Retrodiction as a tool for micromaser field measurements

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Abstract

We use retrodictive quantum theory to describe cavity field measurements by successive atomic detections in the micromaser. We calculate the state of the micromaser cavity field prior to detection of sequences of atoms in either the excited or ground state, for atoms which are initially prepared in the excited state. This provides the POM elements which describe such sequences of measurements.

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I. INTRODUCTION

The advent of quantum information theory, coupled with an increased technological ability to produce and study exotic entangled states of the optical field has stimulated a re-examination of the fundamental principles of quantum mechanics. One example of this is the recent resurgence of interest in retrodictive quantum theory, in which the likelihood of past events is inferred from measurements of the present. This subject was, until relatively recently, a simple problem associated with time-reversal in quantum theory [1]. It has now evolved into a powerful technique for calculating retrodictive conditional probabilities in quantum theory, particularly when the knowledge of the output state produced by a state preparation device is not complete. The crucial step in this was the use of Bayes' theorem [2] coupled with normal predictive quantum theory. This allows a retrodictive state to be defined such that retrodictive and predictive conditional probabilities are consistent [3]. The theory can be applied to both closed systems, in which the full evolution can be followed, and to open ones, in which the system of interest interacts with an unmeasured environment [4–7].

The predictive state of a system at any time during the evolution is the prepared state evolved forwards in time. The 'collapse of the wavefunction' occurs at the measurement device when this evolved state is projected onto the measurement basis. The retrodictive state at any time between preparation and measurement is the measured state evolved backwards in time. At the preparation time this state collapses onto any one of a set of possible initially prepared states. In open systems, where the system must be described by a density operator, we can define a retrodictive density operator which can be evolved backwards in time according to a retrodictive master equation [6].

The micromaser has been useful tool for quantum optical and cavity QED experiments for many years [8]. In the basic device highly-excited Rydberg atoms are passed through a high-Q microwave cavity, one of whose longitudinal modes has a frequency which is close to one of the atomic transition frequencies. The atoms interact with the cavity field, and the

whole device behaves to a very good approximation like a two-level atom interacting with a single-mode quantum optical field. The atoms can be prepared and measured in what amounts to either the excited or ground state. In a further refinement resonant microwave pulses can be applied to the atoms prior to entering and after leaving the cavity. This allows the atoms to be prepared or measured in any desired superposition of the excited or ground states. The micromaser operates in what has been termed the strong-coupling regime of cavity quantum electrodynamics, in which a single photon in the cavity is a strong enough field to cause multiple excitations and de-excitations of the atom.

Usually there is no direct way to measure the cavity field in the micromaser. The only way to tell whether or not the cavity field has been excited by an atom is to send a further atom into the cavity in a known state and to measure the atomic state after it has passed through the cavity. It would therefore be useful to know precisely what measurement an atomic detection in a particular state (or a sequence of detections in various states) makes on the cavity field. This paper is designed to solve this problem for the case that all of the atoms enter the cavity in the excited state. If an atom or sequence of atoms in a known state are passed through the cavity and detected as a sequence of atoms at the output, what does this particular sequence of detections say about the cavity field prior to the passage of the atoms?

This paper is organised as follows. In Section (II) we give a brief overview of retrodictive quantum theory. Section (III) describes how this may be applied to the two-level atom interacting with a single field mode, with particular emphasis on retrodicting micromaser field states form atomic state measurements. Section (IV) discusses the results of particular measurements and in Section (V) contains a summary and discussion of the results.

II. QUANTUM RETRODICTION

In this section we provide brief details of retrodictive quantum theory. A fuller account can be found in references [3–7]. Elsewhere in this volume we provide a more complete

discussion of the quantum theory of preparation and measurement [9]. The analysis which follows, based on preparation operators and positive operator measures (POMs) [10] is necessarily incomplete, but sufficient for our purposes.

Suppose that we have a preparation device which produces output states $\hat{\rho}_i^{\rm pred}$ with prior probabilities $P(i)$. $\hat{\rho}_i^{\rm pred}$ is the usual density operator of predictive quantum mechanics. This state can evolve and interact with other systems until it is measured by a measuring device. A measurement in which none of the measurement results are lost or discarded can be described by a measurement POM [10]. This is a set of non-negative definite, Hermitian elements $\hat{\Pi}_j$ which sum to the unit operator, each element corresponding to a particular measurement outcome. In general there is no requirement that there be the same number of POM elements as there are states which span the system space, but for von Neumann measurement this is so, and the POM elements are simply the projectors of the particular chosen states which span the space. Suppose that preparation takes place at time $t_{\rm p}$ and measurement at a later time $t_{\rm m}$. Within this framework the predictive probability that the measurement outcome $\hat{\Pi}_j$ is obtained given that the state $\hat{\rho}_i^{\rm pred}$ was prepared is

$$P(j|i) = {\rm Tr}\left(\hat{\rho}_i^{\rm pred}(t_{\rm m})\hat{\Pi}_j\right), \tag{1}$$

where

$$\hat{\rho}_i^{\rm pred}(t_{\rm m}) = \hat{U}(\tau)\hat{\rho}_i^{\rm pred}(t_{\rm p})\hat{U}^\dagger(\tau) \tag{2}$$

is the initial density operator evolved forwards in time to the collapse time and

$$\hat{U}(\tau) = \exp\left(-\frac{i}{\hbar}\hat{H}\tau\right) \tag{3}$$

is the evolution operator, which operates for the length of time between preparation and measurement, $\tau = t_{\rm m} - t_{\rm p}$.

Suppose that instead of calculating the predictive probability $P(j|i)$ we wish to calculate the retrodictive conditional probability $P(i|j)$ that the state $\hat{\rho}_i^{\rm pred}$ was prepared, given a particular measurement result $\hat{\Pi}_j$. It is possible to do this by calculating all possible

predictive conditional probabilities for the system, and then using Bayes' theorem. A simpler and more natural approach is to use retrodictive quantum theory, so that the required probability can be written [3,4]

$$P(i|j) = \frac{\mathrm{Tr}\left[\hat{\Lambda}_i \hat{\rho}_j^{\mathrm{retr}}(t_{\mathrm{p}})\right]}{\mathrm{Tr}\left[\hat{\Lambda}\hat{\rho}_j^{\mathrm{retr}}(t_{\mathrm{p}})\right]}. \tag{4}$$

Here the operator $\hat{\Lambda}_i$ is the preparation device operator, and

$$\hat{\Lambda} = \sum_i \hat{\Lambda}_i = \sum_i P(i)\hat{\rho}_i^{\mathrm{pred}}, \tag{5}$$

is the *a priori* density operator, the sum of each possible preparation density operator weighted by its prior probability of production. $\hat{\Lambda}$ is the best description of the state we can give without knowing the outcome of the preparation or measurement. The retrodictive density operator at the preparation time is simply the normalised measurement POM element evolved back from the measurement time to the preparation time,

$$\hat{\rho}_j^{\mathrm{retr}}(t_{\mathrm{p}}) = \hat{U}^{\dagger}(\tau)\hat{\rho}_j^{\mathrm{retr}}(t_{\mathrm{m}})\hat{U}(\tau), \tag{6}$$

with

$$\hat{\rho}_j^{\mathrm{retr}}(t_{\mathrm{m}}) = \frac{\hat{\Pi}_j}{\mathrm{Tr}\hat{\Pi}_j}. \tag{7}$$

The above formulae for the conditional probabilities (eqs. (1) and (4)) apply equally well for open systems, where the system of interest interacts with an unmeasured environment with many degrees of freedom. If this environment causes information to be lost about the system, and the Born-Markov approximation holds, then the the evolved density operators are the solutions of master equations [11]. In eq.(1) the density operator required for such an open system, which for a closed system would be given by eq. (2), is the solution of the usual master equation forwards in time from the preparation time to the measurement time. However, in eq. (4) the solution required instead of eq. (6) is that of the retrodictive master equation, giving the evolution backwards in time from the measurement time to the preparation time. We have recently derived this equation from the general principle that

conditional probabilities should be independent of the time of collapse of the wavefunction [6]. This principle, which applies equally well in open or closed systems, leads to an unusual interpretation of measurements in a quantum system. The measurement POM elements, when evolved back in time, still satisfy the criterion that they form a valid POM, namely that they sum to the unit operator. This means that the evolved POM still represents a valid measurement of the quantum system, and we may regard any amount of the evolution as still forming part of the measurement. The same may be said of the prepared density operator evolved forwards in time.

