathrm{SB}}) = n^n\mathcal{X}_1^{\mathrm{B}}({}^{m,n}\hat{p}^{\mathrm{SB}})$.

6.2 Buyers as Leaders

We now address the model with the reverse timing structure; where the buyers move at the first stage and the sellers move second. The analysis is analogous to that performed above, so details are kept brief and all proofs, which parallel those of the previous analysis, are omitted.

At the first stage the buyers approach the trading post and make their bid $b \in [0, e_h]$. Having observed the buyers' bids the sellers then move at the second stage making their offers $q \in [0, e_h]$. Bids are aggregated to $\mathcal{B}$ and offers to $\mathcal{Q}$. Allocations are then determined in the usual way. We then have a well-defined game, and the equilibrium concept we use is again SPNE.

Let us fix the set of bids from the first stage and consider the second-stage game played by the sellers. If the bids of the buyers total $\mathcal{B}$ then each seller will solve the problem

$$\max_{q\in[0,e_h]} v_h(e_h - q) + \frac{q}{q + \mathcal{Q}_{-h}}\mathcal{B}.$$

As this is precisely the same as that in the simultaneous-move game, we know that in any subgame there will be a unique equilibrium among the sellers and if the aggregate bid in any two subgames is the same then the equilibrium in each of these will be the same. The sellers' optimal behaviour can be represented at the per-replica level by their strategic supply ${}^m\mathcal{X}_1^{\mathrm{S}}(p)$. Thus, the buyers infer that if the aggregate offer is $\mathcal{B}$ then in order to clear the market the price must be such that $\mathcal{B}/p = m^m\mathcal{X}_1^{\mathrm{S}}(p)$. We collect such prices in the correspondence ${}^m\ddot{p}(\mathcal{B})$, and summarise its properties in the following lemma.

Lemma 10. ${}^m\ddot{p}(\mathcal{B})$ *is a function that is continuous and strictly increasing in* $\mathcal{B}$ *with the property that* ${}^m\ddot{p}(\mathcal{B}) \to_{\mathcal{B}\to 0} {}^mP^{\mathrm{S}}$.

Then we can write the payoff to a typical buyer as

$$v_h\left(\frac{b}{{}^m\ddot{p}(b+\mathcal{B}_{-h})}\right)+e_h-b$$

which she will maximise over her choice of bid. In order to ensure this program is concave we require that the price functional is such that $2^m\ddot{p}'(\mathcal{B})({}^m\ddot{p}(\mathcal{B})-\mathcal{B}^m\ddot{p}'(\mathcal{B}))+{}^m\ddot{p}(\mathcal{B})\mathcal{B}^m\ddot{p}''(\mathcal{B})\le 0$. Again we resist deriving sufficient conditions on preferences that imply this condition. When the price functional has this property we can write the best response as

$${}^m\ddot{\mathrm{BR}}_h^{\mathrm{B}}(\mathcal{B}_{-h})=\begin{cases}0 & \text{if } {}^m\ddot{p}(\mathcal{B}_{-h})\ge v_h'(0) \text{ or}\\ \min\{{}^m\ddot{\mathrm{br}}_h^{\mathrm{B}}(\mathcal{B}_{-h}),e_h\} & \text{if } {}^m\ddot{p}(\mathcal{B}_{-h})<v_h'(0)\end{cases}$$

where

$${}^m\ddot{\mathrm{br}}_h^{\mathrm{B}}(\mathcal{B}_{-h})=\left\{b: v_h'\left(\frac{b}{{}^m\ddot{p}(b+\mathcal{B}_{-h})}\right)=\frac{{}^m\ddot{p}(b+\mathcal{B}_{-h})^2}{{}^m\ddot{p}(b+\mathcal{B}_{-h})-b^m\ddot{p}'(b+\mathcal{B}_{-h})}\right\}.$$

We construct the share correspondence of each buyer as

$${}^m\ddot{S}_h^{\mathrm{B}}(\mathcal{B},p)=\begin{cases}0 & \text{if } p\ge v_h'(0) \text{ or}\\ \min\left\{{}^m\ddot{s}_h^{\mathrm{B}}(\mathcal{B},p),\frac{e_h}{\mathcal{B}}\right\} & \text{if } p<v_h'(0)\end{cases}$$

where

$${}^m\ddot{s}_h^{\mathrm{B}}(\mathcal{B},p)=\left\{s: v_h'\left(\frac{s\mathcal{B}}{p}\right)=\frac{p^2}{p-s\mathcal{B}^m\ddot{p}'(\mathcal{B})}\right\}.$$

When multiplied by $\mathcal{B}$ this share correspondence gives the bids of buyer $h\in H^{\mathrm{B}}$ consistent with a SPNE in which the aggregate bid of all buyers is $\mathcal{B}$ and the price is p. In order to find consistent per-replica bids we seek, for each price, those bids which are such that the aggregate share correspondence evaluated at $\mathcal{B}=nB$ is equal to one, and we divide the resulting per-replica bid by the price to determine strategic demand which we denote by ${}^{m,n}\ddot{\mathcal{X}}_1^{\mathrm{B}}(p)$.

Lemma 11. *Suppose the price functional is such that ${}^m\ddot{p}'(\mathcal{B})-\mathcal{B}^m\ddot{p}''(\mathcal{B})\le 0$. Then strategic demand ${}^{m,n}\ddot{\mathcal{X}}_1^{\mathrm{B}}(p)$ is defined for all $0<p<\max_{H^{\mathrm{B}}}\{v_h'(0)\}$ where it is a function that is positive, continuous and strictly decreasing in p.*

[We actually need two conditions, the one stated and ${}^m\ddot{p}'(\mathcal{B})+\mathcal{B}^m\ddot{p}''(\mathcal{B})\ge 0$. However, it can be checked that the one stated implies this second condition.]

It then transpires that there is a SPNE in the two-stage game in which the buyers move first if and only if the strategic demand of the first-stage buyers is equal to the strategic supply of the second-stage sellers at the aggregate level.

Proposition 6. *Suppose the price functional is such that* ${}^m\ddot{p}'(\mathcal{B}) - \mathcal{B}^m\ddot{p}''(\mathcal{B}) \leq 0$. *Then there is a non-autarkic SPNE in the two-stage game in which the buyers move first and the sellers move second in which the price is* p *if and only if*

$$m^m\mathcal{X}_1^{\mathrm{S}}(p) = n^{m,n}\ddot{\mathcal{X}}_1^{\mathrm{B}}(p).$$

Again we use this equivalence between intersections of the appropriate strategic supply and demand functions and equilibria to determine exactly when a non-autarkic SPNE will exist.

Theorem 4. *Suppose the price functional is such that* ${}^m\ddot{p}'(\mathcal{B}) - \mathcal{B}^m\ddot{p}''(\mathcal{B}) \leq 0$. *Then in the economy* ${}^{m,n}\mathcal{E}$, *if* ${}^mP^{\mathrm{S}} \geq \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ *there is no non-autarkic SPNE; the only equilibrium is autarky. Conversely, if* ${}^mP^{\mathrm{S}} < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ *there is a unique non-autarkic SPNE (and no autarkic equilibrium) in which the price is* ${}^{m,n}\hat{p}^{\mathrm{BS}}$ *such that* $m^m\mathcal{X}_1^{\mathrm{S}}({}^{m,n}\hat{p}^{\mathrm{BS}}) = n^{m,n}\ddot{\mathcal{X}}_1^{\mathrm{B}}({}^{m,n}\hat{p}^{\mathrm{BS}})$.

7 Redressing Non-Convergence

Our previous analysis highlighted the fact that when we increased the number of buyers without bound in the simultaneous-move strategic market game the sequence of outcomes remained distinct from that at the Cournot equilibrium even in the limit. Our intuition then suggested that it may be the informational aspects of Cournot competition that prevent convergence, and this in turn suggests that a two-stage strategic market game will be more suited to providing such a foundation. In this section we show that when we increase the number of buyers in the two-stage game in which the sellers move first and the buyers move second the sequence of outcomes associated with the sequence of SPNE do indeed converge to the outcome at the Cournot equilibrium in the limit. Conversely, when the timing structure is reversed, so the buyers move first and the sellers move second, we get a non-convergence result that parallels that of the simultaneous-move game. Thus, in order to approach the Cournot equilibrium in the limit the trading regime must give the sellers the opportunity to commit to their quantity choices before the buyers reveal their strategies.

Let us first look at the two-stage game in which the sellers are leaders. We fix the number of sellers (and drop the ^{m} notation and let $Q = \mathcal{Q}$), and consider the sequence of economies $\{{}^n\mathcal{E}\}_{n=1}^{\infty}$ as the number of buyers increases without bound. In the Cournot oligopoly the equilibrium is identified by the intersection of strategic supply in the Cournot market ${}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ and competitive demand $n\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$. In the two-stage strategic market game in which the sellers are leaders the equilibrium is identified by the intersection of strategic supply derived for this game ${}^n\dot{\mathcal{X}}_1^{\mathrm{S}}(p)$

and strategic demand $n^n\mathcal{X}_1^{\mathrm{B}}(p)$. Recall from Lemma 6 that as $n \to \infty$ strategic demand converges to competitive demand. Thus, to ensure that the equilibrium price and quantity of the consumption commodity traded in the two-stage market game with the sellers as leaders converge to those at the Cournot equilibrium we need only make sure that strategic supply in the market game converges to that in the Cournot oligopoly as the number of buyers increases without bound. This fundamental result we demonstrate in the following lemma.

Lemma 12. *Suppose the price functional ${}^n\dot{p}(Q)$ is such that ${}^n\dot{p}'(Q) + Q^n\dot{p}''(Q) \leq 0$. Then strategic supply in the two-stage market game in which the sellers are leaders converges to strategic supply in the Cournot market as the number of buyers increases without bound, i.e.*

$$\lim_{n\to\infty} {}^n\dot{\mathcal{X}}_1^{\mathrm{S}}(p) = \lim_{n\to\infty} {}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)\, \forall p > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}.$$

Thus, not only does strategic demand converge to competitive demand, but strategic supply in the two-stage game with the sellers as leaders is the same as that in the Cournot oligopoly in the many-buyer limit. As such, we know the intersection point of strategic supply and demand in the two-stage game with the sellers as leaders will converge to that at the Cournot equilibrium. This implies that the price and aggregate quantity of the consumption commodity traded at the two-stage market game equilibrium with the sellers as leaders will converge to that at the Cournot equilibrium as the number of buyers increases without bound, and, as the following theorem demonstrates, this is sufficient to guarantee convergence in equilibrium outcomes.

