Periodic orbits high above the ecliptic plane in the solar sail 3-body problem

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ESTEC Conference 2nd - 5th October

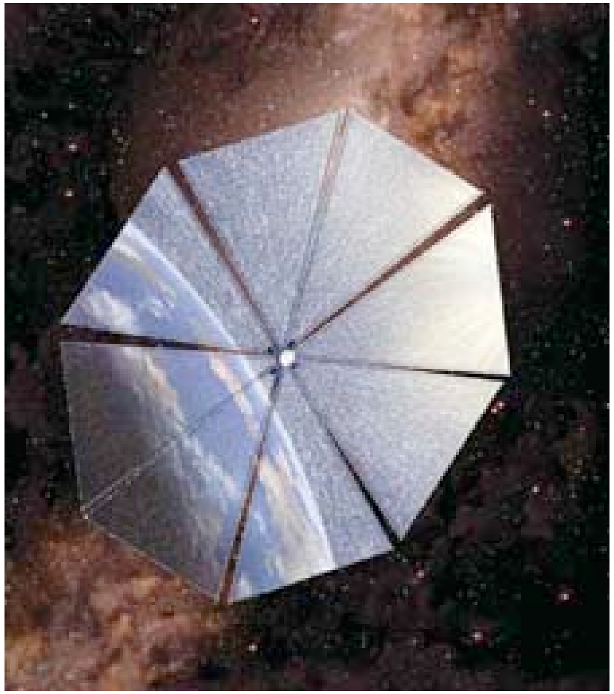
Overview

1. Solar Sails
   - Sail design
2. 3 Body Equations and Equilibria
   - Classical equilibria - the Lagrange points
   - Sail equilibria
3. Periodic Solutions
   - Lindstedt-Poincaré method
   - Families of periodic solutions
4. Applications
   - Polesitter
   - Invariant manifolds

Outline

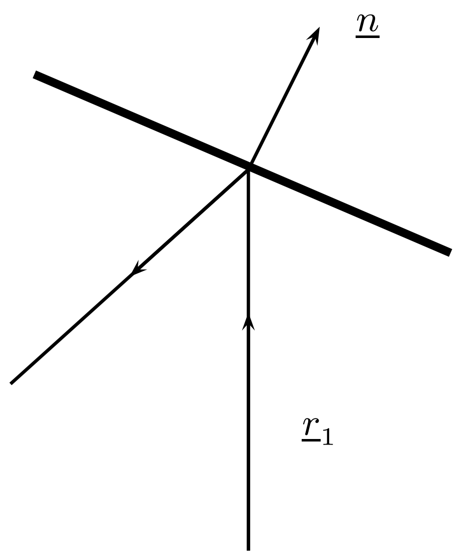
Outline

Sail design



- A solar sail is a spacecraft without an engine, and therefore needs no fuel. It is pushed along by the pressure of photons from the sun hitting the sail.
- Solar sails are typically large square sheets of a highly reflective film supported by booms, although other designs (discs, blades) are popular.
- The material of the sail must be very lightweight and thin, of the order of a couple of microns, and very large, the order of $(50m)x(50m)$.

Solar radiation pressure

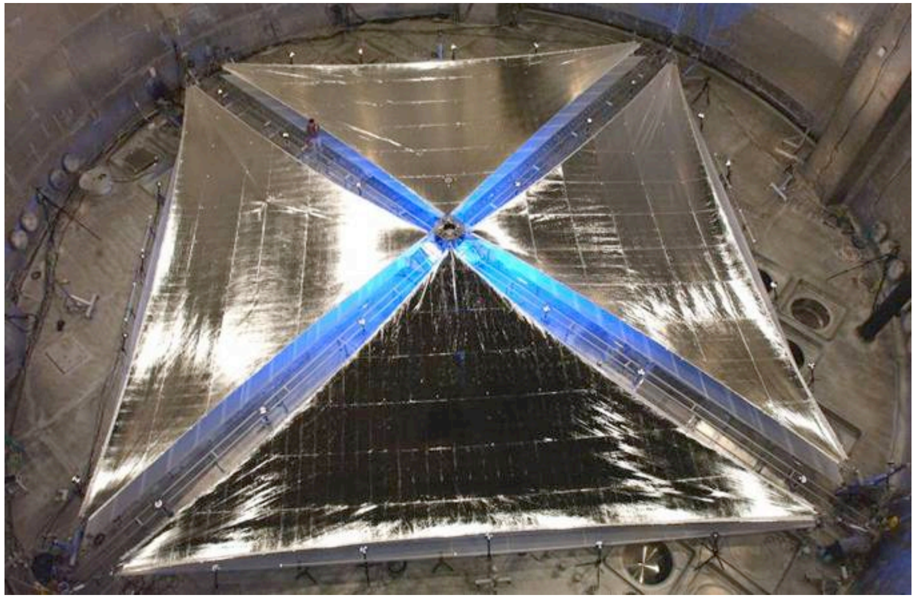


- Photons from the sun hitting the sail impart a small but constant radiation pressure.
- The acceleration on the sail due to this pressure is given by

$$\underline{a} = \beta \frac{M_s}{r_s^2} (\widehat{\underline{r}}_s . \underline{n})^2 \underline{n}.$$

