Simulation of Deposit Formation in Particle Laden Flows: Thermal Properties

Simulazione della Formazione di Deposito in Flussi Carichi di Particelle: Propriet Termiche

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Abstract

Fatty deposits formed on arterial walls lead to atherosclerosis but it is the interplay between these deposits and the vessel walls which govern the growth of plaque formation. Crucially however the vast majority of acute coronary syndromes such as, myocardial infarction, and sudden ischaemic cardiac death are caused by atherosclerotic plaque rupture and not from a stenosis growing and blocking the blood flow. In fact, atherosclerotic plaques expand into the vessel wall during much of their existence and this can make their detection problematic. However inflammation within the necrotic core of the plaque, can be used to detect which plaques may be vulnerable. Thermal mapping of arterial walls can help identify the most likely sites for plaque rupture.This paper aims to provide a direct link between the geometry of these deposits and their thermal properties in order that non-invasive imaging techniques could be used to spot vulnerable plaques. We will discuss a methodology for estimating the thermal conductivity which utilises self-similarity properties using fractal analysis and renormalisation. The self-similar microstructure is captured by a family of random fractals called shuffled Sierpinski carpets (SSC). The thermal conductivity of the SSC can then be predicted both from its box counting fractal dimension and via a generalised real space renormalisation method. This latter approach also affords an analysis of the percolation threshold of two phase fractal media.

Abstract

I depositi di grassi che si formano sulle pareti arteriose provocano l'aterosclerosi ma l'interazione fra questi depositi e le pareti del vaso che governano lo sviluppo della formazione delle placche. Fondamentalmente tuttavia la grande maggioranza delle sindromi coronariche acute come l'infarto miocardico e la morte cardiaca ischemica improvvisa causata dalla rottura della placca aterosclerotica e non da una stenosi che cresce e ostruisce il flusso di sangue. Infatti, le placche aterosclerotiche si espandono nella parete del vaso durante gran parte della loro esistenza e questo pu rendere la loro rilevazione problematica. Tuttavia l'infiammazione nel nucleo necrotico della placca, pu essere usata per rilevare quali placche possono essere vulnerabili. La mappatura termica delle pareti arteriose pu contribuire ad identificare i luoghi pi probabili per la rottura delle placche. Questo articolo intende fornire un legame diretto fra la geometria di questi depositi e le loro propriet termiche per poter utilizzare le tecniche non-invasive visualizzazione nell'identificazione delle placche vulnerabili. Discuteremo una metodologia per la valutazione della conduttivit termica che utilizza le propriet di auto-similitudine usando l'analisi frattale ed la rinormalizzazione. Le microstrutture di auto-similitudine vengono individuate da una famiglia di frattali casuali denominati shuffled Sierpinski carpets (SSC). La conduttivit termica degli SSC pu allora essere prevista sia dalla relativa scatola che conta la dimensione frattale che per mezzo di un metodo generalizzato di rinormalizzazione dello spazio reale. Questo ultimo approccio permette inoltre un'analisi della soglia di percolazione dei mezzi frattali bifase.

1 Introduction

The transport and subsequent deposition of fatty substances on arterial walls leads to atherosclerosis. The rupture of the atherosclerotic plaque is the most common cause of acute coronary syndromes [1]. Detecting unstable plaques at an early enough stage is vital if preventative measures are to be employed. Shear stresses on artery walls caused by the blood flow does play a role but it has been found that the degree of stenosis is a relatively minor factor in predicting which plaques are most prone to rupture. The fatty substances, once deposited in the vessel walls, lead to a series of mechanisms which promote plaque growth and which ultimately trigger the body's defenses. Release of heat from these activated inflammatory cells gives rise to hotter arterial wall regions. Recent experiments have shown that the temperature difference between atherosclerotic plaques and normal vessels is measurable [2]. An infrared angiothermography catheter [3] and a thermistor probe catheter [4, 5] have been employed for generating thermal maps of arterial walls. By identifying regions where the inflammatory cells are most active, these thermal maps can identify the most likely sites for plaque rupture. The vulnerable plaques have a soft lipid core composed of macrophages, full of cholesterol, which is enclosed by a thin fibrous cap. These macrophages contain cholesterol which release matrix-digesting enzymes leading to plaque rupture [6]. In addition, this problem can be exacerbated by insulin deficiencies caused by diabetes. One reason for this is that there is an increase in lipid particles being transported from the body's fat reserves, which lead inevitably to an increase in the rate of deposition on the blood vessel walls.