Theorem 5. *Suppose the price functional is such that ${}^n\dot{p}'(Q) + Q^n\dot{p}''(Q) \leq 0$. Then the allocations and price associated with the SPNE in the two-stage market game in which the sellers are leaders converge to those at the Cournot equilibrium as the number of buyers increases without bound.*

This result implies that the two-stage strategic market game in which the sellers move first is an appropriate fully strategic model in which to provide a foundation for Cournot competition.

We next turn to consider the two-stage strategic market game with the reverse timing structure; that in which the buyers move first and the sellers move second. Recall that the SPNE in this case is identified by the intersection of strategic supply of the second-stage sellers $\mathcal{X}_1^{\mathrm{S}}(p)$ (which is the same as in the simultaneous-move game) and the derived strategic demand $n^n\ddot{\mathcal{X}}_1^{\mathrm{B}}(p)$. We recall that in the Cournot oligopoly the equilibrium is identified by the intersection of strategic supply derived in that game ${}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ and competitive demand $n\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$.

We show in the following lemma that the strategic demand in the two-stage game in which the buyers are leaders converges to the competitive demand as the number of buyers increases without bound.

Lemma 13. *Suppose the price functional is such that $\ddot{p}'(\mathcal{B}) - \mathcal{B}\ddot{p}''(\mathcal{B}) \leq 0$. Then strategic demand in the two-stage strategic market game in which the buyers are leaders converges to competitive demand as the number of buyers increases without bound, i.e.*

$$^{n}\ddot{\mathcal{X}}_1^{\mathrm{B}}(p) \rightarrow_{n\rightarrow\infty} \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\,\forall 0 < p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}.$$

Given the way in which we identify equilibria, this implies that in the many-buyer limit the equilibrium price and quantity of the consumption commodity traded in the two-stage game with the buyers as leaders will converge to the Cournot equilibrium if and only if the strategic supply of the sellers in that game converges to their strategic supply in the Cournot oligopoly (at least in a neighborhood of the Cournot equilibrium) in the many-buyer limit. But strategic supply is the same as that in the simultaneous-move game, and we showed in Lemma 7 that this will occur if and only if the elasticity of competitive demand at the Cournot oligopoly equilibrium is unity. We state this formally in the following proposition.

Proposition 7. *Suppose the price functional is such that $\ddot{p}'(\mathcal{B}) - \mathcal{B}\ddot{p}''(\mathcal{B}) \leq 0$. Then the price and aggregate quantity of the consumption commodity traded at the equilibrium in the two-stage game in which the buyers are leaders converge to those at the Cournot equilibrium if and only if the elasticity of competitive demand at the Cournot equilibrium is unity.*

Thus, unless we are in the very specific circumstance in which the elasticity of competitive demand is unity, the outcomes in the two-stage game in which the buyers are leaders will remain distinct from those at the Cournot oligopoly equilibrium, implying this timing structure is not appropriate in providing a foundation for Cournot oligopoly.

8 Conclusion

We have exploited the aggregative structure of the market game in both its static and dynamic forms to derive strategic versions of supply and demand and showed that equilibria correspond to intersections of these. Using this fact we have been able to compare outcomes in the various strategic market games to those at the Cournot equilibrium, in particular in the many-buyer limit.

In order to provide the Cournot equilibrium concept with a strategic foundation we require those agents that are permitted to behave strategically in the Cournot market to move in the first stage whilst those that are assumed to be price takers move in the second stage. This implies that an essential feature of a firm in a Cournot oligopoly is their ability to commit to their supply decisions. Indeed, if firms cannot commit before the buyers make their choices the Cournot equilibrium is (in general) not supported in the many-buyer limit.

We recognise, however, that the timing order is exogenously specified. Working in a finite economy, there is no justification to assume that, for example, the sellers have a desire to move in the first stage and the buyers want to move in the second stage. The next phase of our research project involves endogenising the order of moves in an attempt to show that, at least as the number of buyers increases, each seller finds it in her own best interests to commit and move at the first stage whilst buyers find it in their own best interests to delay and move at the second stage.

A Proofs

Proof of Lemma 1. Each seller $h \in H^{\mathrm{S}}$ may be seen as solving the problem

$$\max_{q\in[0,e_h]} v_h(e_h - q) + \frac{q}{q+\mathcal{Q}_{-h}}\mathcal{B}.$$

The best response is

$$\mathrm{BR}_h^{\mathrm{S}}(\mathcal{Q}_{-h},\mathcal{B}) = \begin{cases} 0 & \text{if } v_h'(e_h) \geq \frac{\mathcal{B}}{\mathcal{Q}_{-h}} \text{ or } \mathcal{Q}_{-h}=0, \text{ or} \\ \min\{\mathrm{br}_h^{\mathrm{S}}(\mathcal{Q}_{-h},\mathcal{B}), e_h\} & \text{otherwise} \end{cases}$$

where

$$\mathrm{br}_h^{\mathrm{S}}(\mathcal{Q}_{-h},\mathcal{B}) = \left\{ q : v_h'(e_h - q) = \frac{\mathcal{Q}_{-h}}{(q+\mathcal{Q}_{-h})^2}\mathcal{B} \right\}.$$

We seek those offers consistent with a Nash equilibrium in which the aggregate offer of all sellers is $\mathcal{Q}$ and the price is p. Such an offer will be a best response to $\mathcal{Q}$ minus itself and $\mathcal{B} = p\mathcal{Q}$, and they are found by replacing $\mathcal{Q}_{-h}$ with $\mathcal{Q}-q$ and $\mathcal{B}/\mathcal{Q}$ with p in the best response correspondence. This gives the replacement correspondence, and by dividing by $\mathcal{Q}$ we get shares of the aggregate offer. This share correspondence takes the form

$$S_h^{\mathrm{S}}(\mathcal{Q},p) = \begin{cases} 0 & \text{if } p \leq v_h'(e_h) \text{ or} \\ \min\left\{ s_h^{\mathrm{S}}(\mathcal{Q},p), \frac{e_h}{\mathcal{Q}} \right\} & \text{if } p > v_h'(e_h) \end{cases}$$

where

$$s_h^{\mathrm{S}}(\mathcal{Q},p) = \{ s : v_h'(e_h - s\mathcal{Q}) = (1-s)p \}.$$

When multiplied by $\mathcal{Q}$ the share correspondence gives the offers of seller $h \in H^{\mathrm{S}}$ consistent with a Nash equilibrium in which the aggregate offer is $\mathcal{Q}$ and the price is p.

In order to find consistent aggregate offers we must find an aggregate offer that generates individual offers that sum to it, i.e. find where the aggregate share correspondence is equal to one: $\sum_{mH^{\mathrm{S}}} S_h^{\mathrm{S}}(\mathcal{Q},p) = 1$. the aggregate offer is composed of m times the per-replica offer, so a per-replica offer Q is consistent if $m\sum_{H^{\mathrm{S}}} S_h^{\mathrm{S}}(mQ,p) = 1$. Thus,

$${}^m\mathcal{X}_1^{\mathrm{S}}(p) = \{ Q : m\sum_{H^{\mathrm{S}}} S_h^{\mathrm{S}}(mQ,p) = 1 \}.$$

In determining the properties of strategic supply the properties of each $S_h^{\mathrm{S}}(\mathcal{Q},p)$ will be of crucial importance. We show next that this share correspondence is in fact a continuous function that is strictly decreasing in $\mathcal{Q}$ and non-decreasing in p. Recall that $s_h^{\mathrm{S}}(\mathcal{Q},p)$ is those value of s such that $v_h'(e_h - s\mathcal{Q}) = (1-s)p$. When $v_h''(\cdot) < 0$, $v_h'(e_h - s\mathcal{Q})$ is increasing in s and so there can be at most one s such that $v_h'(e_h - s\mathcal{Q}) = (1-s)p$: $S_h^{\mathrm{S}}(\mathcal{Q},p)$ is a function. Continuity is implied by continuity of $v_h'(\cdot)$.

We will now show that $S_h^{\mathrm{S}}(\mathcal{Q},p)$ is strictly decreasing in $\mathcal{Q}$ whenever it is positive. It will suffice to show $s_h^{\mathrm{S}}(\mathcal{Q},p)$ is strictly decreasing in $\mathcal{Q}$. Suppose not, so for $\mathcal{Q}' > \mathcal{Q}$ we have $s' = s_h^{\mathrm{S}}(\mathcal{Q}',p) \geq s_h^{\mathrm{S}}(\mathcal{Q},p) = s$. Then we would have $e_h - s'\mathcal{Q}' < e_h - s\mathcal{Q}$ and $(1-s')p \leq (1-s)p$. But then concavity of $v_h(\cdot)$ implies

$$(1-s')p = v_h'(e_h - s'\mathcal{Q}') > v_h'(e_h - s\mathcal{Q}) = (1-s)p,$$

a contradiction. Thus, $S_h^{\mathrm{S}}(\mathcal{Q},p)$ is strictly decreasing in $\mathcal{Q}$. Next we show that $s_h^{\mathrm{S}}(\mathcal{Q},p)$ is strictly increasing in p implying $S_h^{\mathrm{S}}(\mathcal{Q},p)$ is non-decreasing in p. Suppose, contrarily, that for

$p' > p$ we have $s' = s_h^{\mathrm{S}}(\mathcal{Q}, p') \leq s_h^{\mathrm{S}}(\mathcal{Q}, p) = s$. Then we would have $e_h - s'\mathcal{Q} \geq e_h - s\mathcal{Q}$ and $(1-s')p' > (1-s)p$, but then concavity of $v_h(\cdot)$ implies

$$(1-s')p' = v_h'(e_h - s'\mathcal{Q}) \leq v_h'(e_h - s\mathcal{Q}) = (1-s)p,$$

a contradiction. Moreover, it is easy to discern from the definition that as $\mathcal{Q} \to 0$, $s_h^{\mathrm{S}}(\mathcal{Q}, p) \to 1 - \frac{v_h'(e_h)}{p}$ and so $\lim_{\mathcal{Q} \to 0} S_h^{\mathrm{S}}(\mathcal{Q}, p) = \max\left\{0, 1 - \frac{v_h'(0)}{p}\right\}$.