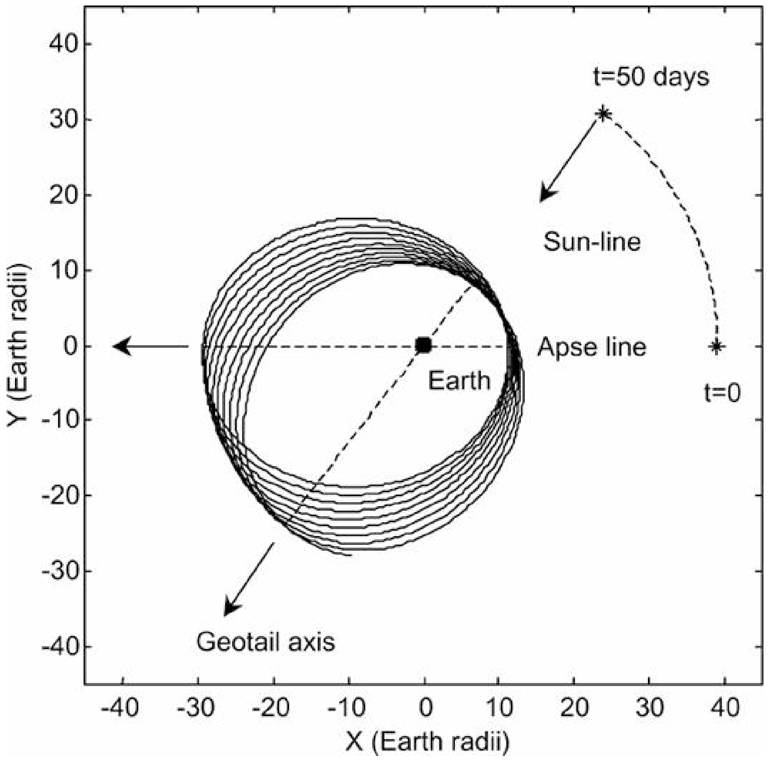
- Here β is the "sail lightness number", the ratio of the radiation pressure acceleration to gravitational acceleration.

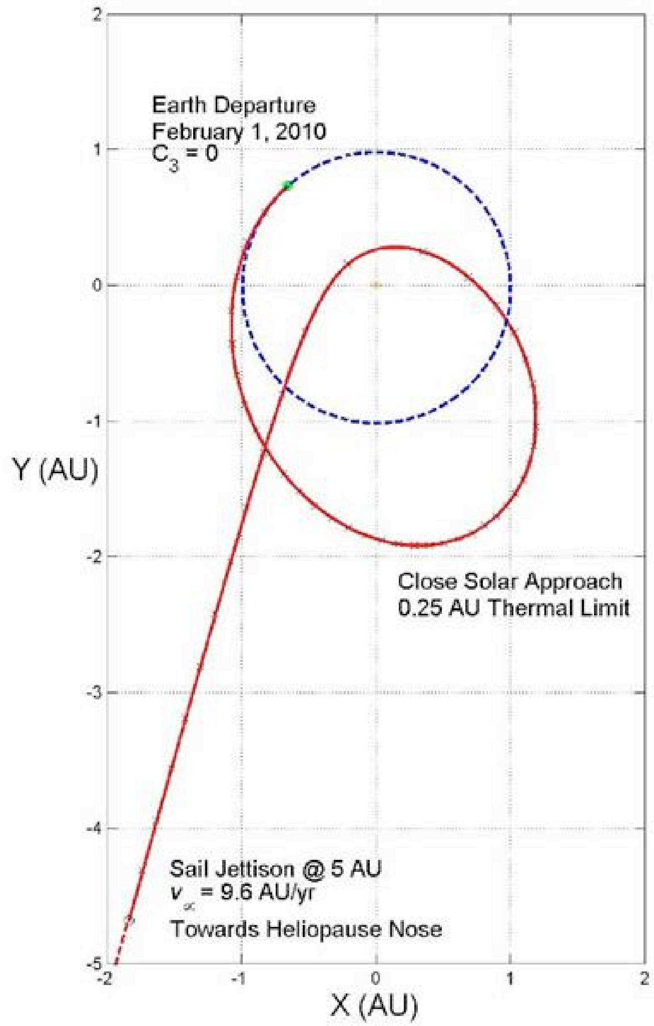
Current designs



- Solar sails are currently being built with a lightness number $\beta \sim 0.2$.
- For the purpose of this presentation we only consider $\beta < 0.1$.

Possible missions





Geosail and Heliopause missions

Outline

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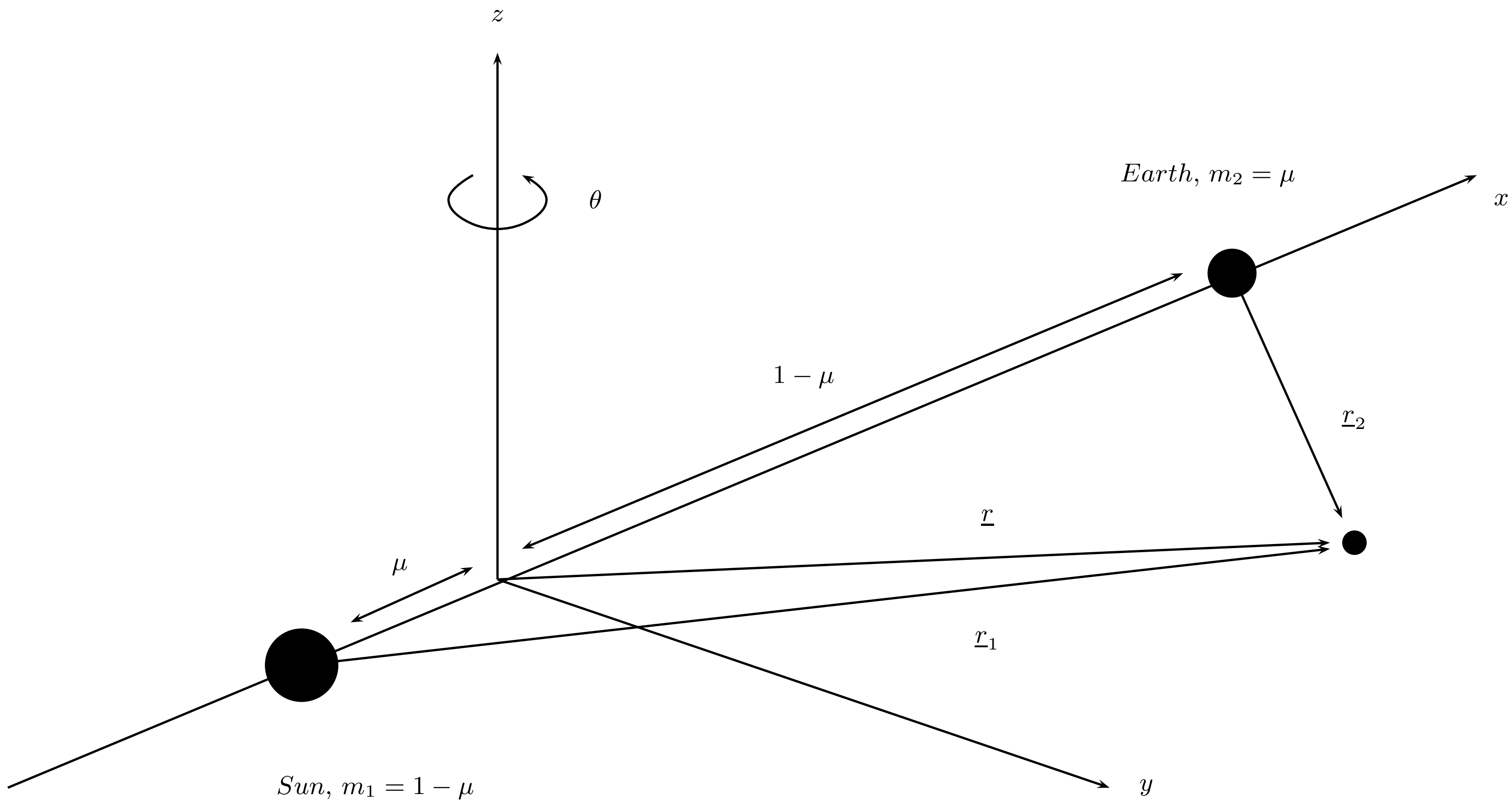
Classical equilibria - the Lagrange points