In the system considered in the present paper, however, the Born-Markov approximation is not made, and so we must consider the full evolution of the coupled atom-field system.

III. PREDICTION AND RETRODICTION FOR THE COUPLED ATOM-FIELD SYSTEM

Here we apply the retrodictive formalism to a coupled system consisting of a two-level atom and a single cavity mode of an electromagnetic field. We have already applied retrodictive quantum mechanics to this system. When the field is excited by a coherent state, predictively there are collapses and revivals of the Rabi oscillations in the atomic excitation probability. If the atom is measured to be in a particular state there also exist earlier revivals or 'previvals' in the atomic excitation at times prior to the measurement [7]. Here we will concentrate specifically on the micromaser, in which atoms pass through a microwave cavity one at a time. The interaction between an atom with upper level $|e\rangle$ and lower level $|g\rangle$, and an electromagnetic field is governed by the Jaynes-Cummings Hamiltonian [11]. In the interaction picture this is

$$\hat{H} = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda\left(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-\right), \tag{8}$$

where Δ is the detuning between the atomic frequency and the cavity mode, $\hat{\sigma}_3 = |e\rangle\langle e| - |g\rangle\langle g|$ is the atomic inversion operator, $\hat{\sigma}_+ = |e\rangle\langle g|$ and $\hat{\sigma}_- = |g\rangle\langle e|$ are the atomic raising

and lowering operators, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for the single mode field, and λ is the coupling constant. The rotating wave approximation, which has been made in deriving this Hamiltonian, ensures that whenever a photon is lost from the field the atomic state must change from $|g\rangle$ to $|e\rangle$. In the standard predictive picture of quantum mechanics a coupled atom-field system evolves forwards in time according to this Hamiltonian from a preparation time $t_{\rm p}$ to a measurement time $t_{\rm m}$. After this has happened the coupled density operator for the whole system is

$$\hat{\rho}_{\rm af}^{\rm pred}(t_{\rm m}) = \hat{U}(\tau)\hat{\rho}_{\rm a}^{\rm pred}(t_{\rm p}) \otimes \hat{\rho}_{\rm f}^{\rm pred}(t_{\rm p})\hat{U}^\dagger(\tau), \tag{9}$$

where $\hat{\rho}_{\rm af}^{\rm pred}(t_{\rm m})$ is the coupled density operator for the atom-field system at the measurement time, $\hat{\rho}_{\rm a}^{\rm pred}(t_{\rm p})$ and $\hat{\rho}_{\rm f}^{\rm pred}(t_{\rm p})$ are the uncoupled atom and field density operators at the preparation time. $\hat{U}(\tau)$ is the evolution operator for the coupled system, given by eq.(3) with Hamiltonian given by eq. (8). In the micromaser we generally wish to infer the state of the cavity field from a measurement of the atom. This is done by conditioning on the atomic measurement outcome and tracing over the atomic states,

$$\hat{\rho}_{\rm f}^{\rm pred}(t_{\rm m}) \propto {\rm Tr}_{\rm a}\left[\hat{\Pi}_{\rm a}\hat{U}(\tau)\hat{\rho}_{\rm a}^{\rm pred}(t_{\rm p}) \otimes \hat{\rho}_{\rm f}^{\rm pred}(t_{\rm p})\hat{U}^\dagger(\tau)\right], \tag{10}$$

where $\hat{\Pi}_{\rm a}$ is the POM element corresponding to the outcome of the measurement performed on the atom. If no measurement is performed on the atom then $\hat{\Pi}_{\rm a} = \hat{1}_{\rm a}$. Similarly we may not have any information about the initial state of the field, in which case $\hat{\rho}_{\rm f}^{\rm pred}(t_{\rm p}) \propto \hat{1}_{\rm f}$.

The retrodictive picture differs from above in that the state of the system before the measurement is assigned on the basis of the measurement outcome. Thus if the measurement POM elements for the atom and the field are $\hat{\Pi}_{\rm a}(t_{\rm m})$ and $\hat{\Pi}_{\rm f}(t_{\rm m})$, the coupled initial density operator corresponding to equation (9) is

$$\begin{aligned}\hat{\rho}_{\rm af}^{\rm retr}(t_{\rm p}) &= \hat{U}^\dagger(\tau)\hat{\rho}_{\rm a}^{\rm retr}(t_{\rm m}) \otimes \hat{\rho}_{\rm f}^{\rm retr}(t_{\rm m})\hat{U}(\tau) \\ &\propto \hat{U}^\dagger(\tau)\hat{\Pi}_{\rm a}(t_{\rm m}) \otimes \hat{\Pi}_{\rm f}(t_{\rm m})\hat{U}(\tau).\end{aligned} \tag{11}$$

In the micromaser the atom will in general have been prepared in a particular initial state,

and this state conditions the coupled density operator similarly to equation (10), to give the retrodictive density operator for the field

$$\hat{\rho}_{\rm f}^{\rm retr}(t_{\rm p}) \propto {\rm Tr}_{\rm a}\left[\hat{\rho}_{\rm a}^{\rm pred}(t_{\rm p})\hat{U}^{\dagger}(\tau)\hat{\Pi}_{\rm a}(t_{\rm m})\otimes\hat{\Pi}_{\rm f}(t_{\rm m})\hat{U}(\tau)\right], \tag{12}$$

where the constant of proportionality is determined by normalisation. If there is no information at all about the preparation of the initial states then $\hat{\rho}_{\rm a}^{\rm pred}(t_{\rm p})$ becomes proportional to the unit operator for the atomic space. Equation (12) is enough to provide all of the information needed for this paper.

IV. RETRODICTION OF MICROMASER FIELD STATES

In this section we apply the formalism derived in the previous section to the micromaser. Two-level atoms in their excited state are passed through an optical cavity and a measurement of their state is performed. We assume that the atomic frequency is resonant with the micromaser field mode. This is a typical set-up of a micromaser experiment [8,12]. The final field state is unmeasured and we wish to know what the initial field state was before the atoms passed through. The formalism can equally well be applied for any other initial atomic state, but we will leave this for a further publication. In order to find the initial field state from a sequence of atomic measurements we first consider the effect of a single atom passing through the cavity. Equation (12) provides the conditioned retrodictive field state given knowledge of the initial and final atomic states. The preparation time $t_{\rm p}$ is effectively the time at which the atom enters the cavity, and as the atomic state does not change after the atom leaves the cavity, the exit time is the measurement time $t_{\rm m}$.

Before their passage through the cavity the atoms are initially in their excited state, so $\hat{\rho}_{\rm a}(t_{\rm p}) = |e\rangle\langle e|$. The trace over the atomic states is trivial, and the initial field state is

$$\hat{\rho}_{\rm f}^{\rm retr}(t_{\rm p}) \propto \langle e|\hat{U}^{\dagger}(\tau)\hat{\Pi}_{a}(t_{\rm m})\otimes\hat{\Pi}_{f}(t_{\rm m})\hat{U}(\tau)|e\rangle. \tag{13}$$

We assume that the cavity POM element at $t_{\rm m}$ is given by

$$\Pi_{\rm f}(t_{\rm m}) \propto \hat{\rho}_{\rm f}(t_{\rm m}) = \sum_n P_n(t_{\rm m})|n\rangle\langle n|, \tag{14}$$

where $P_n(t_{\rm m})$ is the probability that the field contains n photons at $(t_{\rm m})$. If the atom is detected in the excited state then the POM element (and retrodictive density operator) is simply the excited state projector. The unitary evolution operator is given by equation (3). A little algebra gives the result

$$\hat{\rho}_{\rm f}^{\rm retr}(t_{\rm p}) \propto \sum_n P_n(t_{\rm m}) \cos^2[\Omega(n+1)\tau/2]|n\rangle\langle n|, \tag{15}$$

where $\Omega(n) = 2\lambda\sqrt{n}$. Thus the initial state photon number probabilities are equal to the final state probabilities multiplied by a cosine function whose argument is the product of the Rabi frequency and the interaction time. If the atom is detected in the ground state a similar calculation gives

$$\hat{\rho}_{\rm f}^{\rm retr}(t_{\rm p}) \propto \sum_n P_{n+1}(t_{\rm m}) \sin^2[\Omega(n+1)\tau/2]|n\rangle\langle n|. \tag{16}$$

In this case the probability that the initial state contains n photons depends upon the final state probability for $n+1$ photons and a sine function which contains the $n+1$-photon Rabi frequency. The occurrence of $P_{n+1}(t_{\rm m})$ in the above formula is easy to explain, as when the atom exits in the ground state there is one more photon in the final field than the initial.