The above plaque detection methodology could be enhanced if a non-invasive technique could be developed. If a link between the plaque geometry and its thermal properties could be established then one could infer the location of vulnerable plaques. Indeed, recent experimental evidence suggests that plaque geometry can be used to predict plaque instability [7]. We have recently derived a methodology which directly relates the thermal conductivity of a randomly aggregated deposit to its geometry [8-10]. The *in situ* determination of the physical and geometrical properties of plaques is intrinsically difficult and so we have simulated the material growth using a Monte Carlo type method. We have simulated the deposition from a particle laden flow in a two-dimensional rectangular channel. We have shown that it is possible to model the transport and deposition in a particle laden flow by coupling together the Euler-Lagrangian bulk flow calculation with a Monte Carlo simulation near the wall boundaries [11]. An alternative approach using Lattice Boltzmann Simulations to model the deposition processes in blood flow is the subject of current research [12].

In this paper we discuss a class of random fractals known as shuffled Sierpinski carpets (SSC) (see Figure 1) which capture the self-similar microstructure of the deposit material. The thermal conductivity of these fractals can be determined both by a real space renormalisation approach and from its box counting fractal dimension. The former approach also affords an analysis of the percolation threshold of two phase fractal media.

2 Methodology

In this section we detail the real space renormalisation group (RSRG) approach to the calculation of the thermal conductivity and percolation properties of randomly aggregated deposits. We will first use a standard RSRG approach [13] to examine how well this can estimate the effective conductivity of a monodispersed random aggregate. To simplify the following discussion we will restrict ourselves to a two phase composite, embedded in a two dimensional Euclidean plane, but these ideas extend in a natural way to multiple phases in higher dimensions.

2.1 Monodispersed Aggregates

Suppose we have a two phase composite consisting of phase A and phase B and let p_B be the volume fraction of B. If we denote the conductances of the pure substances A and B as σ_A and σ_B respectively,then the initial probability distribution of the conductances is $I\!P_0(\sigma) = (1 - p_B)\delta(\sigma - \sigma_A) + p_B\delta(\sigma - \sigma_B)$. A square grid is placed over the structure where the size of the grid elements corresponds to the smallest monophase particle. The grid is then divided up into groups of 2×2 cells (again this can be done more generally) or renormalisation clusters to form the next tessellation β_1. In general β_w is renormalised to β_{w+1} by replacing every renormalisation cluster in β_w by a cell whose effective conductivity is equivalent to the effective conductivity of that renormalisation cluster. The mean cluster size is denoted by ξ_w and, after m renormalisation steps, $\xi_m \to 0$ and the process is complete. The final probability distribution is then $I\!P_m(\sigma) = \delta(\sigma - \sigma^e)$ where σ^e is the effective conductivity of the structure. Now the number of possible configurations for a two phase, 2×2 cluster is 16 and so the renormalisation transformation will result in a multi-phase composite with a range of effective conductances lying between σ_A and σ_B. To gain an analytic hold we maintain the two phase nature by segregating the effective conductivity distribution into two classes. Each class is assigned a single effective conductivity and the process continues. We can measure the effect that this averaging has on the estimated conductivity by simulating on computer a large number of two-phase composite realisations and calculating their effective conductivity by allowing the multi-phase distribution to persist. To provide an estimate of the effective conductivity of a cluster an equivalent conductor network is used and the effective conductance of any configuration is given by Kirchoff's laws. The segregation of the multi-phases at step $w+1$ uses the geometric mean of the effective conductances at step w as the watershed. The new effective conductivity of these two new classes is given by a probability-weighted geometric mean of the class constituents. This process continues until one of the stable fixed points of the renormalisation group transformation is reached. This has three fixed points, two of which are stable, 0 and 1, and one unstable $p_B^* = (-1 + \sqrt{5})/2$ (the golden section). The latter of these corresponds to the percolation threshold which is defined as the minimum volume fraction of phase B for which there exists a phase B cluster connecting opposite edges of the unit square. In Figure 2(a) we have estimated the effective conductivity σ_2^e of a mono-dispersed random two phase composite, using RSRG, as a function of p_B. Also shown is the calculation obtained by renormalising with no conductivity