The CR3BP in a rotating coordinate frame:



Classical equilibria - the Lagrange points

- In the inertial frame, the equations of motion of the third body are:

$$\frac{d^2\underline{r}}{dt^2} = -\nabla V, \qquad V = -\left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right).$$

- However, in the rotating frame two additional forces are introduced: the Corioli force and the centrifugal force,

$$\frac{d^2\underline{r}}{dt^2} + \left\{2\underline{\theta} \times \frac{d\underline{r}}{dt} + \underline{\theta} \times (\underline{\theta} \times \underline{r})\right\} = -\nabla V.$$

- Equilibrium points now exist in this coordinate system.

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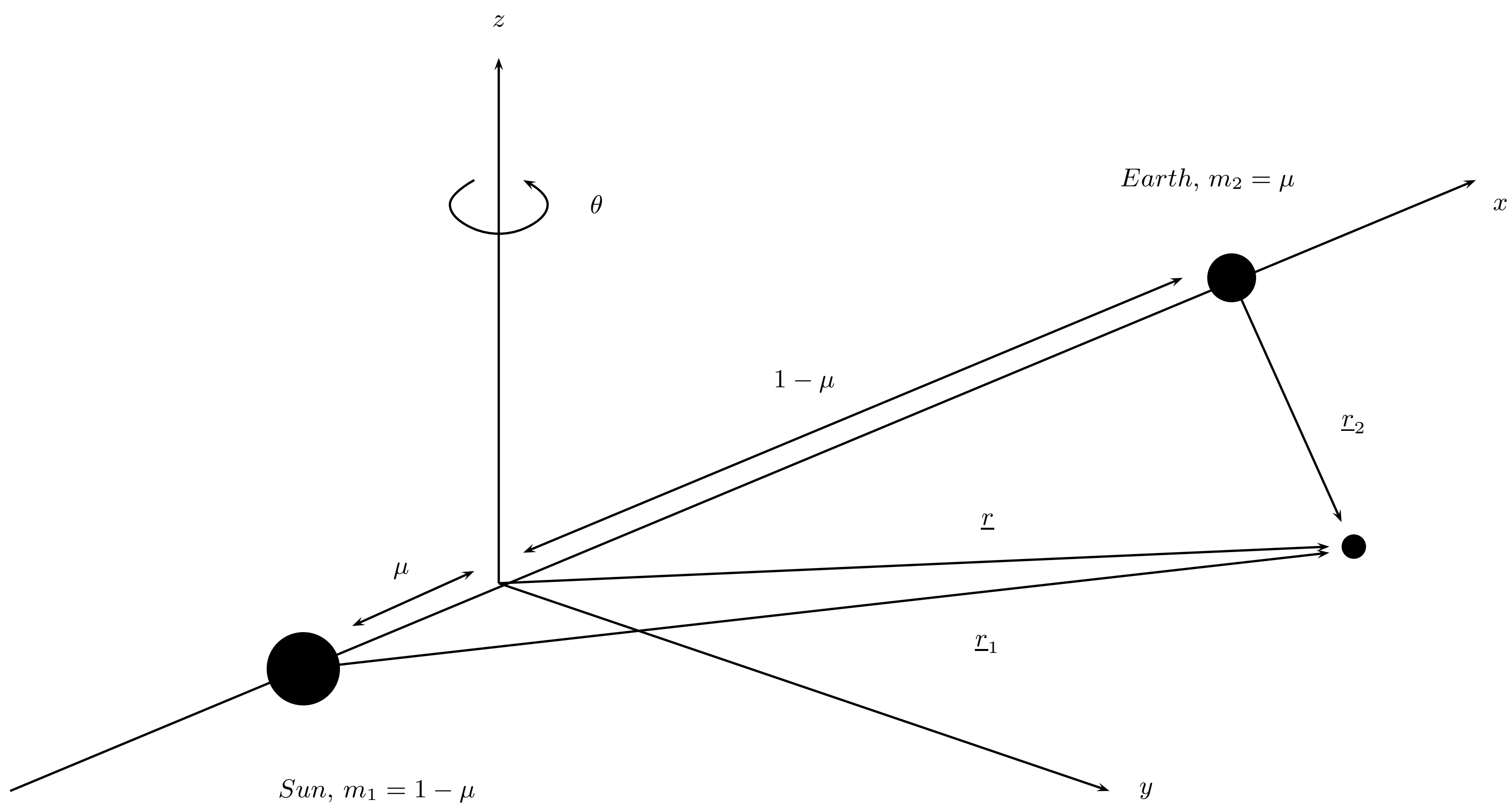
Classical equilibria - the Lagrange points

There are five equilibrium points in the rotating coordinate frame, called Lagrange points. All are in the plane of the primaries' mutual orbit.