We now seek to aggregate the share functions and determine the property of the solution in $Q = \mathcal{Q}/m$ to this function being equal to one. When $m \sum_{H^{\mathrm{S}}} \max\left\{0, 1 - \frac{v_h'(e_h)}{p}\right\} \leq 1$ we know that as $Q \to 0$ the aggregate share function approaches something less than one. Since the aggregate share function is also strictly decreasing in Q this implies there is no $Q > 0$ such that the aggregate share function is equal to one. Thus, for all $p \leq {}^m P^{\mathrm{S}}$ there is no $Q > 0$ such that $m \sum_{H^{\mathrm{S}}} S_h^{\mathrm{S}}(mQ, p) = 1$.

Conversely, when $p > {}^m P^{\mathrm{S}}$ the aggregate share function will exceed one when Q is small, and, when $Q = \sum_{H^{\mathrm{S}}} e_h$ it will not exceed one as each $S_h^{\mathrm{S}}(mQ, p) \leq \max\left\{1, \frac{e_h}{mQ}\right\}$ implying $m \sum_{H^{\mathrm{S}}} S_h^{\mathrm{S}}(m \sum_{H^{\mathrm{S}}} e_h, p) \leq 1$. Since the aggregate share function is also strictly decreasing in Q this implies there is exactly one $Q \in (0, \sum_{H^{\mathrm{S}}} e_h]$ such that $m \sum_{H^{\mathrm{S}}} S_h^{\mathrm{S}}(mQ, p) = 1$. Thus, strategic supply is a function.

Higher values of p mean each individual share function will be no lower than before, and so the aggregate share function will be no lower than before. Since it is also strictly decreasing in Q this implies that for higher values of p the value of Q consistent with the aggregate share function being equal to one can be no lower. Thus, strategic supply is non-decreasing in the price. □

Proof of Lemma 2. We derive the share correspondence of each buyer by operations analogous to those performed for the sellers. Written as being dependent on $\mathcal{B}$ and p, this takes the form

$$S_h^{\mathrm{B}}(\mathcal{B}, p) = \begin{cases} 0 & \text{if } p \geq v_h'(0) \text{ or} \\ \min\left\{s_h^{\mathrm{B}}(\mathcal{B}, p), \frac{e_h}{\mathcal{B}}\right\} & \text{if } p < v_h'(0) \end{cases}$$

where

$$s_h^{\mathrm{B}}(\mathcal{B}, p) = \left\{s : v_h'\left(\frac{s\mathcal{B}}{p}\right) = \frac{1}{1-s}p\right\}.$$

It is more convenient, however, to write this in terms of the demand for the first commodity $\mathcal{V} = \mathcal{B}/p$, and the price. In this way, we get the share correspondence as

$$\breve{S}_h^{\mathrm{B}}(\mathcal{V}, p) = \begin{cases} 0 & \text{if } p \geq v_h'(0) \text{ or} \\ \min\left\{\breve{s}_h^{\mathrm{B}}(\mathcal{V}, p), \frac{e_h}{\mathcal{V}p}\right\} & \text{if } p < v_h'(0) \end{cases}$$

where

$$\breve{s}_h^{\mathrm{B}}(\mathcal{V}, p) = \left\{s : v_h'(s\mathcal{V}) = \frac{1}{1-s}p\right\}.$$

[Note that $\breve{S}_h^{\mathrm{B}}(\mathcal{V}, p)$ is simply $S_h^{\mathrm{B}}(\mathcal{V}p, p)$.] Strategic demand (consistent levels of $V = \mathcal{V}/n$) is the solution in V to $n \sum_{H^{\mathrm{B}}} \breve{S}_h^{\mathrm{B}}(nV, p) = 1$, so it is the properties of such a share correspondence that will be of crucial importance. In fact, as $v_h'(s\mathcal{V})$ is decreasing in s under our concavity assumption whilst $\frac{1}{1-s}p$ is increasing in s this correspondence will be a function. Continuity is implied by continuity of $v_h'(\cdot)$.

We show next that the share function is strictly decreasing in both $\mathcal{V}$ and p wherever it is positive. It is sufficient to show that $\breve{s}_h^{\mathrm{B}}(\mathcal{V}, p)$ is strictly decreasing in both $\mathcal{V}$ and p. First for $\mathcal{V}$:

suppose, contrarily, that for $\mathcal{V}' > \mathcal{V}$ we have $s' = \breve{s}^{\mathrm{B}}_h(\mathcal{V}', p) \geq \breve{s}^{\mathrm{B}}_h(\mathcal{V}, p) = s$. Then we would have $s'\mathcal{V}' > s\mathcal{V}$ and $\frac{1}{1-s'}p \geq \frac{1}{1-s}p$. But then concavity of $v_h(\cdot)$ implies

$$\frac{1}{1-s'}p = v_h'(s'\mathcal{V}') < v_h'(s\mathcal{V}) = \frac{1}{1-s}p,$$

a contradiction. Now for p: suppose, again to the contrary, that for $p' > p$ we have $s' = \breve{s}^{\mathrm{B}}_h(\mathcal{V}, p') \geq \breve{s}^{\mathrm{B}}_h(\mathcal{V}, p) = s$. Then we would have $s'\mathcal{V} \geq s\mathcal{V}$ and $\frac{1}{1-s'}p' > \frac{1}{1-s}p$. But then concavity implies

$$\frac{1}{1-s'}p = v_h'(s'\mathcal{V}) \leq v_h'(s\mathcal{V}) = \frac{1}{1-s}p,$$

a contradiction.

Note, moreover, that when $\mathcal{V} \to 0$, $\breve{s}^{\mathrm{B}}_h(\mathcal{V}, p) \to 1 - \frac{p}{v_h'(0)}$ and this implies that $\lim_{\mathcal{V}\to 0} \breve{S}^{\mathrm{B}}_h(\mathcal{V}, p) = \max\left\{0, 1 - \frac{p}{v_h'(0)}\right\}$.

Now, strategic demand ${}^n\mathcal{X}^{\mathrm{B}}_1(p)$ is the solution in V to $n\sum_{H^{\mathrm{B}}} \breve{S}^{\mathrm{B}}_h(nV, p) = 1$. When $n\sum_{H^{\mathrm{B}}} \max\left\{0, 1 - \frac{p}{v_h'(0)}\right\} \leq 1$ we know that when $V \to 0$ the aggregate share function is no greater than one. In addition, since individual share functions are strictly decreasing in V the aggregate will inherit this property and so for all $V > 0$ the aggregate share function will be less than one, and strategic demand is undefined in such a case. This occurs for all $p \geq {}^nP^{\mathrm{B}}$ which is defined such that $n\sum_{H^{\mathrm{B}}} \max\left\{0, 1 - \frac{{}^nP^{\mathrm{B}}}{v_h'(0)}\right\} = 1$.

When $0 < p < {}^nP^{\mathrm{B}}$ the aggregate share function will exceed one when V is close to zero. When $V = \sum_{H^{\mathrm{B}}} e_h/p$ the aggregate share function will be less than one since each individual share function has an upper bound $e_h/n\sum_{H^{\mathrm{S}}} e_h$ at this level of V. Since the aggregate share function is strictly decreasing in V this implies there is a single $V \in (0, \sum_{H^{\mathrm{B}}} e_h/p]$ such that $n\sum_{H^{\mathrm{B}}} \breve{S}_h(nV, p) = 1$, so strategic demand will be a function for all $0 < p < {}^nP^{\mathrm{B}}$. To show that it is strictly decreasing in p we note that each individual share function is decreasing in p and hence, so is the aggregate. This, together with the fact that the aggregate share function is decreasing in V also implies that for higher values of p the V such that $n\sum_{H^{\mathrm{B}}} \breve{S}_h(nV, p) = 1$ will be lower, which gives the desired result. $\square$

Proof of Proposition 1. First we show that for every intersection of strategic supply and demand there is a Nash equilibrium. For, suppose $m\,{}^m\mathcal{X}^{\mathrm{S}}_1(\hat{p}) = n\,{}^n\mathcal{X}^{\mathrm{B}}_1(\hat{p})$. Then the aggregate offer is $\hat{\mathcal{Q}} = m\,{}^m\mathcal{X}^{\mathrm{S}}_1(\hat{p})$ and the aggregate bid is $\hat{\mathcal{B}} = n\hat{p}\,{}^n\mathcal{X}^{\mathrm{B}}_1(\hat{p})$, and the individual strategies are

$$\begin{aligned} \hat{q}_h &= \hat{\mathcal{Q}} S^{\mathrm{S}}_h(\hat{\mathcal{Q}}, \hat{p})\,\forall h \in H^{\mathrm{S}} \text{ and} \\ \hat{b}_h &= \hat{\mathcal{B}} S^{\mathrm{B}}_h(\hat{\mathcal{B}}, \hat{p})\,\forall h \in H^{\mathrm{B}}. \end{aligned}$$

But then by definition we have that

$$\begin{aligned} \hat{q}_h &= \mathrm{BR}^{\mathrm{S}}_h(\hat{\mathcal{Q}}_{-h}, \hat{\mathcal{B}})\,\forall h \in H^{\mathrm{S}} \text{ and} \\ \hat{b}_h &= \mathrm{BR}^{\mathrm{B}}_h(\hat{\mathcal{B}}_{-h}, \hat{\mathcal{Q}})\,\forall h \in H^{\mathrm{B}} \end{aligned}$$

implying the strategies form a Nash equilibrium in the game.