Successive atoms entering the cavity can be treated one at a time. If two atoms enter the cavity in sequence then the initial retrodictive state into which the cavity is projected by the detection of the second atom becomes the final state in an identical calculation to the above for the first atom entering the cavity. Thus every time an atom enters the cavity and leaves in its excited state the photon number probabilities undergo the transformation

$$P_n(t_{\rm p}) \propto P_n(t_{\rm m}) \cos^2[\Omega(n+1)\tau/2], \tag{17}$$

and when an atom leaves in the ground state the transition is

$$P_n(t_{\rm p}) \propto P_{n+1}(t_{\rm m}) \sin^2[\Omega(n+1)\tau/2]. \tag{18}$$

The exact retrodictive density operator can be found by normalisation at the end of the calculation by simply requiring all of the probabilities at $t_{\rm p}$ for the first atom to sum to unity.

As a simple example consider two atoms passing through the cavity. There are four possible pairs of detection events which are summarised in the table below.

detection events	initial photon number relative probability
$\|e\rangle$ then $\|e\rangle$	$P_n \cos^4[\Omega(n+1)\tau/2]$
$\|g\rangle$ then $\|g\rangle$	$P_{n+2} \sin^2[\Omega(n+2)\tau/2] \sin^2[\Omega(n+1)\tau/2]$
$\|e\rangle$ then $\|g\rangle$	$P_{n+1} \cos^2[\Omega(n+1)\tau/2] \sin^2[\Omega(n+1)\tau/2]$
$\|g\rangle$ then $\|e\rangle$	$P_{n+1} \cos^2[\Omega(n+2)\tau/2] \sin^2[\Omega(n+1)\tau/2]$

Table 1. Relative probability that the cavity contains n photons prior to the passage and detection of two atoms. P_n is the final state probability for n photons after both atoms have been detected.

It is clear from the last two lines of the table that the order of detection matters. Every time an atom is detected in its ground state the number of photons in the cavity increases, and the Rabi frequencies increases correspondingly. Whether this occurs before or after the detection of an atom in the excited state will affect the Rabi factor imposed on the initial state probability.

Measurement outcomes are characterised by their corresponding POM elements [10]. The atom detection experiments make a measurement of the cavity field, so there must be a POM element corresponding to this measurement. Different numbers and orderings of atomic detections in either the excited or ground states correspond to different POM elements [10]. For s atomic detections in either the excited or ground state there are a possible 2^s POM elements $\hat{\Pi}_{1..s}$. These POM elements can be used to calculate the probability for any given sequence of atom measurements given a known initial field state density operator $\hat{\rho}_{\rm f}(t_{\rm p})$. The POM elements may be found by using an unmeasured state as the final cavity state that, in the usual predictive formalism, would have the unit operator for the cavity state

space as its associated POM element. This implies no knowledge of the final cavity state. It is convenient to set an upper limit N on the allowed photon number distribution. This allows us to determine a well-behaved retrodictive field state. A derivation of the field POM elements without this cut-off is given in the appendix. Thus in place of the unit operator we can use the operator

$$\hat{1}_{\rm f} = \sum_{n=0}^{N} |n\rangle\langle n| \tag{19}$$

corresponding to a final state $\hat{1}_{\rm f}/(N+1)$. This is the unit operator acting on the $N+1$-dimensional subspace spanned by the photon number states up to and including $|N\rangle$. Backward time evolution including the passage through the cavity of each of the s atoms yields the POM elements $\hat{\Pi}_{1..s}$ in the form

$$\hat{\Pi}_{1..s} = \sum_{n=0}^{N} C_{n,1..s}|n\rangle\langle n| \tag{20}$$

where the coefficients $C_{n,1..s}$ will be proportional to the corresponding coefficients in the expression for $\hat{\rho}_{\rm f}^{\rm retr}(t_{\rm p})$. The POM elements must be normalised to sum to the unit operator or, in this case, the unit operator $\hat{1}_{\rm f}$ that acts on the restricted photon subspace. For the two-atom case the coefficients $C_{n,1..s}$ for the four POM elements are just the trigonometric factors given in Table 1. For example $C_{n,ee}$ for the detection of $|e\rangle$ then $|e\rangle$ is $\cos^4[\Omega(n+1)\tau/2]$.

Figures 1-4 show photon number probabilities (and thus give the POM elements) corresponding to particular sequences of atomic detections. In each figure the interaction strength and time have been set so that $\lambda\tau = \pi$.

Figure 1 shows the relative photon number probabilities for a sequence of atoms detected in the excited state. If a single atom passes through the cavity, and is detected the initial photon number distribution of the cavity has the cosinusoidal form shown in figure 1(a). The distribution has maxima at photon numbers which are one less than perfect squares, corresponding to Rabi phase shifts of integral multiples of π. If the cavity contains one of these numbers of photons then an atom in the excited state must exit in the excited state. This is what is known as the trapping state condition, and it has been used to prepare

photon number states of the electromagnetic field [12]. If a whole series of atoms pass through and are detected in the excited state the troughs in the distribution become wider as the passage of one more atom means that each photon number probability gets multiplied by a cosinusoidal factor smaller than unity for all photon numbers except those one less than perfect squares. This is shown for five atoms in figure 1(b).

Detecting atoms in the ground state gives rise to a different cavity POM element. In this case the minima of the sinusoidal distribution fall at photon numbers which are one less than perfect squares. One of the photons in the cavity when the atom is measured had to come from atomic de-excitation so there had to be one fewer photon in the cavity prior to the passage of the atom. Thus the whole distribution at $t_{\rm m}$ is shifted by one photon at $t_{\rm p}$. The distribution for one atom detected in the ground state is shown in figure 2(a).

Detecting atoms in the ground state thus has two effects: (1) it shifts the photon number probability distribution downwards by one photon for each atom detected and, (2) for a system which satisfies the trapping state condition it introduces zeros into the distribution . The zeros occur because the Rabi phase shifts for excitation numbers one greater than these photon numbers are π, and so atoms prepared in the excited state can not exit in the ground state. These two effects combine to allow the measurement of photon number distributions with large gaps in them. For example, if three atoms are detected in the ground state then the cavity field cannot have contained fewer than four photons, as shown by figure 2(b). This POM element also has a photon number gap from 6-8 photons, and there are also gaps at higher photon number. These gaps extend their range downwards by one photon with each extra ground state detection. This is shown in figure 2(c), which is the photon number distribution prior to six atoms detected in the ground state. The initial cavity photon distribution must have contained at least nine photons. This also gives information about the number of photons in the cavity after the detection. There must be at least 15. Six of these come from the passing atoms and at least nine were in the cavity initially.

If we have a little extra prior information about the cavity photon number distribution this measurement can allow us to measure and prepare specific photon number states. For

example, say two atoms are sent through the cavity and detected in the ground state. If we know that there are not more than five photons in the cavity after the atoms have passed through (not more than three prior to the atomic passage) then these two atoms are sufficient to give an initial cavity distribution of one photon for certain (figure 3). This measures the one photon number state. Furthermore, the fact that the cavity initially contained one photon means that the de-excitation of the two atoms has now created a three photon state. Other possible measurements and preparations are possible, with the restriction on photon number a little higher.

In general a set of atoms passing through the cavity will not all be detected in the excited or ground state. Detection of any atoms in the ground state will introduce zeros into the photon number distribution, and these will move towards lower photon numbers with each successive atom detected in the ground state. Otherwise the distribution will be peaked around photon numbers where the product of the cosinusoidal and sinusoidal distributions is maximised. Examination of figures 1(a) and 2(a) shows that this will be around five and ten photons. This is borne out by figure 4 which shows two typical results for six detected atoms, alternating between detection in the excited and ground states. The only significant probabilities are for 4-5 and 9-12 photons. Other orderings give slightly different ratios of probability between these numbers, but the results are similar.