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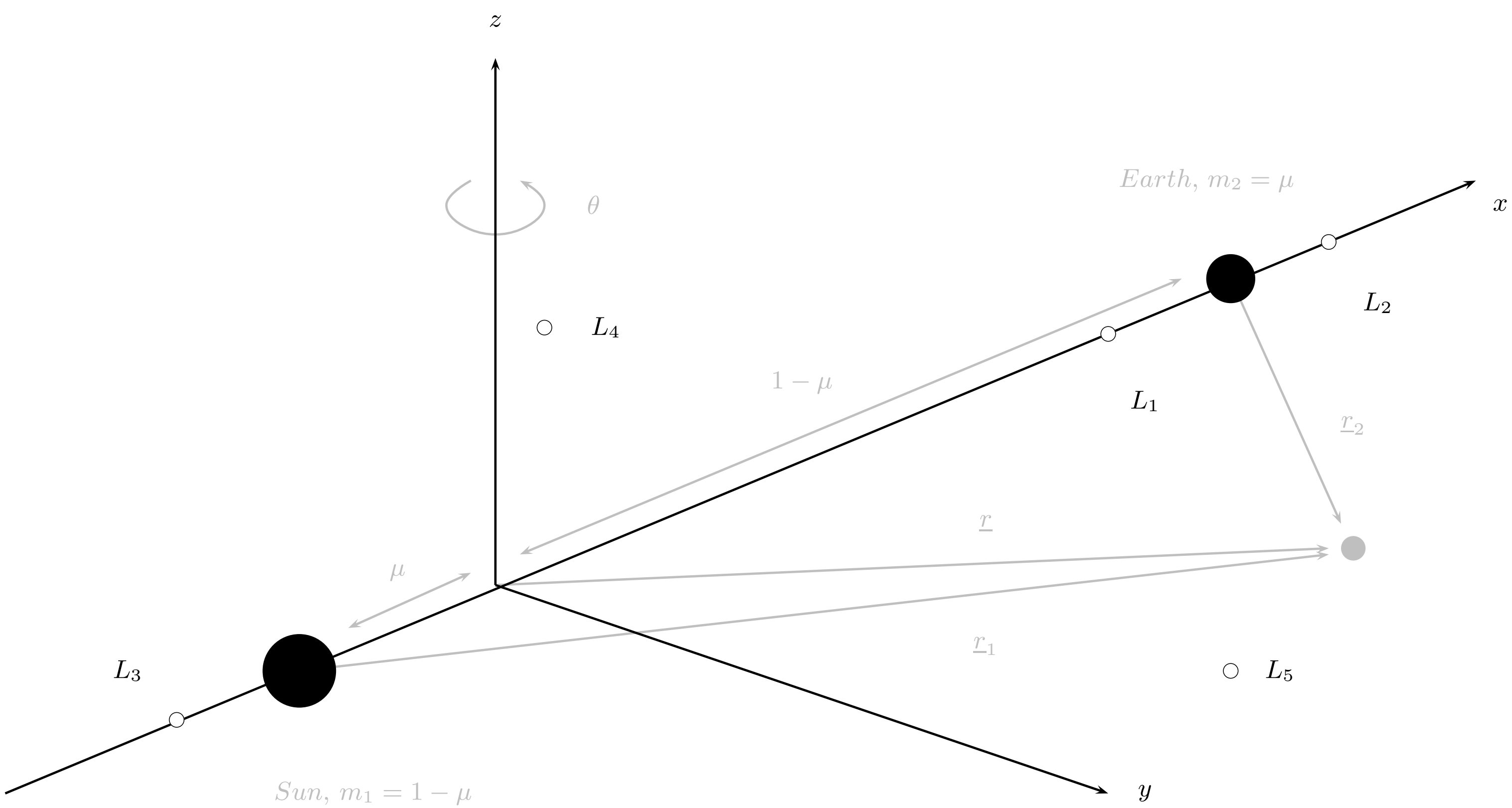


Outline

Sail equilibria

- In the rotating coordinate frame the equations of motion of the solar sail are

$$\frac{d^2\underline{r}}{dt^2} + 2\underline{\theta} \times \frac{d\underline{r}}{dt} = \underline{a} - \underline{\theta} \times (\underline{\theta} \times \underline{r}) - \nabla V \equiv \underline{F},$$