Next we show that if there is a Nash equilibrium then strategic supply and demand must intersect. So suppose there is a Nash equilibrium $(\hat{\mathbf{q}}, \hat{\mathbf{b}})$ with aggregate offer $\hat{\mathcal{Q}} = m\hat{Q}$, aggregate bid $\hat{\mathcal{B}} = n\hat{B}$ and price $\hat{p} = n\hat{B}/m\hat{Q}$. Since $\hat{q}_h = \mathrm{BR}^{\mathrm{S}}_h(\hat{\mathcal{Q}}_{-h}, \hat{\mathcal{B}})\,\forall h \in H^{\mathrm{S}}$ and $\hat{b}_h = \mathrm{BR}^{\mathrm{B}}_h(\hat{\mathcal{B}}_{-h}, \hat{\mathcal{Q}})\,\forall h \in H^{\mathrm{B}}$ we must have that

$$\begin{aligned} \hat{q}_h &= m\hat{Q} S^{\mathrm{S}}_h(m\hat{Q}, \hat{p})\,\forall h \in H^{\mathrm{S}} \text{ and} \\ \hat{b}_h &= n\hat{B} S^{\mathrm{B}}_h(n\hat{B}, \hat{p})\,\forall h \in H^{\mathrm{B}}. \end{aligned}$$

But then $m\sum_{H^{\mathrm{S}}} S_h^{\mathrm{S}}(m\hat{Q},\hat{p}) = 1 \Rightarrow \hat{Q} = {}^m\mathcal{X}_1^{\mathrm{S}}(\hat{p})$ and $n\sum_{H^{\mathrm{B}}} S_h^{\mathrm{B}}(n\hat{B},\hat{p}) = 1 \Rightarrow \hat{B}/\hat{p} = {}^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$. Then since $\hat{p} = n\hat{B}/m\hat{Q}$, $m\hat{Q} = n\hat{B}/\hat{p}$ and it follows that $m^m\mathcal{X}_1^{\mathrm{S}}(\hat{p}) = n^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$. □

Proof of Theorem 1. We know that under the stated conditions strategic supply is a continuous function defined for all $p > {}^mP^{\mathrm{S}}$ where it is non-decreasing in the price (Lemma 1) and strategic demand is a continuous function defined for all $0 < p < {}^nP^{\mathrm{B}}$ where it is strictly decreasing in the price (Lemma 2). Moreover, non-autarkic Nash equilibria are in one-to-one correspondence with intersections of strategic supply and demand (Proposition 1). When ${}^mP^{\mathrm{S}} \geq {}^nP^{\mathrm{B}}$ there is no price where strategic supply and strategic demand are both defined, so they cannot intersect. In this case, there is no non-autarkic equilibrium: the only equilibrium is autarky, which always exists. When ${}^mP^{\mathrm{S}} < {}^nP^{\mathrm{B}}$ there is an ϵ^{S} such that $m^m\mathcal{X}_1^{\mathrm{S}}(p) < n^n\mathcal{X}_1^{\mathrm{B}}(p)$ when $p = {}^mP^{\mathrm{S}} + \epsilon^{\mathrm{S}}$, and an ϵ^{B} such that $m^m\mathcal{X}_1^{\mathrm{S}}(p) > n^n\mathcal{X}_1^{\mathrm{B}}(p)$ when $p = {}^nP^{\mathrm{B}} - \epsilon^{\mathrm{B}}$. By continuity, therefore, strategic supply and demand (at the aggregate level) must intersect. Since the former is non-decreasing in the price whilst the latter is strictly decreasing in p, they can intersect only once. Thus, there is a single non-autarkic Nash equilibrium (accompanied, of course, by the autarkic no-trade equilibrium). □

Proof of Lemma 3. Each buyer can be seen as solving the problem

$$\max_{b\in[0,e_h]} v_h\left(\frac{b}{p}\right) + e_h - b$$

the solution to which is

$$\tilde{\mathfrak{b}}_h(p) = \begin{cases} 0 & \text{if } p \geq v_h'(0) \text{ or} \\ \min\{\tilde{b}_h(p), e_h\} & \text{if } p < v_h'(0) \end{cases}$$

where

$$\tilde{b}_h(p) = \left\{b : v_h'\left(\frac{b}{p}\right) = p\right\}.$$

Competitive demand is simply the summation of individual demands at each price:

$$\tilde{\mathcal{X}}_1^{\mathrm{B}}(p) = \sum_{H^{\mathrm{B}}} \frac{\tilde{\mathfrak{b}}_h(p)}{p}.$$

When $p \geq v_h'(0)\,\forall h \in H^{\mathrm{B}}$, i.e. when $p \geq \max_{H^{\mathrm{B}}}\{v_h'(0)\}$, the demand from each buyer, hence at the per-replica and aggregate levels, will be zero. When $p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ there will be some buyers who have positive demand, given by $\tilde{v}_h(p) = \tilde{b}_h(p)/p$. This is the solution in v to $v_h'(v) = p$, which, since $v_h''(\cdot) < 0$, is strictly decreasing in p. Thus, $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$ will be strictly decreasing in p. Continuity derives from continuity of $v_h'(\cdot)$. □

Proof of Lemma 4. The price ${}^n\tilde{p}(\mathcal{Q})$ is that which is consistent with $\mathcal{Q} = n\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$. We know from Lemma 3 that $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$ is strictly decreasing in p when $v_h''(\cdot) < 0$ and so it directly follows that for higher values of $\mathcal{Q}$ the price consistent with $\mathcal{Q} = n\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$ must be lower. Thus, ${}^n\tilde{p}(\mathcal{Q})$ is strictly decreasing in $\mathcal{Q}$. The limit is a consequence of continuity of $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$ and the easily discernable fact that $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p) \to 0$ as $p \to \max_{H^{\mathrm{B}}}\{v_h'(0)\}$. □

Proof of Lemma 5. Each seller $h \in H^{\mathrm{S}}$ may be seen as solving the problem $\max_{q\in[0,e_h]} v_h(e_h - q) + q\,{}^n\tilde{p}(q + \mathcal{Q}_{-h})$. The best response is

$${}^n\tilde{\mathrm{BR}}_h^{\mathrm{S}}(\mathcal{Q}_{-h}) = \begin{cases} 0 & \text{if } {}^n\tilde{p}(\mathcal{Q}_{-h}) \leq v_h'(e_h) \text{ or} \\ \min\{{}^n\tilde{\mathrm{br}}_h^{\mathrm{S}}(\mathcal{Q}_{-h}), e_h\} & \text{if } {}^n\tilde{p}(\mathcal{Q}_{-h}) > v_h'(e_h) \end{cases}$$

where

$$ {}^{n}\tilde{\mathrm{br}}_h^{\mathrm{S}}(\mathcal{Q}_{-h}) = \left\{q : v_h'(e_h - q) = {}^{n}\tilde{p}(q + \mathcal{Q}_{-h}) + q\,{}^{n}\tilde{p}'(q + \mathcal{Q}_{-h})\right\}. $$

Now consider those offers consistent with a Nash equilibrium in which the aggregate offer and price take certain values. By replacing $\mathcal{Q}_{-h}$ with $\mathcal{Q} - q$ and ${}^{n}\tilde{p}(\mathcal{Q})$ with p we find such offers, and by considering shares of the aggregate offer we find the share correspondence of each seller in a Cournot oligopoly. This takes the form

$$ {}^{n}\tilde{S}_h^{\mathrm{S}}(\mathcal{Q}, p) = \begin{cases} 0 & \text{if } p \le v_h'(e_h) \text{ or} \\ \min\left\{{}^{n}\tilde{s}_h^{\mathrm{S}}(\mathcal{Q}, p), \frac{e_h}{\mathcal{Q}}\right\} & \text{if } p > v_h'(e_h) \end{cases} $$

where

$$ {}^{n}\tilde{s}_h^{\mathrm{S}}(\mathcal{Q}, p) = \left\{s : v_h'(e_h - s\mathcal{Q}) = p + s\mathcal{Q}\,{}^{n}\tilde{p}'(\mathcal{Q})\right\}. $$

When multiplied by $\mathcal{Q}$, this correspondence gives those offers of seller $h \in H^{\mathrm{S}}$ consistent with a Cournot equilibrium in which the aggregate offer is $\mathcal{Q}$ and the price is p.

At any given price we then seek those aggregate offers that are consistent, in that they generate individual offers that sum to the aggregate offer. Alternatively, we look for those values of $\mathcal{Q}$ where the sum of all sellers' share correspondences are equal to one. In a type-symmetric economy the aggregate offer will be $\mathcal{Q} = mQ$ and strategic supply is those levels of Q at each price such that the aggregate share function evaluated at mQ is equal to one:

$$ {}^{m,n}\tilde{\mathcal{X}}_1^{\mathrm{S}}(p) = \{Q : m \sum_{H^{\mathrm{S}}} {}^{n}\tilde{S}_h^{\mathrm{S}}(mQ, p) = 1\}. $$

In order to determine the properties of strategic supply we must first determine the properties of individual share correspondences. Since $v_h''(\cdot) < 0$ we know $v_h'(e_h - s\mathcal{Q})$ is increasing in s. Moreover, since ${}^{n}\tilde{p}'(\mathcal{Q}) < 0$ we know $p + s\mathcal{Q}\,{}^{n}\tilde{p}'(\mathcal{Q})$ is decreasing in s. As such, for any p and $\mathcal{Q}$ there will be only a single s such that $v_h'(e_h - s\mathcal{Q}) = p + s\mathcal{Q}\,{}^{n}\tilde{p}'(\mathcal{Q})$, so ${}^{n}\tilde{s}_h^{\mathrm{S}}(\mathcal{Q}, p)$, hence ${}^{n}\tilde{S}_h^{\mathrm{S}}(\mathcal{Q}, p)$, will be a function.