V. CONCLUSIONS

In this paper we have analysed field measurements in the micromaser. The field is not directly measurable, but knowledge of the prepared initial atomic state and the measured state after the interaction between the atoms and the cavity gives information about the initial cavity photon number distribution. If nothing is known *a priori* about the cavity field this allows us to calculate the POM elements for the field corresponding to particular sequences of atomic detections in either the excited or ground states.

We have considered the case where the atoms are all initially prepared in the excited

state. If an atom is detected in the excited state the calculated initial photon number distribution is simply the distribution after the measurement multiplied by a cosinusoidal Rabi factor which depends upon one more than the final photon number. This happens for each repeated detection in the excited state. Atoms detected in the ground state give the initial field a sinusoidal distribution which depends upon the one more than the final photon number.

If the atoms are measured in the excited state the cavity is much more likely to have been in a photon number state corresponding to a trapping state, for which the Rabi phase shift is a multiple of π. Conversely, if the atom is measured in the ground state the cavity field cannot have been in a trapping state. The probability of states with these photon numbers is zero. Repeated measurement in the ground state therefore measures photon number distributions with large gaps. This can be used both to measure and prepare photon number states, particularly where some information about the field also exists.

Earlier work on retrodictive master equations [6], and on the derivation of the POM elements corresponding to homodyne detection (the line states) [13], has shown that the boundaries between preparation, evolution and measurement in quantum theory are arbitrary choices. This is emphasised in the micromaser where the measurement *and* preparation of particular atomic states corresponds to a single measurement of the field. A measurement of a quantum system necessarily includes all of the available information about the system. If any of this information should change so does the measurement POM. If the atoms had been prepared in a superposition of the excited and ground states then measurement in the excited or ground state would correspond to different field POM elements to those described here. We will explore preparation and measurement of the atom in superpositions in further work.

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APPENDIX

The sequence of detections of atoms in their ground or excited states represents a measurement of the initial field. Our analysis has concentrated on the retrodictive field state prior to interaction with the first atom. It is not necessary, however, to introduce the retrodictive state in order to analyse the field measurement. It can be calculated simply by normalising the derived POM element. In calculating the POM elements we have placed a limit on the photon number of the final cavity field. This is also not strictly necessary provided that we are only interested in the field measurement and its associated POM. If we know the initial field state, $\hat{\rho}_{\rm f}^{\rm pred}(t_{\rm p})$, then these POM elements give us the probabilities for any given sequence of atom detections.

In constructing the POM, we start with the unmeasured field state after the final ($s^{\rm th}$) atom has passed throught the cavity. We associate with this the trivial POM element

$$\hat{\Pi}_{\rm f}^{\rm final} = \hat{1}_{\rm f} = \sum_{n=0}^{\infty} |n\rangle\langle n|. \tag{A.1}$$

The POM element associated with immediately prior to detection of the $s^{\rm th}$ atom (denoted $\hat{\Pi}_{\rm f}^{\rm s}$) will depend upon whether the atom was detected in the excited or ground state:

$$\begin{aligned}\hat{\Pi}_{\rm f}^{\rm s}(e) &= \langle e|\hat{U}^{\dagger}|e\rangle\langle e| \otimes \hat{1}_{\rm f}\hat{U}|e\rangle \\ &= \cos\left[\Omega(\hat{n}+1)\tau/2\right]\hat{1}_{\rm f}\cos\left[\Omega(\hat{n}+1)\tau/2\right] \\ &= \cos^2\left[\Omega(\hat{n}+1)\tau/2\right], \end{aligned} \tag{A.2}$$

$$\begin{aligned}\hat{\Pi}_{\rm f}^{\rm s}(g) &= \langle e|\hat{U}^\dagger|g\rangle\langle g|\otimes\hat{1}_{\rm f}\hat{U}|e\rangle \\ &= \sin\left[\Omega(\hat{n}+1)\tau/2\right]\hat{E}\hat{1}_{\rm f}\hat{E}^\dagger\sin\left[\Omega(\hat{n}+1)\tau/2\right] \\ &= \sin^2\left[\Omega(\hat{n}+1)\tau/2\right]. \end{aligned} \tag{A.3}$$

Here $\hat{n}$ is the photon number operator and $\hat{E}(\hat{E}^\dagger)$ is the bare lowering (raising) operator [14],

$$\hat{E} = \sum_{n=0}^{\infty}|n\rangle\langle n+1| \tag{A.4}$$

$$\hat{E}^\dagger = \sum_{n=0}^{\infty}|n+1\rangle\langle n|. \tag{A.5}$$

These operators have the properties

$$\hat{E}f(\hat{n}) = f(\hat{n}+1) \tag{A.6}$$

$$f(\hat{n}) = \hat{E}^\dagger f(\hat{n}+1) \tag{A.7}$$

$$\hat{E}\hat{E}^\dagger = \hat{1} \tag{A.8}$$

$$\hat{E}^\dagger\hat{E} = \hat{1} - |0\rangle\langle 0|. \tag{A.9}$$

We can use these operators to write the POM elements associated with any given sequence of atom detections by induction. This means starting with the last atom detected and writing the POM element $\cos^2\left[\Omega(\hat{n}+1)\tau/2\right]$ if the atom was in its excited state and $\sin^2\left[\Omega(\hat{n}+1)\tau/2\right]$ if it was in its ground state. For each preceding detection we pre- and postmultiply the POM element by $\cos\left[\Omega(\hat{n}+1)\tau/2\right]$ if the atom was measured in its excited state. If the atom was found in its ground state then we premultiply by $\sin\left[\Omega(\hat{n}+1)\tau/2\right]\hat{E}$ and postmultiply by $\hat{E}^\dagger\sin\left[\Omega(\hat{n}+1)\tau/2\right]$.

As an example, we consider a sequence of three atoms. The corresponding POM elements $\hat{\Pi}_{\rm f}(i,j,k)$ $(i,j,k=e,g)$ are associated with the atoms interacting with the cavity in the order i,j then k. The eight possible POM elements are.

$$\hat{\Pi}_{\rm f}(e,e,e) = \hat{c}^6(1), \tag{A.10}$$

$$\hat{\Pi}_{\rm f}(e,e,g) = \hat{c}^2(1)\hat{s}(1)\hat{E}\hat{E}^\dagger\hat{s}(1)\hat{c}^2(1) = \hat{c}^4(1)\hat{s}^2(1), \tag{A.11}$$

$$\hat{\Pi}_{\rm f}(e,g,e) = \hat{c}(1)\hat{s}(1)\hat{E}\hat{c}^2(1)\hat{E}^\dagger\hat{s}(1)\hat{c}(1) = \hat{c}^2(1)\hat{s}^2(1)\hat{c}^2(2), \tag{A.12}$$

$$\hat{\Pi}_{\rm f}(e,g,g) = \hat{c}(1)\hat{s}(1)\hat{E}\hat{s}(1)\hat{E}\hat{E}^\dagger\hat{s}(1)\hat{E}^\dagger\hat{s}(1)\hat{c}(1) = \hat{c}^2(1)\hat{s}^2(1)\hat{s}^2(2), \tag{A.13}$$

$$\hat{\Pi}_{\rm f}(g,e,e) = \hat{s}(1)\hat{E}\hat{c}^4(1)\hat{E}^\dagger\hat{s}(1) = \hat{s}^2(1)\hat{c}^4(2), \tag{A.14}$$

$$\hat{\Pi}_{\rm f}(g,e,g) = \hat{s}(1)\hat{E}\hat{c}(1)\hat{s}(1)\hat{E}\hat{E}^\dagger\hat{s}(1)\hat{c}(1)\hat{E}^\dagger\hat{s}(1) = \hat{s}^2(1)\hat{c}^2(2)\hat{s}^2(2), \tag{A.15}$$

$$\hat{\Pi}_{\rm f}(g,g,e) = \hat{s}(1)\hat{E}\hat{s}(1)\hat{E}\hat{c}^2(1)\hat{E}^\dagger\hat{s}(1)\hat{E}^\dagger\hat{s}(1) = \hat{s}^2(1)\hat{s}^2(2)\hat{c}^2(3), \tag{A.16}$$

$$\hat{\Pi}_{\rm f}(g,g,g) = \hat{s}(1)\hat{E}\hat{s}(1)\hat{E}\hat{s}(1)\hat{E}\hat{E}^\dagger\hat{s}(1)\hat{E}^\dagger\hat{s}(1)\hat{E}^\dagger\hat{s}(1) = \hat{s}^2(1)\hat{s}^2(2)\hat{s}^2(3). \tag{A.17}$$

Here we have written

$$\hat{c}(m) = \cos\left[\Omega(\hat{n}+m)\tau/2\right], \tag{A.18}$$

$$\hat{s}(m) = \sin\left[\Omega(\hat{n}+m)\tau/2\right]. \tag{A.19}$$

We can also summarise the construction of the field POM elements by iteration starting from the first atom to enter the cavity:

(1) Start with the first atom to interact with the cavity.