where

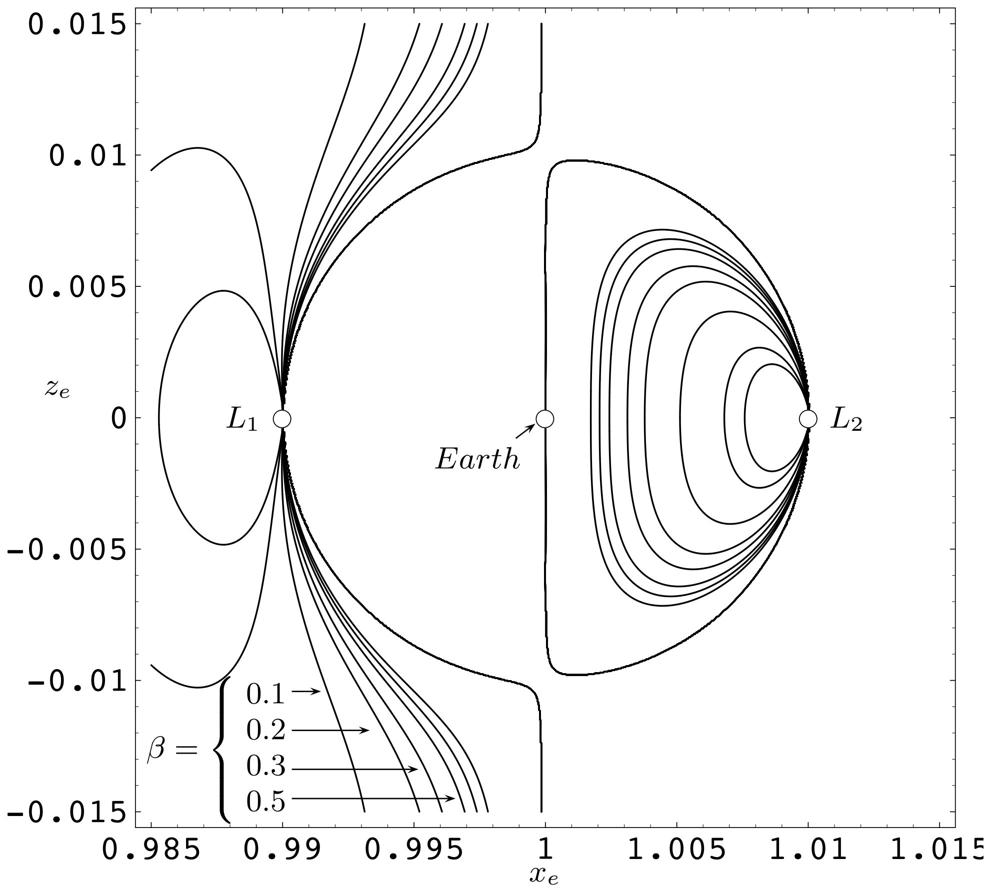
$$\underline{a} = \beta \frac{1-\mu}{r_1^2} (\underline{\widehat{r}}_1 . \underline{n})^2 \underline{n}.$$

- At equilibrium, $\dot{r}$ and $\ddot{r}$ vanish, so an equilibrium point is a zero of $\underline{F}$.
- We seek equilibria in the x-z plane, thus we let

$$\underline{n} = \cos(\gamma)\underline{\widehat{x}} + \sin(\gamma)\underline{\widehat{z}}.$$

Sail equilibria

We find continuous surfaces of equilibria:



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3. **Periodic Solutions**
   - Lindstedt-Poincaré method
   - Families of periodic solutions
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   - Polesitter
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Periodic solutions

- We use the Lindstedt-Poincaré method to find periodic solutions.
- We Taylor expand around an equilibrium point

$$\begin{pmatrix} \ddot{x} - 2\dot{y} \\ \ddot{y} + 2\dot{x} \\ \ddot{z} \end{pmatrix} = \delta r^a (\partial_a \underline{F})|_e + \tfrac{1}{2}\delta r^a \delta r^b (\partial_a \partial_b \underline{F})|_e$$

$$+\tfrac{1}{6}\delta r^a \delta r^b \delta r^c (\partial_a \partial_b \partial_c \underline{F})|_e + O(\delta \underline{r}^4)$$

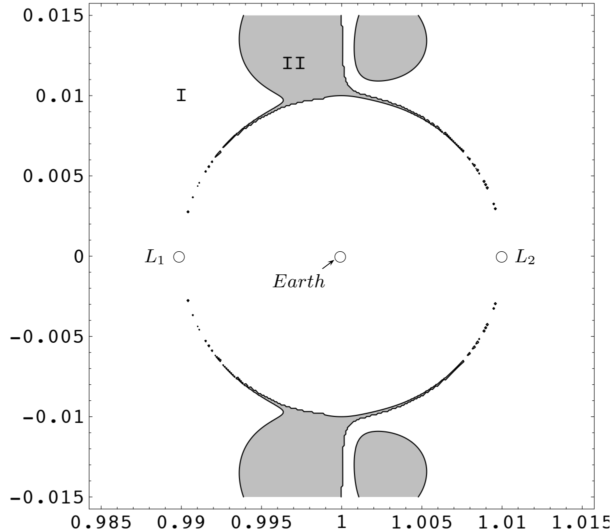
- At linear order

$$\begin{aligned} \ddot{x} - 2\dot{y} &= ax + bz \\ \ddot{y} + 2\dot{x} &= \quad cy \\ \ddot{z} \quad &= dx + ez \end{aligned}$$

Periodic solutions

- There are two distinct regions, depending on the eigenvalues of the linear system.
- In region I, the linear spectrum is

$$I : \{ \pm \lambda_1 i, \pm\lambda_2 i, \pm\lambda_r \}$$



- Thus periodic solutions exist at linear order, by suppressing unwanted modes:

$$x, y, z = A\cos(\lambda_1 t) + B\sin(\lambda_1 t) + C\cos(\lambda_2 t) + D\sin(\lambda_2 t) + Ee^{\lambda_r t} + Fe^{-\lambda_r t}$$

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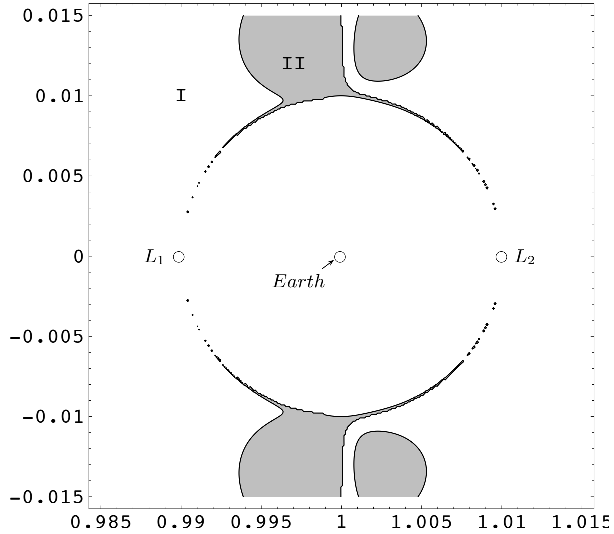

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Lindstedt-Poincaré method

- We let ε be a small parameter (typically the amplitude of a periodic orbit), and we let

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \varepsilon^3\omega_3 + O(\varepsilon^4).$$