We show next that it is strictly decreasing in $\mathcal{Q}$ and non-decreasing in p. Sufficient is to show that ${}^{n}\tilde{s}_h^{\mathrm{S}}(\mathcal{Q}, p)$ is strictly decreasing in $\mathcal{Q}$ and strictly increasing in p. First for $\mathcal{Q}$: suppose, contrarily, that for $\mathcal{Q}' > \mathcal{Q}$ we have $s' = {}^{n}\tilde{s}_h^{\mathrm{S}}(\mathcal{Q}', p) \ge {}^{n}\tilde{s}_h^{\mathrm{S}}(\mathcal{Q}, p) = s$. Then we would have $e_h - s'\mathcal{Q}' < e_h - s\mathcal{Q}$ and so concavity of $v_h(\cdot)$ implies

$$ p + s'\mathcal{Q}'\,{}^{n}\tilde{p}'(\mathcal{Q}') = v_h'(e_h - s'\mathcal{Q}') > v_h'(e_h - s\mathcal{Q}) = p + s\mathcal{Q}\,{}^{n}\tilde{p}'(\mathcal{Q}). $$

However,

$$ \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\mathcal{Q}}\{s\mathcal{Q}\,{}^{n}\tilde{p}'(\mathcal{Q})\} &= s({}^{n}\tilde{p}'(\mathcal{Q}) + \mathcal{Q}\,{}^{n}\tilde{p}''(\mathcal{Q})) + \frac{\mathrm{d}s}{\mathrm{d}\mathcal{Q}}\{\mathcal{Q}\,{}^{n}\tilde{p}'(\mathcal{Q})\} \\ &\le 0 \end{aligned} $$

as ${}^{n}\tilde{p}'(\mathcal{Q}) + \mathcal{Q}\,{}^{n}\tilde{p}''(\mathcal{Q}) \le 0$, ${}^{n}\tilde{p}'(\mathcal{Q}) < 0$ and $\frac{\mathrm{d}s}{\mathrm{d}\mathcal{Q}} \ge 0$ by presumption, which is a contradiction as the first inequality implies $s'\mathcal{Q}'\,{}^{n}\tilde{p}'(\mathcal{Q}') > s\mathcal{Q}\,{}^{n}\tilde{p}'(\mathcal{Q})$.

Next to show that ${}^{n}\tilde{s}_h^{\mathrm{S}}(\mathcal{Q}, p)$ is strictly increasing in p. In order to demonstrate this we note that

$$ \begin{aligned} \frac{\partial\,{}^{n}\tilde{s}_h^{\mathrm{S}}(\mathcal{Q}, p)}{\partial p} &= -\frac{\frac{\partial}{\partial p}\{v_h'(e_h - s\mathcal{Q}) - p - s\mathcal{Q}\,{}^{n}\tilde{p}'(\mathcal{Q})\}}{\frac{\partial}{\partial s}\{v_h'(e_h - s\mathcal{Q}) - p - s\mathcal{Q}\,{}^{n}\tilde{p}'(\mathcal{Q})\}} \\ &= -\frac{1}{\mathcal{Q}v_h''(e_h - s\mathcal{Q}) + \mathcal{Q}\,{}^{n}\tilde{p}'(\mathcal{Q})} > 0 \end{aligned} $$

as $v_h''(\cdot) < 0$ and $\tilde{p}'(\cdot) < 0$, which gives the result.

Now, when $p > v_h'(e_h)$ and $\mathcal{Q}$ is small we may have situations in which the share function exceeds one. To avoid this economically meaningless case we restrict the domain of the share function to $\mathcal{Q} > \underline{\mathcal{Q}}^h(p)$ where $\underline{\mathcal{Q}}^h(p)$ is such that $v_h'(e_h - \underline{\mathcal{Q}}^h(p)) = p + \underline{\mathcal{Q}}^h(p)^n\tilde{p}'(\underline{\mathcal{Q}}^h(p))$. By definition, ${}^n\tilde{S}_h^{\mathrm{S}}(\underline{\mathcal{Q}}^h(p), p) = 1$.

If $p \leq \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ all sellers' share functions are zero, as is the aggregate. When $p > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ there is a set of sellers, denoted H_*^{S}, for whom $p > v_h'(e_h)$ and who have positive share functions defined for $\mathcal{Q} > \underline{\mathcal{Q}}^h(p)$ which are continuous, bounded above by $\min\left\{1, \frac{e_h}{\mathcal{Q}}\right\}$, strictly decreasing in $\mathcal{Q}$, non-decreasing in p and such that ${}^n\tilde{S}_h^{\mathrm{S}}(\underline{\mathcal{Q}}^h(p), p) = 1$. We take the aggregate share function to be defined for all $\mathcal{Q} \geq \max_{H_*^{\mathrm{S}}}\{\underline{\mathcal{Q}}^h(p)\}$. When $\mathcal{Q} = \max_{H_*^{\mathrm{S}}}\{\underline{\mathcal{Q}}^h(p)\}$ the aggregate share function is no lower than one. When $\mathcal{Q} = m\sum_{H^{\mathrm{S}}} e_h$ it is no higher than one. As it is strictly decreasing in $\mathcal{Q}$ there is a single $Q \in [\max_{H_*^{\mathrm{S}}}\{\underline{\mathcal{Q}}^h(p)\}, \sum_{H^{\mathrm{S}}} e_h]$ such that $m\sum_{H^{\mathrm{S}}} {}^n\tilde{S}_h^{\mathrm{S}}(mQ, p) = 1$. Thus, strategic supply is a function. As individual share functions are non-decreasing in p, so is $\sum_{H^{\mathrm{S}}} {}^n\tilde{S}_h^{\mathrm{S}}(mQ, p)$ and, since this function is strictly decreasing in Q, the value of Q such that $m\sum_{H^{\mathrm{S}}} {}^n\tilde{S}_h^{\mathrm{S}}(mQ, p) = 1$ can be no lower for higher levels of p: ${}^{m,n}\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ is non-decreasing in p. □

Proof of Proposition 2. First we show that if $m^{m,n}\tilde{\mathcal{X}}_1^{\mathrm{S}}(\hat{p}) = n\tilde{\mathcal{X}}_1^{\mathrm{B}}(\hat{p})$ then there must be a Nash equilibrium with price $\hat{p}$. When ${}^n\tilde{p}'(\mathcal{Q}) + \mathcal{Q}^n\tilde{p}''(\mathcal{Q}) \leq 0$ the aggregate offer at price $\hat{p}$ is exactly $\hat{\mathcal{Q}} = m^{m,n}\tilde{\mathcal{X}}_1^{\mathrm{S}}(\hat{p})$ and we will have $\hat{p} = {}^n\tilde{p}(\hat{\mathcal{Q}})$ by definition. For each seller $h \in H^{\mathrm{S}}$ we know $\hat{q}_h = \hat{\mathcal{Q}}^n\tilde{S}_h^{\mathrm{S}}(\hat{\mathcal{Q}}, \hat{p})$ and this implies $\hat{q}_h = {}^n\tilde{\mathrm{BR}}_h(\hat{\mathcal{Q}}_{-h})\,\forall h \in H^{\mathrm{S}}$ in turn implying the strategies $\{\hat{q}_h\}_{h\in H^{\mathrm{S}}}$ form a Nash equilibrium.

Next, suppose the strategies $\{\hat{q}_h\}_{h\in H^{\mathrm{S}}}$ form a Nash equilibrium with aggregate offer $\hat{\mathcal{Q}}$. Then the price will be $\hat{p} = {}^n\tilde{p}(\hat{\mathcal{Q}})$ and demand from the buyers will be $n\tilde{\mathcal{X}}_1^{\mathrm{B}}(\hat{p})$. Since the strategies form a Nash equilibrium we have that $\hat{q}_h = {}^n\tilde{\mathrm{BR}}_h(\hat{\mathcal{Q}}_{-h})\,\forall h \in H^{\mathrm{S}}$ and this implies $\hat{q}_h = \hat{\mathcal{Q}}^n\tilde{S}_h^{\mathrm{S}}(\hat{\mathcal{Q}}, \hat{p})\,\forall h \in H^{\mathrm{S}}$, in turn implying $\hat{Q} = {}^{m,n}\tilde{\mathcal{X}}_1^{\mathrm{S}}(\hat{p})$ because it follows that $m\sum_{H^{\mathrm{S}}} {}^n\tilde{S}_h^{\mathrm{S}}(m\hat{Q}, \hat{p}) = 1$. As $\hat{p} = {}^n\tilde{p}(\hat{\mathcal{Q}})$, this implies $m\hat{Q} = n\tilde{\mathcal{X}}_1^{\mathrm{B}}(\hat{p})$ in turn implying $m^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(\hat{p}) = n\tilde{\mathcal{X}}_1^{\mathrm{B}}(\hat{p})$. □

Proof of Theorem 2. We know from Lemma 3 that competitive demand is positive only when $0 < p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ where it is a continuous strictly decreasing function. From Lemma 5 we know that strategic supply is defined only for $p > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ where it is a function that is positive, continuous and non-decreasing in p. Thus, if $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} \geq \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ strategic supply and demand never intersect at a positive level and the only equilibrium is autarky. Conversely, when $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ they intersect once and only once by arguments analogous to those presented in the proof of the former uniqueness theorem. Then applying Proposition 2 we get our result. Unlike in the simultaneously-played strategic market game, autarky is not always an equilibrium in a Cournot market: when a non-autarkic equilibrium exists, it is the only equilibrium. □

Proof of Lemma 6. We will first show that for each $h \in H^{\mathrm{B}}$

$$nBS_h^{\mathrm{B}}(nB, p) \to_{n\to\infty} \mathfrak{b}_h(p)\,\forall B > 0,\,\forall p.$$

The magnitude $nBS_h^{\mathrm{B}}(nB, p)$ is equivalent to the replacement function $R_h^{\mathrm{B}}(\mathcal{B}, p)$ which gives the bid of a buyer consistent with a Nash equilibrium in which the aggregate bid is $\mathcal{B}$ and the price is p. It is found by replacing $\mathcal{B}_{-h}$ with $\mathcal{B} - b$ and $\mathcal{B}/\mathcal{Q}$ with p in the best response function, and is such that

$$R_h^{\mathrm{B}}(\mathcal{B}, p) = \begin{cases} 0 & \text{if } p \geq v_h'(0) \text{ or} \\ \min\{r_h^{\mathrm{B}}(\mathcal{B}, p), e_h\} & \text{if } p < v_h'(0) \end{cases}$$

where

$$r_h^{\mathrm{B}}(\mathcal{B},p)=\left\{b : v_h'\left(\frac{b}{p}\right)=\frac{1}{1-\frac{b}{\mathcal{B}}}p\right\}.$$

It will suffice to show that $r_h^{\mathrm{B}}(nB,p)\to_{n\to\infty}\tilde{b}_h(p)\,\forall B>0,\,\forall p$. This is obviously true as $\frac{1}{1-\frac{b}{nB}}p\to_{n\to\infty}p\,\forall B>0,\,\forall p$ implying $r_h^{\mathrm{B}}(nB,p)$ tends to the solution in b to $v_h'\left(\frac{b}{p}\right)=p$, which is precisely $\tilde{b}_h(p)$.