(2) If the atom is measured in the excited state then write $\hat{c}^2(1)$. If the atom is measured in the ground state them write $\hat{s}^2(1)$ and add 1 to the arguments of all subsequent $\hat{c}$ and $\hat{s}$ operators.

(3) Follow the same prescription for each subsequent atom in turn.

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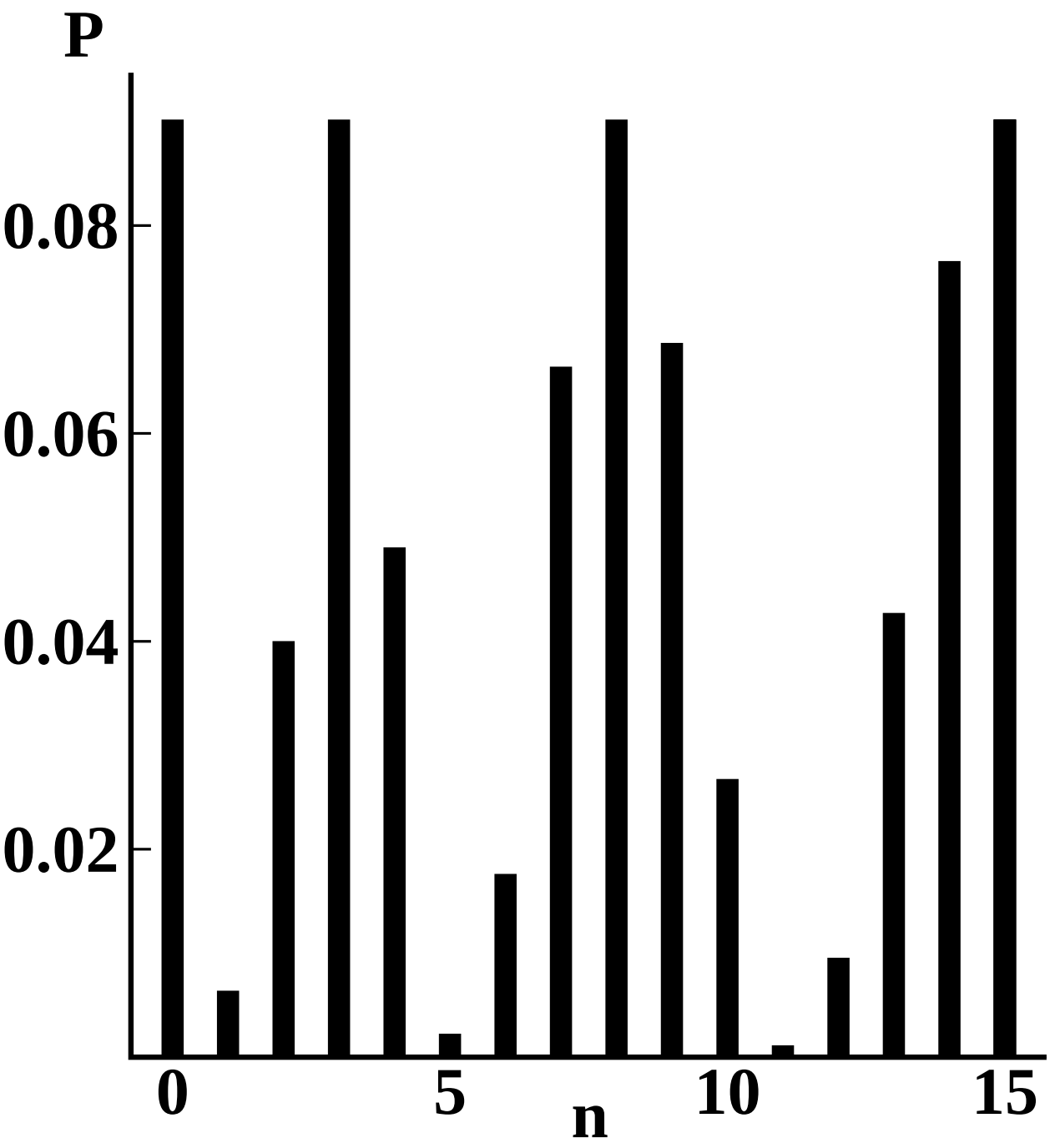
Figure Captions

Figure 1. Measured cavity photon number probability distributions for atoms detected in the excited state (a) one atom, (b) five atoms.

Figure 2. Measured cavity photon number distributions for atoms detected in the ground state (a) one atom, (b) three atoms, (c) six atoms.

Figure 3. Measured cavity photon number probability distributions for two atoms detected in the ground state given that there were no more than three photons in the cavity prior to the passage of the atoms.