- The idea is, if the frequency of a periodic solution in the linear system is λ, then this method lets you find approximations to periodic solutions in the nonlinear system with frequency $\omega\lambda$.
- We define a new time coordinate, $\tau = \omega t$, and let $x_n = x_n(\tau)$ etc. Then the following statements are equivalent:

The periodic solution has frequency $\omega\lambda$ in t-seconds $\Leftrightarrow$ The periodic solution has frequency λ in τ-seconds

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Lindstedt-Poincaré method

- We find the solutions for x and z are trigonometric cosine series and y is a sine series. For example when n is odd

$$x_n = p_{n3}\cos(3T) + \ldots + p_{nn}\cos(nT),$$

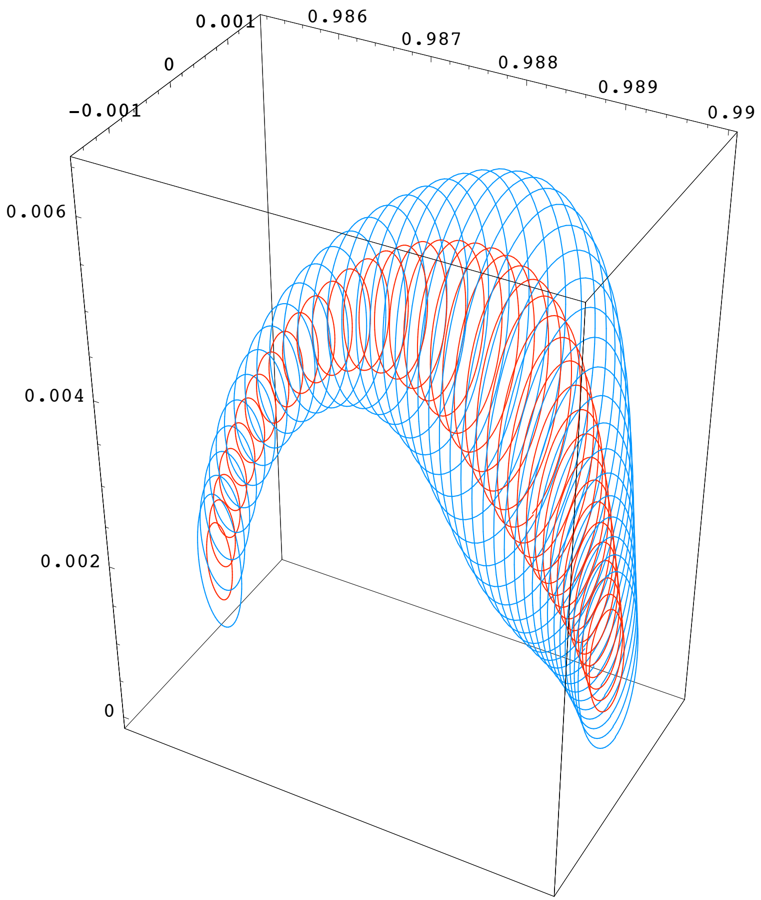
and when n is even

$$x_n = p_{n0} + p_{n2}\cos(2T) + \ldots + p_{nn}\cos(nT).$$

- The problem reduces to solving systems of algebraic equations for the coefficients p_{ni} and ω_i.
- For large amplitude periodic solutions high above the ecliptic plane, we need to include up to the 7th term in the series (7th order approximation).

Outline

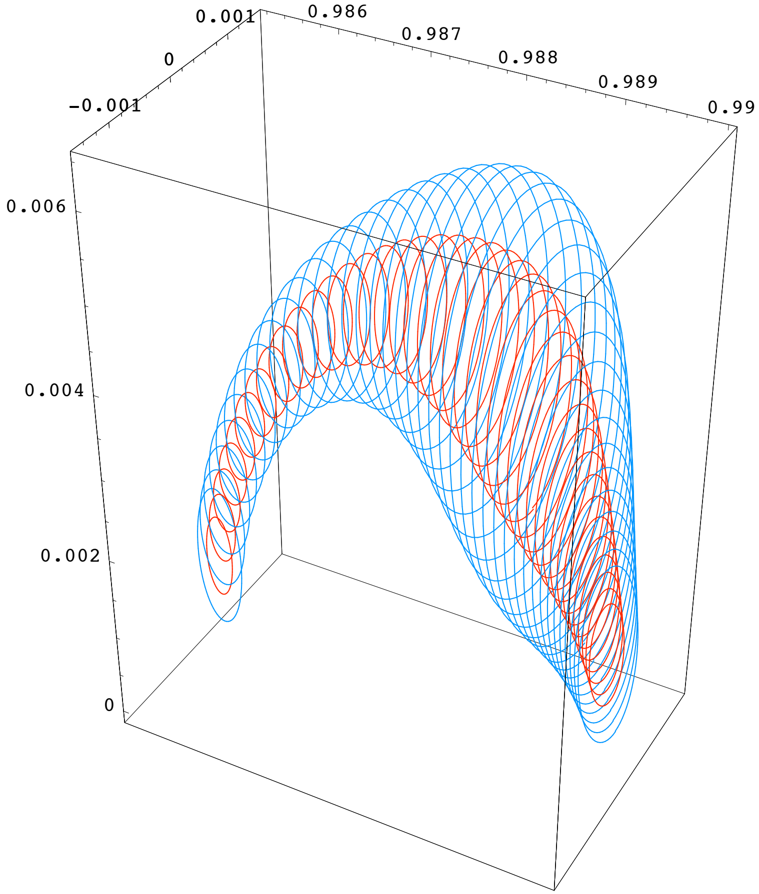
Families of periodic solutions



- The Lindstedt-Poincaré series provides our first guess for the initial data of a periodic solution of the full nonlinear system.
- We then use a differential corrector to fine tune the initial data.

The picture on the left shows some of the periodic orbits possible for a solar sail with $\beta = 0.03$.