Then we have that

$$nB\sum_{H^{\mathrm{B}}}S_h^{\mathrm{B}}(nB,p)\to_{n\to\infty}\sum_{H^{\mathrm{B}}}\tilde{\mathfrak{b}}_h(p)\,\forall B>0,\,\forall p.$$

B is positive when $0<p<{}^nP^{\mathrm{B}}$ and one can check that $\lim_{n\to\infty}{}^nP^{\mathrm{B}}=\max_{H^{\mathrm{B}}}\{v_h'(0)\}$. Then setting $B=p^n\mathcal{X}_1^{\mathrm{B}}(p)$ and dividing by p we find

$$n^n\mathcal{X}_1^{\mathrm{B}}(p)\sum_{H^{\mathrm{B}}}S_h^{\mathrm{B}}(np^n\mathcal{X}_1^{\mathrm{B}}(p),p)\to_{n\to\infty}\sum_{H^{\mathrm{B}}}\frac{\tilde{\mathfrak{b}}_h(p)}{p}=\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\,\forall 0<p<\max_{H^{\mathrm{B}}}\{v_h'(0)\}.$$

But $n\sum_{H^{\mathrm{B}}}S_h^{\mathrm{B}}(np^n\mathcal{X}_1^{\mathrm{B}}(p),p)=1$ by definition, so

$${}^n\mathcal{X}_1^{\mathrm{B}}(p)\to_{n\to\infty}\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\,\forall 0<p<\max_{H^{\mathrm{B}}}\{v_h'(0)\},$$

which is the desired result. $\square$

Proof of Lemma 7. When ${}^n\eta(Q,p)=\left|\frac{p}{Q}\frac{1}{{}^n\tilde{p}'(Q)}\right|\lesseqgtr 1$ we have

$$\begin{aligned}-p&\gtreqless Q^n\tilde{p}'(Q)\Rightarrow\\(1-s)p&\gtreqless p+sQ^n\tilde{p}'(Q)\end{aligned}$$

by multiplying the first inequality by s and adding p to each side. Now, $s_h^{\mathrm{S}}(Q,p)$ is that s where $v_h'(e_h-sQ)=(1-s)p$, whilst ${}^n\tilde{s}_h^{\mathrm{S}}(Q,p)$ is that s where $v_h'(e_h-sQ)=p+sQ^n\tilde{p}'(Q)$. Since $v_h'(e_h-sQ)$ is increasing in s (by concavity of $v_h(\cdot)$) it follows that when $(1-s)p\gtreqless p+sQ^n\tilde{p}'(Q)$ we have that $s_h^{\mathrm{S}}(Q,p)\gtreqless {}^n\tilde{s}_h(Q,p)$. Thus, it follows that

$${}^n\eta(Q,p)\lesseqgtr 1\Leftrightarrow\sum_{H^{\mathrm{S}}}S_h^{\mathrm{S}}(Q,p)\gtreqless\sum_{H^{\mathrm{S}}}{}^n\tilde{S}_h(Q,p).$$

Setting $Q={}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ we get that

$${}^n\eta({}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p),p)\lesseqgtr 1\Leftrightarrow\sum_{H^{\mathrm{S}}}S_h^{\mathrm{S}}({}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p),p)\gtreqless\sum_{H^{\mathrm{S}}}{}^n\tilde{S}_h({}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p),p)=1.$$

When $\sum_{H^{\mathrm{S}}}S_h^{\mathrm{S}}({}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p),p)\gtreqless 1$ the value of Q that ensures equality with one, which is precisely $\mathcal{X}_1^{\mathrm{S}}(p)$, must be $\gtreqless {}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ since the function $\sum_{H^{\mathrm{S}}}S_h^{\mathrm{S}}(Q,p)$ is strictly decreasing in Q. As such,

$${}^n\eta({}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p),p)\lesseqgtr 1\Leftrightarrow\mathcal{X}_1^{\mathrm{S}}(p)\gtreqless{}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p).$$

$\square$

Proof of Proposition 3. If the elasticity of competitive demand at the Cournot equilibrium is unity we know that strategic supply in the strategic market game is the same as that in the Cournot market in a neighborhood of the price at the Cournot equilibrium. Moreover, we know from Lemma 6 that strategic demand converges to competitive demand as $n \to \infty$. This implies that the price and aggregate quantity of the consumption commodity traded in the strategic market game will converge to those at the Cournot equilibrium. Let $\lim_{n\to\infty} {}^n\hat{p}^{\mathrm{C}} = \hat{p}^{\mathrm{C}}$ and $\lim_{n\to\infty} {}^n\hat{Q}^{\mathrm{C}} = \hat{Q}^{\mathrm{C}}$. Then ${}^n\hat{p} \to_{n\to\infty} \hat{p}^{\mathrm{C}}$ and ${}^n\hat{Q} \to_{n\to\infty} \hat{Q}^{\mathrm{C}}$. Moreover, we know that when the elasticity of competitive demand is one $S_h^{\mathrm{S}}(Q,p)$ and ${}^n\tilde{S}_h^{\mathrm{S}}(Q,p)$ coincide for each $h \in H^{\mathrm{S}}$ and this implies that at a given price and aggregate offer individual offers will coincide. As the price and aggregate offer converge we thus find ${}^n\hat{q}_h = S_h^{\mathrm{S}}({}^n\hat{Q}, {}^n\hat{p}) \to_{n\to\infty} \tilde{S}_h^{\mathrm{S}}(\hat{Q}^{\mathrm{C}}, \hat{p}^{\mathrm{C}}) = \hat{q}_h^{\mathrm{C}}$ (where $\tilde{S}_h^{\mathrm{S}}(Q,p) = \lim_{n\to\infty} {}^n\tilde{S}_h^{\mathrm{S}}(Q,p)$). In addition, for the buyers we recall from the proof of Lemma 6 that $nBS_h^{\mathrm{B}}(nB,p) \to_{n\to\infty} \tilde{\mathfrak{b}}_h(p)$ and this, combined with the fact that the price converges and ${}^n\hat{B} = {}^n\hat{p}\,{}^n\hat{Q} \to_{n\to\infty} \hat{p}^{\mathrm{C}}\hat{Q}^{\mathrm{C}} = \hat{B}^{\mathrm{C}}$ implies individual bids will converge: ${}^n\hat{b}_h \to_{n\to\infty} \hat{b}_h^{\mathrm{C}}\,\forall h \in H^{\mathrm{B}}$. Since the allocation mechanism is the same we thus see convergence in allocations and prices in the many-buyer limit.

Conversely, when the elasticity of competitive demand is not unity the equilibrium price and aggregate offer will not converge in the limit, and so generically we will see a discrepancy between offers and bids in the limit, implying allocations and prices will not converge. □

Proof of Proposition 4. We could infer this from our previous analysis (in the proof of Lemma 2) concerning share functions, but for completeness we show it directly. Fix the offers of the sellers at $\{q_h\}_{h\in H^{\mathrm{S}}}$ so we specify the subgame. In this subgame the aggregate offer is $\mathcal{Q}$. As the maximisation problem of each buyer is the same as in the simultaneous-move game we know her best response will be $\mathrm{BR}_h^{\mathrm{B}}(\mathcal{B}_{-h}, \mathcal{Q})$. Let us consider her bids consistent with an equilibrium in this subgame in which the aggregate bid is $\mathcal{B}$. Such a bid will be a best response to $\mathcal{B}$ minus itself (and $\mathcal{Q}$) and these (or rather their ratio to $\mathcal{B}$) can be represented by the share correspondence

$$\check{S}_h^{\mathrm{B}}(\mathcal{B}, \mathcal{Q}) = \begin{cases} 0 & \text{if } v_h'(0) \le \frac{\mathcal{B}}{\mathcal{Q}} \text{ or} \\ \min\left\{\check{s}_h^{\mathrm{B}}(\mathcal{B}, \mathcal{Q}), \frac{e_h}{\mathcal{B}}\right\} & \text{if } v_h'(0) > \frac{\mathcal{B}}{\mathcal{Q}} \end{cases}$$

where

$$\check{s}_h^{\mathrm{B}}(\mathcal{B}, \mathcal{Q}) = \left\{ s : v_h'(s\mathcal{Q}) = \frac{1}{1-s}\frac{\mathcal{B}}{\mathcal{Q}} \right\}.$$

The properties of this share correspondence are outlined in the following lemma.

Lemma 14. *The share correspondence $\check{S}_h^{\mathrm{B}}(\mathcal{B}, \mathcal{Q})$ is a function that is continuous and, where positive, strictly decreasing in $\mathcal{B} > 0$. When $\mathcal{B} > \bar{\mathcal{B}}^h(\mathcal{Q})$ (which is equal to $v_h'(0)\mathcal{Q}$) it is identically zero. When $0 < \mathcal{B} < \bar{\mathcal{B}}^h(\mathcal{Q})$ it is positive, bounded above by $\min\left\{1, \frac{e_h}{\mathcal{B}}\right\}$, strictly decreasing in $\mathcal{B} > 0$ and is such that $\lim_{\mathcal{B}\to 0} \check{S}_h^{\mathrm{B}}(\mathcal{B}, \mathcal{Q}) = 1$.*