Figure 4. Measured cavity photon number probability distributions for six detected atoms, three in the ground state, three in the excited state in the order (a) g, e, g, e, g, e and (b) e, g, e, g, e, g.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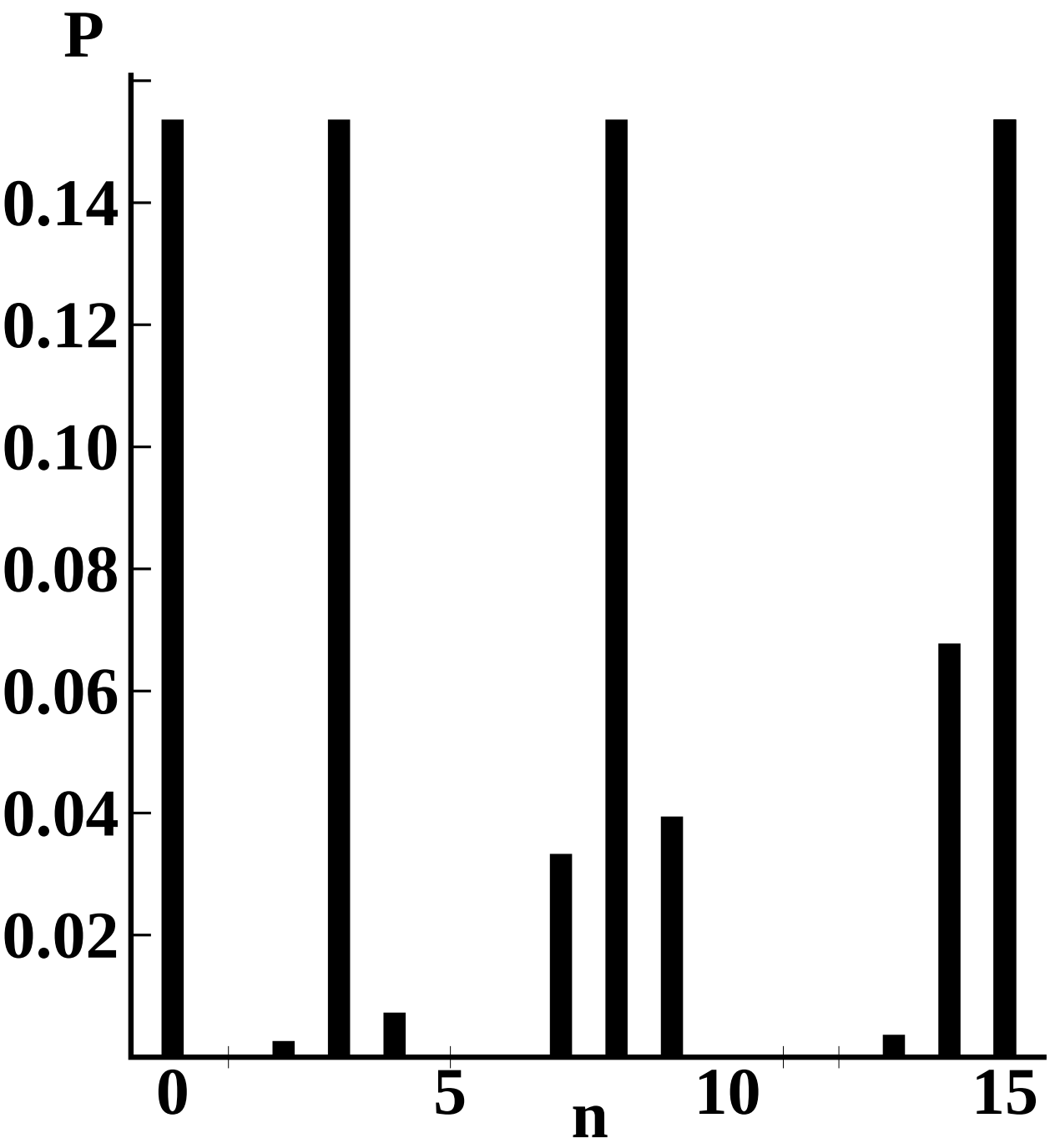