Families of periodic solutions



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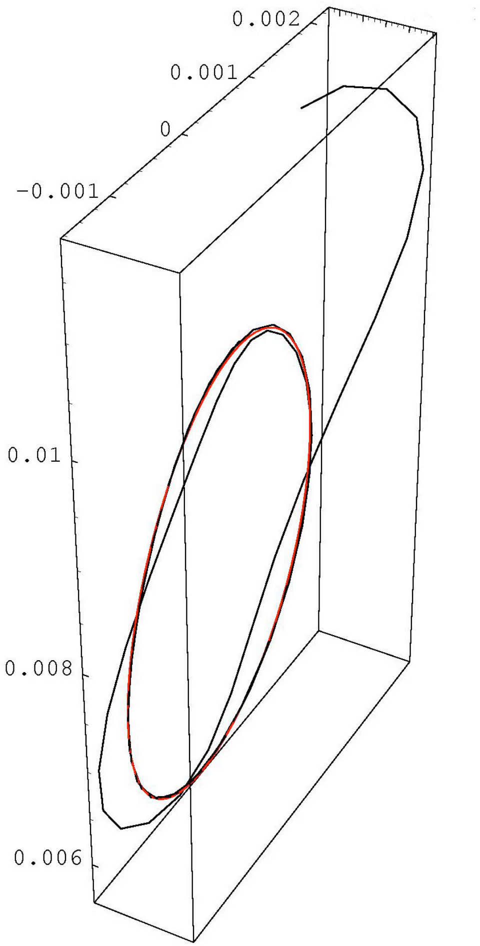
Stability - Control



- The orbital stability of these periodic solutions is found by calculating the eigenvalues of the monodromy matrix.
- The spectrum is typically $\{1, 1, \lambda_r, 1/\lambda_r, \lambda_c, \bar{\lambda}_c\}$
- However, we can easily control to the nominal orbit with variations in the orientation of the sail normal.

Outline

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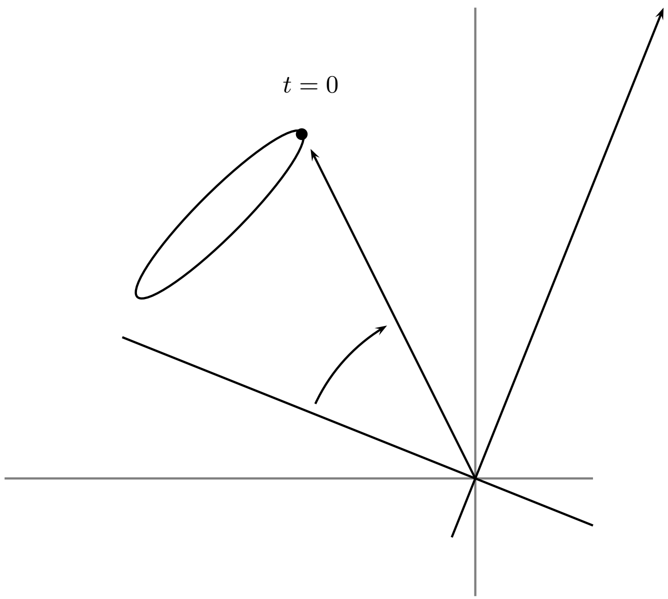
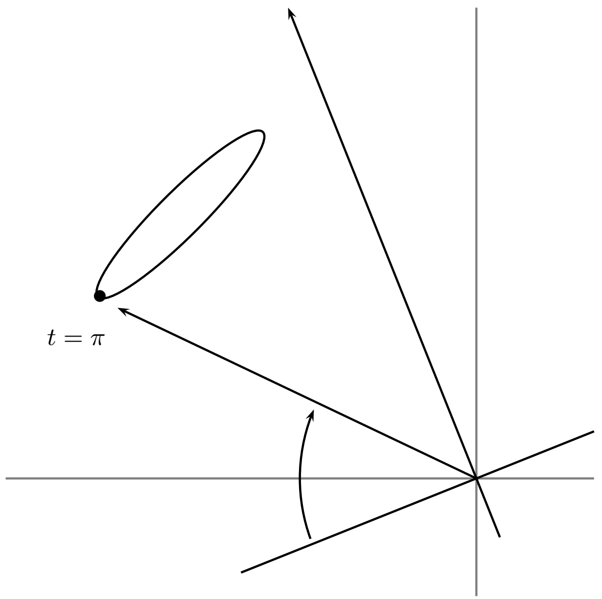
Polesitter

- We may use these orbits to provide a constant view of one of the poles.