Proof. That it is a function follows from realising that $v_h'(s\mathcal{Q})$ is strictly decreasing in s (by concavity) whilst $\frac{1}{1-s}\frac{\mathcal{B}}{\mathcal{Q}}$ is strictly increasing in s so there can be at most one s consistent with equality between the two. For any given $\mathcal{Q} > 0$ there will be some cutoff value $\bar{\mathcal{B}}^h(\mathcal{Q}) = v_h'(0)\mathcal{Q}$. When $\mathcal{B} \ge \bar{\mathcal{B}}^h(\mathcal{Q})$, $v_h'(0) \le \frac{\mathcal{B}}{\mathcal{Q}}$ and so $\check{S}_h^{\mathrm{B}}(\mathcal{B}, \mathcal{Q}) = 0$. When $\mathcal{B} < \bar{\mathcal{B}}^h(\mathcal{Q})$, $v_h'(0) > \frac{\mathcal{B}}{\mathcal{Q}}$ and $\check{S}_h^{\mathrm{B}}(\mathcal{B}, \mathcal{Q}) = \min\left\{\check{s}_h^{\mathrm{B}}(\mathcal{B}, \mathcal{Q}), \frac{e_h}{\mathcal{B}}\right\}$. To show that the function is decreasing in $\mathcal{B}$ suppose, to the contrary that when $\mathcal{B} > \mathcal{B}'$ we have $s' = \check{s}_h^{\mathrm{B}}(\mathcal{B}', \mathcal{Q}) \ge \check{s}_h^{\mathrm{B}}(\mathcal{B}, \mathcal{Q}) = s$. Then $s'\mathcal{Q} \ge s\mathcal{Q}$ and $\frac{1}{1-s'}\frac{\mathcal{B}'}{\mathcal{Q}} > \frac{1}{1-s}\frac{\mathcal{B}}{\mathcal{Q}}$. But concavity of $v_h(\cdot)$ implies

$$\frac{1}{1-s'}\frac{\mathcal{B}'}{\mathcal{Q}} = v_h'(s'\mathcal{Q}) \le v_h'(sQ) = \frac{1}{1-s}\frac{\mathcal{B}}{\mathcal{Q}},$$

a contradiction. When $\mathcal{B} \to 0$, $\frac{1}{1-s}\frac{\mathcal{B}}{\mathcal{Q}}$ approaches a $\lrcorner$ shape (where s is on the horizontal axis) with the corner at $(1, 0)$. As such, intersection between $v_h'(s\mathcal{Q})$ and $\frac{1}{1-s}\frac{\mathcal{B}}{\mathcal{Q}}$ tends to occur when $s = 1$. $\square$

When multiplied by $\mathcal{B}$ the share function gives the bid of the buyer consistent with an equilibrium in which the aggregate bid of all buyers is $\mathcal{B}$ in the subgame in which the aggregate offer is $\mathcal{Q}$. In order to identify an equilibrium in this subgame we need only find a consistent aggregate bid, i.e. such that the individual bids generated by it add up to the aggregate bid, or where the aggregate share function is equal to one. Indeed, one can check in the routine way that there is a Nash equilibrium in the subgame in which the aggregate offer is $\mathcal{Q}$ with per-replica bid B (aggregate bid nB) if and only if $n\sum_{H^{\mathrm{S}}} \check{S}_h^{\mathrm{B}}(nB, \mathcal{Q}) = 1$. Now, we know that when B is arbitrarily close to zero, $n\sum_{H^{\mathrm{S}}} \check{S}_h^{\mathrm{B}}(nB, \mathcal{Q}) > 1$, and when $B = \sum_{H^{\mathrm{B}}} e_h$ the aggregate share function will not exceed one due to the upper bound on individual share functions. Since individual share functions are strictly decreasing in B the aggregate inherits this property and there will be a unique $B \in (0, \sum_{H^{\mathrm{S}}} e_h]$ such that $n\sum_{H^{\mathrm{S}}} \check{S}_h^{\mathrm{B}}(nB, \mathcal{Q}) = 1$, *ergo* a unique Nash equilibrium in which the strategy of each buyer is $nB\check{S}_h^{\mathrm{B}}(nB, \mathcal{Q})$.

The magnitude of of this per-replica bid, hence the nature of individual strategies, is only dependent on the aggregate offer $\mathcal{Q}$, not its composition. As such, in any subgame in which the aggregate offer is the same, the optimal responses of the buyers will be the same. $\square$

Proof of Lemma 9. This proof exactly parallels that of Lemma 5 but where the strategic supply function ${}^{m,n}\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ is replaced with ${}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(p)$, the share function ${}^n\tilde{S}_h(\mathcal{Q}, p)$ is replaced with ${}^n\dot{S}_h^{\mathrm{S}}(\mathcal{Q}, p)$ and the price functional ${}^n\tilde{p}(\mathcal{Q})$ is replaced with ${}^n\dot{p}(\mathcal{Q})$. The details are thus omitted. $\square$

Proof of Proposition 5. Given any aggregate offer $\mathcal{Q}$ we know that the second-stage buyers will behave optimally and use Nash equilibrium strategies in the second stage of the game. As such, the price will be ${}^n\dot{p}(\cdot)$ and the best response of each first-stage seller will be ${}^n\dot{\mathrm{BR}}_h^{\mathrm{S}}(\cdot)$. A SPNE is identified by a set of mutually consistent best responses for the sellers (and the corresponding optimal responses from the buyers).

First we show that whenever $m^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(\hat{p}) = n^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$ there is a SPNE in which the price is $\hat{p}$. The per-replica offer at price $\hat{p}$ is $\hat{Q} = {}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(\hat{p})$ and the per-replica bid is $\hat{B} = \hat{p}^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$. We know that for each buyer $h \in H^{\mathrm{B}}$, $\hat{b}_h = n\hat{B}S_h^{\mathrm{B}}(n\hat{B}, \hat{p}) = \mathrm{BR}_h^{\mathrm{B}}(\hat{\mathcal{B}} - \hat{b}_h, \hat{\mathcal{Q}})$ by construction and so buyers are playing optimal responses when the aggregate offer is $\hat{\mathcal{Q}} = m\hat{Q}$, i.e. there is a Nash equilibrium in the subgame. Moreover, since the price is $\hat{p} = {}^n\dot{p}(\hat{\mathcal{Q}})$ we know that for each seller $h \in H^{\mathrm{S}}$, $\hat{q}_h = \hat{\mathcal{Q}}^n\dot{S}_h^{\mathrm{S}}(\hat{\mathcal{Q}}, \hat{p}) = {}^n\dot{\mathrm{BR}}_h^{\mathrm{S}}(\hat{\mathcal{Q}} - \hat{q}_h)$ and so each seller is using a mutually consistent best response. Thus, there is a SPNE in which the price is $\hat{p}$.

Next suppose we have a SPNE in which the price is $\hat{p}$, then we need to show that aggregate strategic supply and demand are equal at this price. So, suppose the strategies $(\{\hat{q}_h\}_{h\in H^{\mathrm{S}}}, \{\hat{b}_h\}_{h\in H^{\mathrm{B}}})$ form a SPNE. For each buyer $h \in H^{\mathrm{B}}$, $\hat{b}_h = \mathrm{BR}_h^{\mathrm{B}}(\hat{\mathcal{B}} - \hat{b}_h, \hat{\mathcal{Q}})$ and so $\hat{b}_h = n\hat{B}S_h^{\mathrm{B}}(n\hat{B}, \hat{p})$ implying $n\sum_{H^{\mathrm{B}}} S_h^{\mathrm{B}}(n\hat{B}, \hat{p}) = 1$ in turn implying $\frac{\hat{B}}{\hat{p}} = {}^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$ and moreover that $\hat{p} = {}^n\dot{p}(\hat{\mathcal{Q}})$. Then for each seller $h \in H^{\mathrm{S}}$ we must have $\hat{q}_h = {}^n\dot{\mathrm{BR}}_h^{\mathrm{S}}(\hat{\mathcal{Q}}_{-h})$ and so it follows that $\hat{q}_h = \hat{\mathcal{Q}}^n\dot{S}_h^{\mathrm{S}}(\hat{\mathcal{Q}}, \hat{p})\,\forall h \in H^{\mathrm{S}}$ by definition. But then $m\sum_{H^{\mathrm{S}}} {}^n\dot{S}_h^{\mathrm{S}}(m\hat{Q}, \hat{p}) = 1$ implying $\hat{Q} = {}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(\hat{p})$. Then since $\hat{p} = {}^n\dot{p}(\hat{\mathcal{Q}})$ and ${}^n\dot{p}(\hat{\mathcal{Q}})$ is such that $\hat{\mathcal{Q}} = n^n\mathcal{X}_1^{\mathrm{B}}({}^n\dot{p}(\hat{\mathcal{Q}}))$ it follows that $m^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(\hat{p}) = n^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$. $\square$

Proof of Theorem 3. We recall from Lemma 2 that strategic demand is positive only for $0 < p < {}^nP^{\mathrm{B}}$ where it is a function that is continuous and strictly decreasing in p, and from Lemma 9 that strategic supply in the two-stage market game in which the sellers move first is positive only for $p > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ where it is a continuous function that is non-decreasing in p. Moreover, Proposition 5 tells us that non-autarkic SPNE are in one-to-one correspondence with intersections of strategic supply and demand at the aggregate level. When $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} \geq {}^nP^{\mathrm{B}}$ there are no prices where both strategic supply and demand are defined, so they cannot intersect. Thus, there is no non-autarkic SPNE; the only equilibrium is autarky. Conversely, when $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < {}^nP^{\mathrm{B}}$ strategic supply and demand must intersect, and due to their monotonicity properties they intersect only once. As such, there is a unique non-autarkic SPNE.

In this latter case, contrary to the simultaneous-move market game, autarky is not also an equilibrium. If the aggregate offer from the sellers in the first stage is positive, any buyer, even if she is acting alone, has the incentive to make a positive bid so the aggregate bid in the second stage will generically be strictly positive if the aggregate offer is positive. Given this, individual sellers have an incentive to make a positive offer in an attempt to acquire the whole of this bid if it is individually rational to do so. If $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < {}^nP^{\mathrm{B}}$ there will indeed be such an offer and no autarkic equilibrium will exist. $\square$

Proof of Lemma 12. We first show that $\lim_{n\to\infty} {}^n\dot{S}_h^{\mathrm{S}}(Q,p) = \lim_{n\to\infty} {}^n\tilde{S}_h^{\mathrm{S}}(Q,p)\,\forall Q > 0$, $\forall p$, noting that it will suffice to show $\lim_{n\to\infty} {}^n\dot{s}_h^{\mathrm{S}}(Q,p) = \lim_{n\to\infty} {}^n\tilde{s}_h^{\mathrm{S}}(Q,p)$. The former is that level of s where $v_h'(e_h - sQ) = p + sQ^n\dot{p}'(Q)$ whilst the latter is where $v_h'(e_h - sQ) = p + sQ^n\tilde{p}'(Q)$. But we know from Lemma 6 that ${}^n\mathcal{X}_1^{\mathrm{B}}(p) \to_{n\to\infty} \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$ and this implies $\lim_{n\to\infty} {}^n\dot{p}(Q) = \lim_{n\to\infty} {}^n\tilde{p}(Q)\,\forall Q > 0$, which implies the desired result.