averaging $< \sigma_M^e >$. This of course has to be done for finite size composites simulated on computer. It can be seen that the averaging process does not greatly affect the final estimate and the variation in effective conductivities when using the multi-phase approach is very small (see Figure 2(b)).

If however we apply RSRG to the deposits which have a non-uniform particle size distribution the method does not perform quite as well (see Figure 3(c)). The deposit structures are composed of a random agglomeration of particles whose diameters follow a power law distribution rather than being mono-dispersed. To undertand the thermal properties of these materials we will analyse a random structure which has a predetermined size distribution of particles. The basis of this structure is a deterministic fractal, the Sierpinski Carpet (SC) [14], whose constituent particles have their spatial locations randomised to produce a Shuffled Sierpinski Carpet (SSC) [10].

2.2 Shuffled Sierpinski Carpet

For clarity we start by defining the deterministic Sierpinski carpet (SC) set. The initial configuration or pre-fractal is the unit square which we denote by $E_0 \in I\!R^2$. We divide E_0 into nine squares of side length one third and remove the middle square. The union of the eight remaining squares we denote by E_1. Each element of E_1 is treated in a similar fashion, whereby a central square of side length one ninth is removed from each of them. Repeating this process we get a decreasing sequence of nonempty compact sets $E_{n+1} \subset E_n$. The Sierpinski Carpet set F is then given by, $F = \bigcap_{n=0}^{\infty} E_n$. We denote the length scale of the remaining squares at generation level n by $\delta = (1/3)^n$ and the number of such squares as $N(\delta) = 8^n$. The box counting dimension can be shown to equal $dim_B F = \lim_{\delta \to 0} log(N(\delta))/log(1/\delta)$.

This basic model can then be generalised by allowing the tremas to occupy (non-overlapping) random locations, that is we *shuffle* or mix the conducting particles in the insulating matrix F. By varying δ and n a range of particle size distributions can be easily prescribed. The area of F is given by the sum of an infinite series as $1 - \delta^2/(1 - N\delta^2)$ for $N\delta^2 < 1$, and so by requiring this to equal zero, as in the deterministic SC, we obtain

$$N(\delta) = \frac{1}{\delta^2} - 1. \tag{1}$$

with the corresponding box-counting dimension given by

$$dim_B F = Log(\frac{1}{\delta^2} - 1)/Log(1/\delta) \tag{2}$$

and we get the following relationship between $dim_B F$ and δ,

$$\delta^2 + \delta^{2-dim_B F} - 1 = 0. \tag{3}$$

It can be seen therefore that we get the full range of dimensions $dim_B F \in [1,2]$ for $\delta \in [0,(-1+\sqrt{5})/2]$ (the golden section again !).