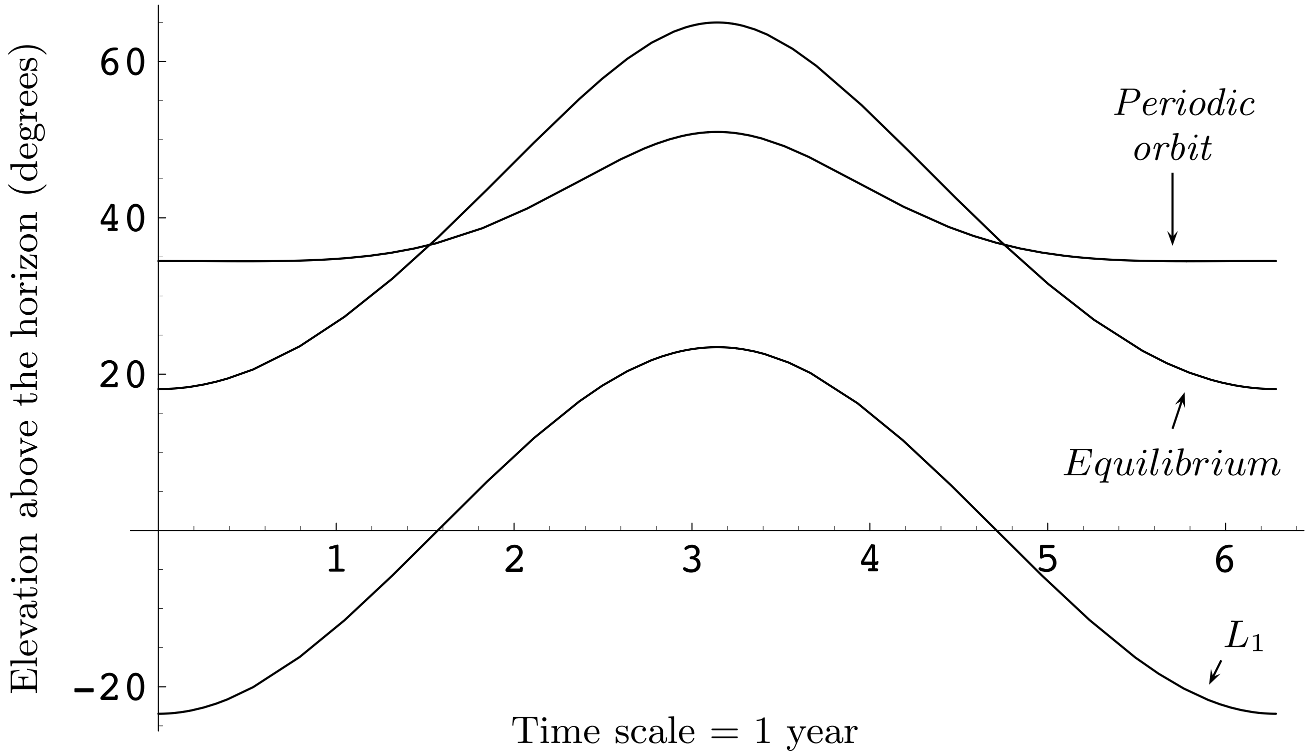
Wintertime, Northern Hemisphere



Summertime, Northern Hemisphere

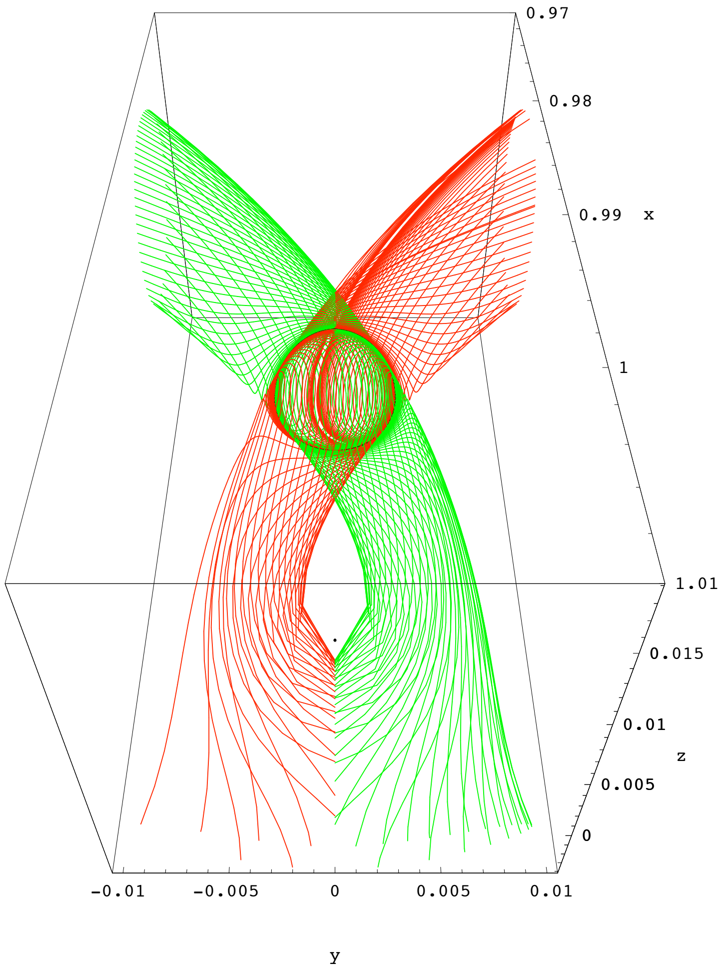
View from the pole

By timing the orbit well we can narrow the angle subtended by the sail:



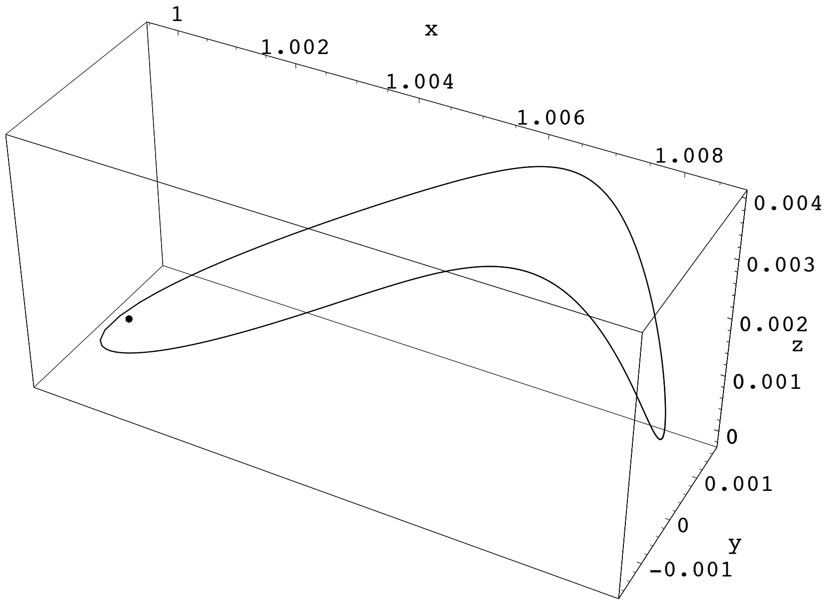
Outline

Invariant manifolds



- Associated with each periodic orbit are a set of invariant manifolds.
- We find these by integrating in the direction of the stable and unstable eigenvectors of the monodromy matrix.
- These manifolds provide us with a mechanism to transfer between various periodic orbits.

Unusual orbits



- Unexpected families of periodic solutions arise.
- Interestingly, this particular orbit has monodromy matrix with spectrum

$$\{1, 1, \lambda_{r1}, 1/\lambda_{r1}, \lambda_{r2}, 1/\lambda_{r2}\}$$