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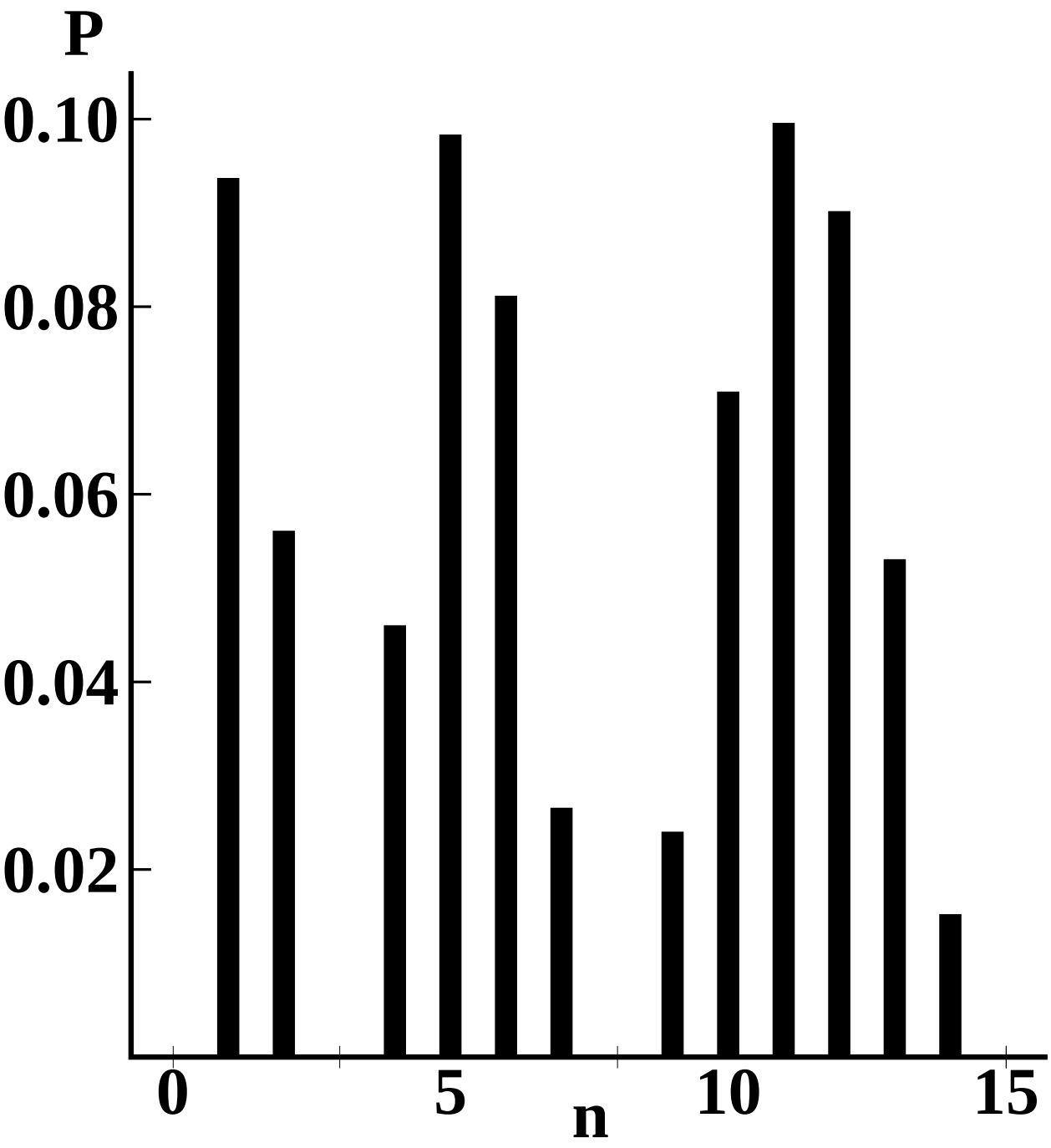
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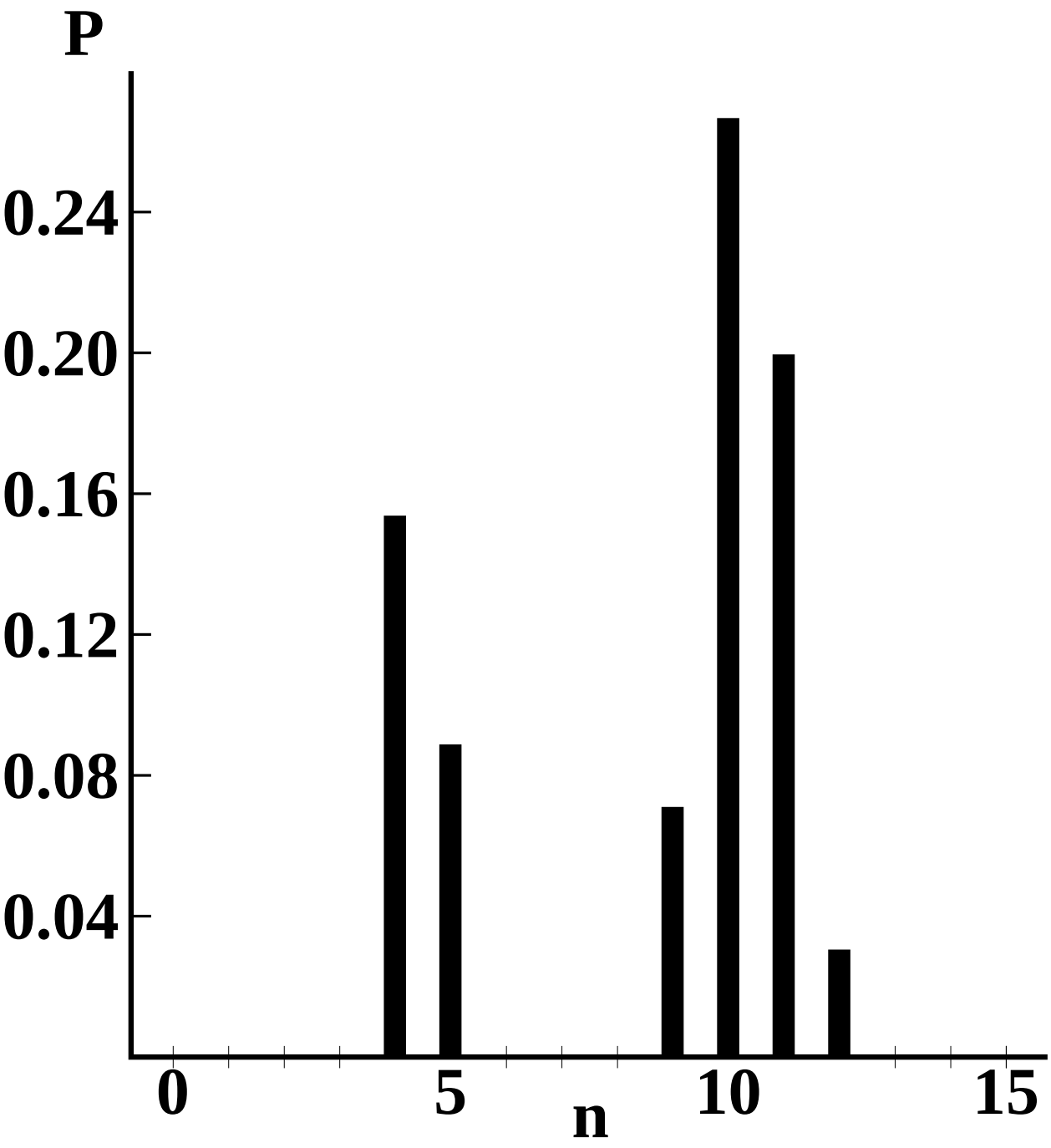
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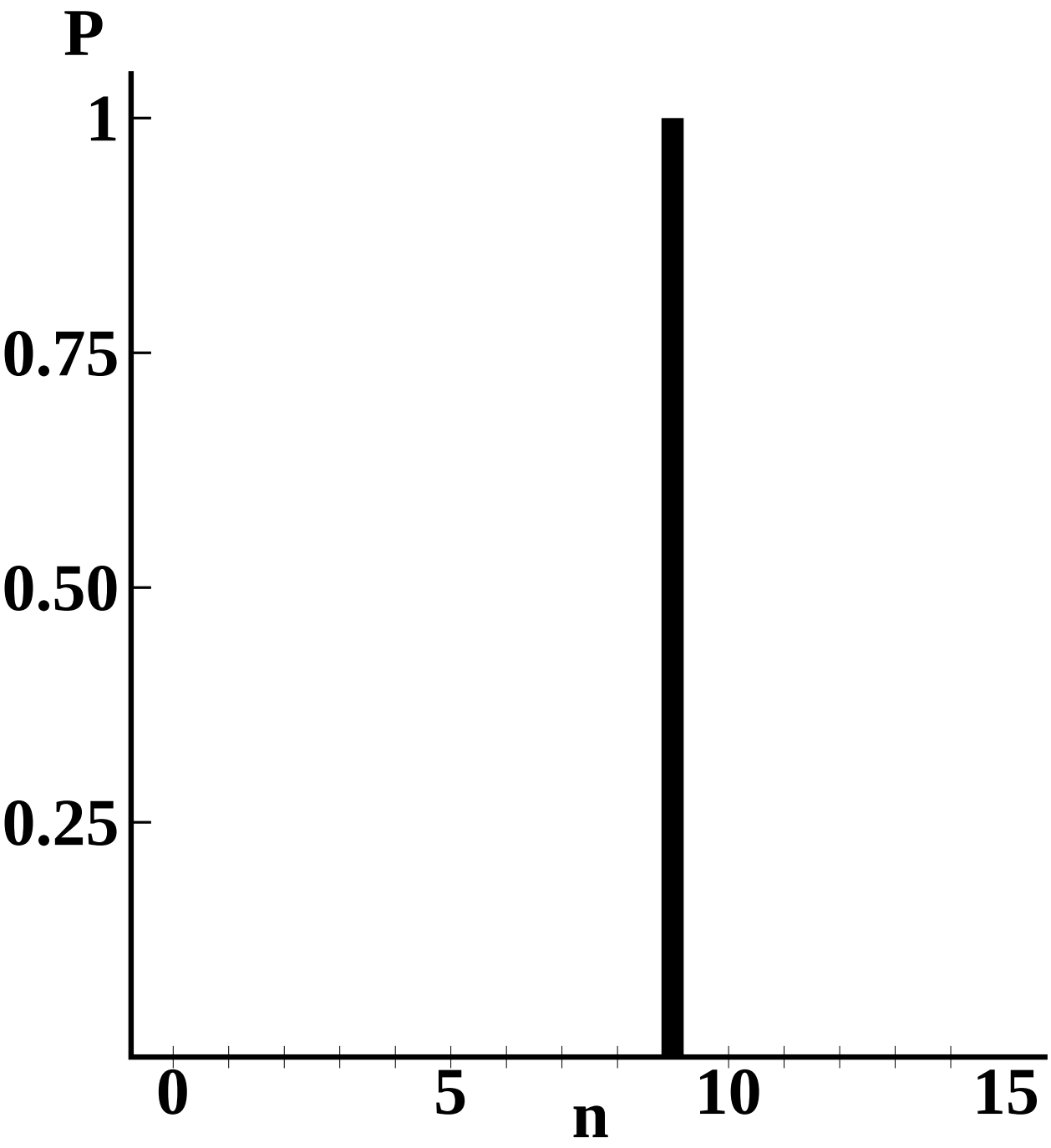