As a consequence, we know that

$$\lim_{n\to\infty} \sum_{H^{\mathrm{S}}} {}^n\dot{S}_h^{\mathrm{S}}(Q,p) = \lim_{n\to\infty} \sum_{H^{\mathrm{S}}} {}^n\tilde{S}_h^{\mathrm{S}}(Q,p)\,\forall Q > 0,\,\forall p.$$

Setting $Q = \lim_{n\to\infty} {}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$, which is positive for all $p > \min_{H^{\mathrm{S}}}\{v_h'(0)\}$ we get that

$$\lim_{n\to\infty} \sum_{H^{\mathrm{S}}} {}^n\dot{S}_h^{\mathrm{S}}(\lim_{n\to\infty} {}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p),p) = \lim_{n\to\infty} \sum_{H^{\mathrm{S}}} \tilde{S}_h^{\mathrm{S}}(\lim_{n\to\infty} {}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p),p) = 1\,\forall p > \min_{H^{\mathrm{S}}}\{v_h'(0)\}.$$

Since $\sum_{H^{\mathrm{S}}} {}^n\dot{S}_h^{\mathrm{S}}(Q,p)$ is strictly decreasing in Q under the stated conditions this implies that the only value of Q consistent with $\lim_{n\to\infty} \sum_{H^{\mathrm{S}}} {}^n\dot{S}_h^{\mathrm{S}}(Q,p) = 1$ (which is precisely $\lim_{n\to\infty} {}^n\dot{\mathcal{X}}_1^{\mathrm{S}}(p)$) is $Q = \lim_{n\to\infty} {}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$. Thus, $\lim_{n\to\infty} {}^n\dot{\mathcal{X}}_1^{\mathrm{S}}(p) = \lim_{n\to\infty} {}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)\,\forall p > \min_{H^{\mathrm{S}}}\{v_h'(0)\}$. $\square$

Proof of Theorem 5. Since ${}^n\mathcal{X}_1^{\mathrm{B}}(p) \to_{n\to\infty} \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\,\forall 0 < p < \min_{H^{\mathrm{B}}}\{v_h'(0)\}$ and $\lim_{n\to\infty} {}^n\dot{\mathcal{X}}_1^{\mathrm{S}}(p) = \lim_{n\to\infty} {}^n\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)\,\forall p > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ we know $\lim_{n\to\infty} {}^n\hat{p}^{\mathrm{SB}} = \lim_{n\to\infty} {}^n\hat{p}^{\mathrm{C}}$ and $\lim_{n\to\infty} {}^n\hat{Q}^{\mathrm{SB}} = \lim_{n\to\infty} {}^n\hat{Q}^{\mathrm{C}}$. A direct consequence is that $\lim_{n\to\infty} {}^n\hat{B}^{\mathrm{SB}} = \lim_{n\to\infty} {}^n\hat{B}^{\mathrm{C}}$. It only remains to show that individual bids and offers converge. For the sellers, their individual offers in the two-stage game are ${}^n\hat{Q}^{\mathrm{SB}n}\dot{S}_h^{\mathrm{S}}({}^n\hat{Q}^{\mathrm{SB}}, {}^n\hat{p}^{\mathrm{SB}})$ whilst in the Cournot oligopoly they are ${}^nQ^{\mathrm{C}n}\tilde{S}_h^{\mathrm{S}}({}^nQ^{\mathrm{C}}, {}^np^{\mathrm{C}})$. In the proof of Lemma 12 we showed that for each $h \in H^{\mathrm{S}}$, $\lim_{n\to\infty} {}^n\dot{S}_h^{\mathrm{S}}(Q,p) = \lim_{n\to\infty} {}^n\tilde{S}_h^{\mathrm{S}}(Q,p)\,\forall Q > 0$, $\forall p$. As such, since $\lim_{n\to\infty} {}^n\hat{Q}^{\mathrm{SB}} = \lim_{n\to\infty} {}^n\hat{Q}^{\mathrm{C}}$ and $\lim_{n\to\infty} {}^n\hat{p}^{\mathrm{SB}} = \lim_{n\to\infty} {}^n\hat{p}^{\mathrm{C}}$ it follows that $\lim_{n\to\infty} {}^n\hat{q}_h^{\mathrm{SB}} = \lim_{n\to\infty} {}^n\hat{Q}^{\mathrm{SB}n}\dot{S}_h^{\mathrm{S}}({}^n\hat{Q}^{\mathrm{SB}}, {}^n\hat{p}^{\mathrm{SB}}) = \lim_{n\to\infty} {}^nQ^{\mathrm{C}n}\tilde{S}_h^{\mathrm{S}}({}^nQ^{\mathrm{C}}, {}^np^{\mathrm{C}}) = \lim_{n\to\infty} {}^n\hat{q}_h^{\mathrm{C}}$. The buyers' individual bids in the two-stage game are $n^n\hat{B}^{\mathrm{SB}}S_h^{\mathrm{B}}(n^n\hat{B}^{\mathrm{SB}}, {}^n\hat{p}^{\mathrm{SB}})$ whilst in the Cournot oligopoly they are $\tilde{\mathfrak{b}}_h({}^n\hat{p}^{\mathrm{C}})$. We showed in the proof of Lemma 6 that $nBS_h(nB,p) \to_{n\to\infty} \tilde{\mathfrak{b}}_h(p)\,\forall B > 0$, $\forall p$. As such, since $\lim_{n\to\infty} {}^n\hat{p}^{\mathrm{SB}} = \lim_{n\to\infty} {}^n\hat{p}^{\mathrm{C}}$, it follows that for each $h \in H^{\mathrm{B}}$, $\lim_{n\to\infty} {}^n\hat{b}_h^{\mathrm{SB}} = \lim_{n\to\infty} n^n\hat{B}^{\mathrm{SB}}S_h^{\mathrm{B}}(n^n\hat{B}^{\mathrm{SB}}, {}^n\hat{p}^{\mathrm{SB}}) = \lim_{n\to\infty} \tilde{\mathfrak{b}}_h({}^n\hat{p}^{\mathrm{C}}) = \lim_{n\to\infty} {}^n\hat{b}_h^{\mathrm{C}}$. $\square$

Proof of Lemma 13. Write the share correspondence of each buyer in terms of the ratio $\mathcal{V} = \frac{\mathcal{B}}{p}$, in which case it takes the form

$$\breve{\ddot{S}}_h^{\mathrm{B}}(\mathcal{V}, p) = \begin{cases} 0 & \text{if } p \geq v_h'(0) \text{ or} \\ \min\left\{\breve{\ddot{s}}_h^{\mathrm{B}}(\mathcal{V}, p), \frac{e_h}{\mathcal{V}p}\right\} & \text{if } p < v_h'(0) \end{cases}$$

where

$$\breve{\ddot{s}}_h^{\mathrm{B}}(\mathcal{V}, p) = \left\{ s : v_h'(s\mathcal{V}) = \frac{p^2}{p - s\mathcal{V}p\ddot{p}'(\mathcal{V}p)} \right\}.$$

The magnitude of a buyer's bid consistent with a SPNE in which the ratio of aggregate bid to price is $\mathcal{V}$ and the price is p is given by $\mathcal{V}p\breve{\ddot{S}}_h^{\mathrm{B}}(\mathcal{V}, p)$, and in order to find the consistent level of $\mathcal{V} = nV$, i.e. strategic demand, we look for that level of V such that $n\sum_{H^{\mathrm{B}}} \breve{\ddot{S}}_h^{\mathrm{B}}(nV, p) = 1$. [One can verify in the usual way that $\breve{\ddot{S}}_h^{\mathrm{B}}(\mathcal{V}, p)$ is a function that is strictly decreasing in $\mathcal{V}$.]

Now, the first task is to show $nVp\breve{\ddot{S}}_h^{\mathrm{B}}(nV, p) \to_{n\to\infty} \tilde{\mathfrak{b}}_h(p)\, \forall V > 0, \forall p$. The magnitude $nVp^n\breve{\ddot{S}}_h^{\mathrm{B}}(nV, p)$ is equivalent to the replacement value

$$\breve{\ddot{R}}_h^{\mathrm{B}}(nV, p) = \begin{cases} 0 & \text{if } p \geq v_h'(p)\text{or} \\ \min\{\breve{\ddot{r}}_h^{\mathrm{B}}(nV, p), e_h\} & \text{if } p < v_h'(0) \end{cases}$$

where

$$\breve{\ddot{r}}_h^{\mathrm{B}}(nV, p) = \left\{ b : v_h'\left(\frac{b}{p}\right) = \frac{p^2}{p - b\ddot{p}'(nVp)} \right\}$$

and it will suffice to show $\breve{\ddot{r}}_h^{\mathrm{B}}(nV, p) \to_{n\to\infty} \tilde{b}_h(p)\, \forall V > 0, \forall p$. But the former is that b where $v_h'\left(\frac{b}{p}\right) = \frac{p^2}{p - b\ddot{p}'(nVp)}$ whilst the latter is where $v_h'\left(\frac{b}{p}\right) = p$. As $n \to \infty$, $\ddot{p}'(nVp) \to 0$ (the marginal effect of any buyer on the price diminishes to zero as their number increases) and so $\frac{p^2}{p - b\ddot{p}'(nVp)} \to p$ and the desired result follows.

Then we have

$$nVp\sum_{H^{\mathrm{B}}} \breve{\ddot{S}}_h^{\mathrm{B}}(nV, p) \to_{n\to\infty} \sum_{H^{\mathrm{B}}} \tilde{\mathfrak{b}}_h(p)\, \forall V > 0, \forall p.$$

Setting $V = {}^n\ddot{\mathcal{X}}_1^{\mathrm{B}}(p)$ which is positive for all $0 < p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ and dividing by p we get

$$n^n\ddot{\mathcal{X}}_1^{\mathrm{B}}(p)\sum_{H^{\mathrm{B}}} \breve{\ddot{S}}_h^{\mathrm{B}}(n^n\ddot{\mathcal{X}}_1^{\mathrm{B}}(p), p) \to_{n\to\infty} \sum_{H^{\mathrm{B}}} \frac{\tilde{\mathfrak{b}}_h(p)}{p} = \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\, \forall 0 < p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}.$$

But $n\sum_{H^{\mathrm{S}}} {}^n\breve{\ddot{S}}_h^{\mathrm{B}}(n^n\ddot{\mathcal{X}}_1^{\mathrm{B}}(p), p) = 1$, which gives the desired result. □

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