3 Results

We can use this framework to simulate a family of SSC's , governed by equation (1), on computer. We generate the SSC on a square grid of dimensions $W \times W$, where $W = 2^\gamma$, as this allows easy identification of the renormalisation clusters at each renormalisation level γ. The smallest trema in the SSC dictates the initial grid size and so $\gamma = \lceil -nlog\delta/log2 \rceil$ (where $\lceil x \rceil$ denotes the least integer greater than or equal to x). The overall side length of the SSC is $[W\delta^n]$ grid cells (where $[x]$ denotes the nearest integer to x). For each $\{\delta, n\}$ pair we will generate r realisations so that we can examine the effect that the shuffling process has on the thermal conductivity. Due to computational cost we have restricted attention to $\{\delta, n\}$ pairs which give rise to $\gamma \leq 9$. This will in practice set a lower bound $\delta_L^{(n)}$ on δ for each generation level. The set of SSC arrays we will use below are generated for $n = \{1, 2, \ldots, 12\}$ and $\delta = \{\delta_L^{(n)}, \delta_L^{(n)} + \Delta\delta, \ldots, (-1+\sqrt{5})/2\}$.

In Figure 3(a) the RSRG multi-phase, thermal conductivity $< \sigma_M^e >$, averaged over the r realisations, is shown as a function of the length scale generator δ for various generation levels n. For a particular length scale, as n increases, $< \sigma_M^e >$ increases. Since $\sigma_B > \sigma_A$ then as n increases the volume fraction of phase B increases and so therefore does the effective thermal conductivity. There is however a large variation in the mean thermal conductivity for a given $\{\delta, n\}$ pair. Particularly for large n and large δ (see Figure 3(b)). This suggests that an RSRG theory parameterised by δ and n could provide estimates for $< \sigma_M^e >$ but as the particles move the variation about this value will be significant. In fact $< \sigma_M^e >$ compares well with the analytic (two-phase) value σ_2^e for low values of δ (see Figure 3(c)) but can have an error as much as 20% at high values. Comparing the mono-dispersed aggregate with the SSC we can see that, at intermediate to high volume fractions of phase B, p_B, the variation in the estimated thermal conductivity is as much as 20 times greater (see Figures 2(b) and 3(d)). In the SSC as the constituent particles are shuffled they can form a wide range of cluster distributions and subsequently a wide variance in the thermal conductivity arises. This is highlighted again if we plot $\{< \sigma_M^e >, \sigma_2^e\}$ pairs parameterised by the volume fraction of phase B (see Figure 4).

The results show that the RSRG approach deviates quite substantially from the non-averaged value. This is because the RSRG approach is appropriate for materials where each of the 16 cluster configurations occurs with equal probability. However the size distribution of the tremas in the SSC greatly affects these probabilities. We have revised the underlying probability structure to reflect the non-uniform particle size distribution to obtain a more specific renormalisation scheme which we have shown provides improved estimates [9, 10]. This revised RSRG theory is used in the next section to enable us to state a result about the percolation threshold in an SSC.

3.1 Percolation Threshold

We can utilise this probability structure to examine the dependence of the percolation threshold of the SSC on δ and n. In the percolation analysis we stipulate that the removed squares (or *tremas*) are conducting particles and that the set F forms an insulating pore network. As we shuffle, the tremas can come into contact with each other and hence a conducting (or percolation) path through this two-phase medium can be established.

Result 1. *The number of generation levels n required to achieve percolation threshold in a SSC, with length generator δ is*

$$n = 1 - \frac{log(2 - \delta^2 - 2\delta^4)}{log(1 - \delta^2)} \tag{4}$$

where the volume fraction of phase B is,

$$\begin{aligned} p_B^{SSC} = &(\frac{5}{2}\delta^4 - \delta^8 - \delta^2 - \frac{1}{2})(1 - \delta^2)^{2n-2} \\ &-(\frac{1}{2} - \delta^2 + \frac{1}{2}\delta^4)(1 - \delta^2)^{n-1} + 1. \end{aligned} \tag{5}$$

(For the proof of this result see [9]).