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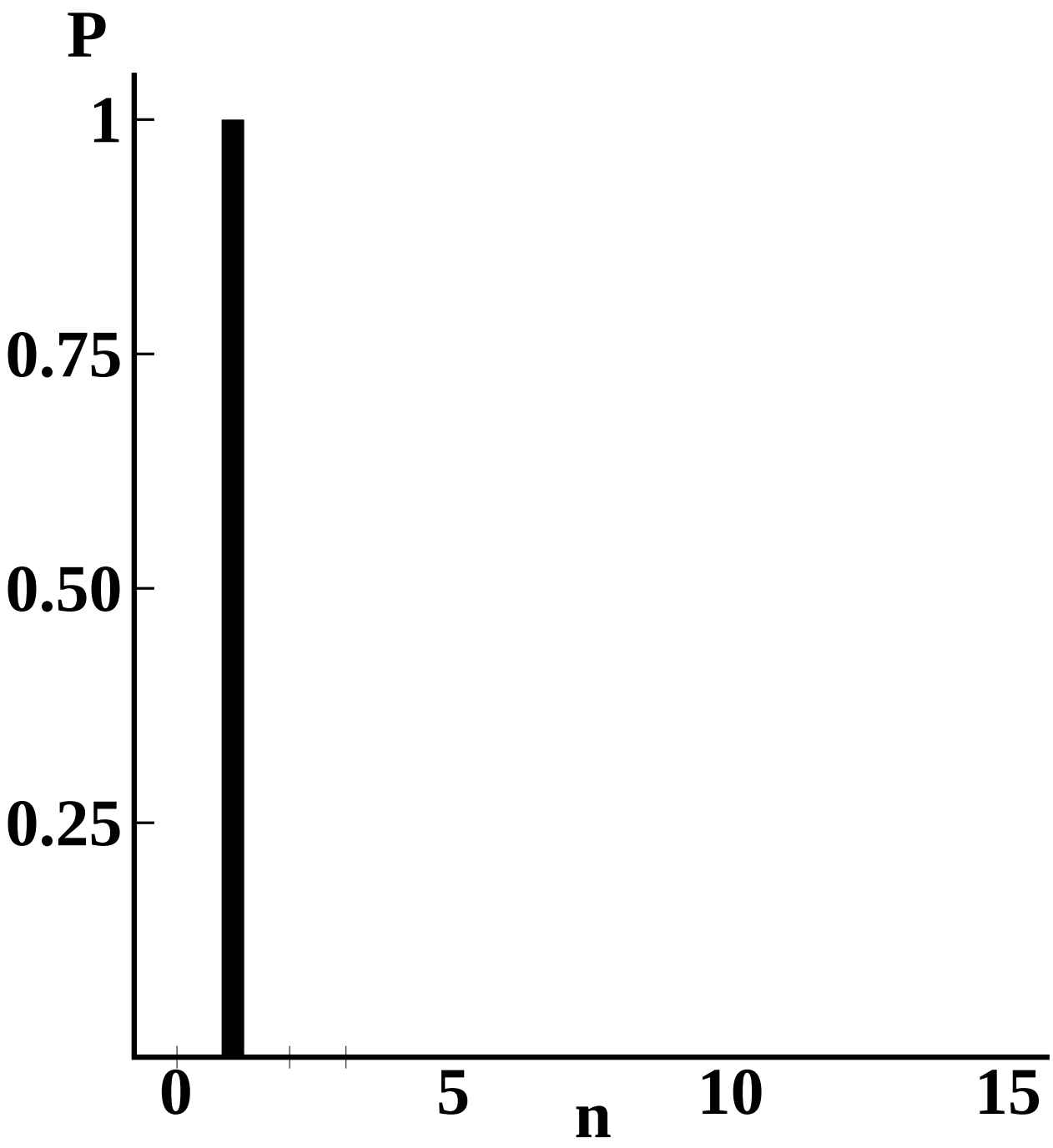
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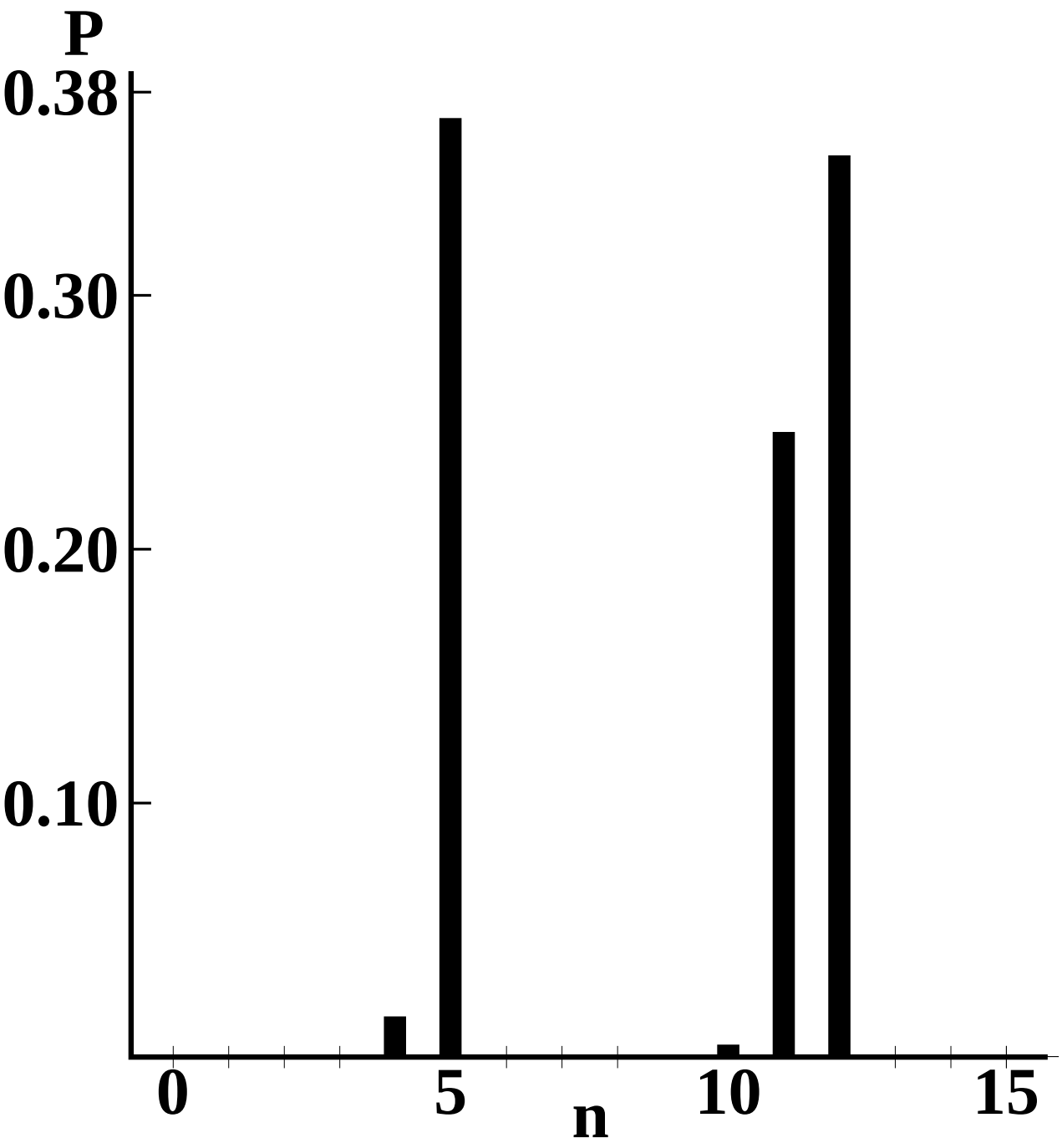
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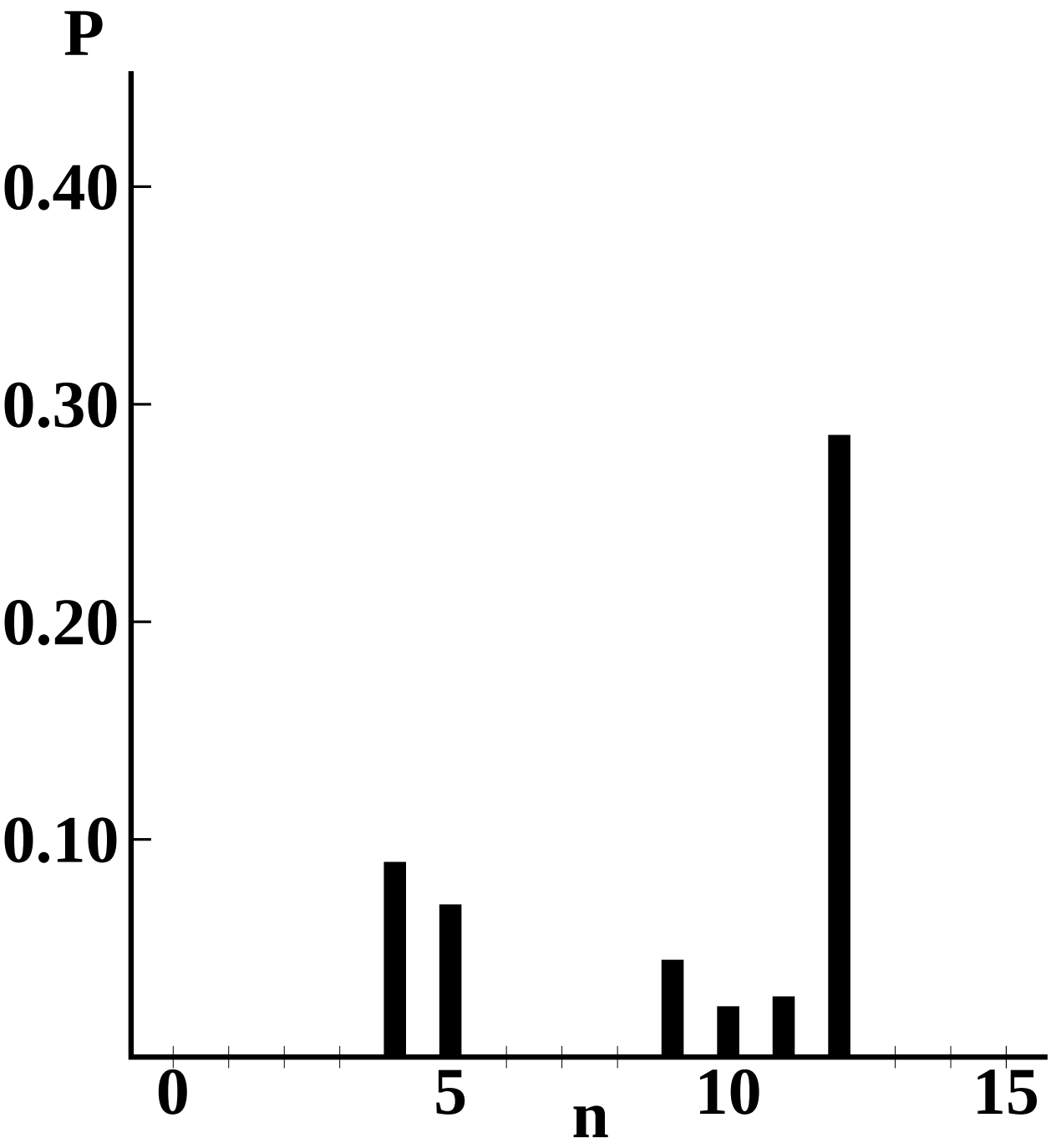
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