We can test this result for our sample set of finite generation SSC arrays. Here the SSC structure is a binary matrix with the pores represented by zero. We can use the Hoshen-Kopelman algorithm to assign a separate integer to each of the pores in the SSC [15]. If the top and bottom row of this coded matrix contain the same integer then we know a single conducting cluster spans the SSC. In Figure 5(a) we have plotted the dependency of the generation level n on the length scale generator δ at percolation threshold using equation (4). We have also plotted the $\{\delta, n\}$ pairs which gave rise to a positive probability of achieving a percolation path over the r realisations. There were only a few points which satisfied this criterion since the number of generation levels required for most values of δ is computationally prohibitive. There is however a very good agreement between the theory and the numerical calculations. If we calculate the corresponding phase B volume fraction for these data points we can test equation (5). There is quite a wide variation in these points about the theoretical value (see Figure 5(b)) although they appear to centre themselves about this curve. Note that as δ becomes very small we are approaching the mono-dispersed case and as such the curve approaches the theoretical value of $(\sqrt{(}5) - 1)/2$ detailed in section 2.1.

From the coded SSC matrix we can also extract the size distribution of the pores $\psi = \{(\delta_i, N_i) : i = 1, \ldots, d\}$ where δ_i and N_i are the length scale and number of pores at level i respectively. Hence the

remaining conducting area at level i is given by,

$$A_i = 1 - \sum_{j=1}^{i} \delta_j^2 N_j. \tag{6}$$

From a log-log plot of A_i versus δ_i we can utilise the scaling law $A_i \propto \delta_i^{2-dim_B F}$ to estimate $dim_B F$.

For the SSC the set F is phase set A and Figure 6(a) shows the dependency of the averaged value $< dim_B A >$ on δ, for each generation level n. The theory predicts that as δ increases then $dim_B A$ will decrease. The numerical values adhere to this for low values of δ but for large values, with a high generation level, there is a large discrepancy. There is also a corresponding trend in the variation in the calculated $< dim_B A >$ values for each $\{\delta, n\}$ pair (see Figure 6(b)). These pairs correspond to high volume fractions of B and so concur with our earlier findings.

For regular fractals the parameters δ, $dim_B F$ and n are known but for random deposits these have to be determined. From our estimate of $dim_B F$ we can use equation (3) for the inverse problem of recovering the length scale generator $\delta^\star$ and hence $n^\star = log(\delta^\star Min\{\delta_i\}/Max\{\delta_i\})/log(\delta^\star)$ provides an estimate of the generation level. In Figure 7(a) the recovered length scale generator, averaged for each $\{\delta, n\}$ pair, is compared with the actual value, δ. The recovery is very good for the lower values of δ and n. The recovery algorithm predominantly predicts a lower value for the length generator especially for the higher generation levels. The variance in the recovered length scale generator is also high for these cases (see Figure 7(b)). It appears that the shuffling of the particles, in an SSC which has a large range of length scales, can lead to a wide range of cluster distributions. This is also highlighted when we compare the recovered length scale generator to the actual generation level (see Figures 7(c), (d)). The mean prediction is predominantly lower and has the largest error and variance when δ is large.

3.2 Thermal Conductivity

By utilising bounds on the thermal conductivity of random media in the limit of low porosity Thovert and co-workers [16] were able to derive an expression for the thermal conductivity of regular fractals. Here the constituent particles of the conducting medium, that is phase A, have unit conductivity, whilst those of phase B are perfect insulators. Using the SSC as our model structure the remaining conducting surface ϕ is given by $\phi = \delta^{(2-dim_B A)}$. We can therefore derive the following result [10].

Result 2.

The thermal conductivity σ of an SSC is given by

$$\sigma = (1 - \frac{1 - \delta^{(2-dim_B A)}}{1 - 2g})^n. \tag{7}$$

where n is the number of generation levels in the construction of the fractal and g is a morphological

parameter characterising the pore shape.

We can test the above result using the SSC data set generated in the previous section. In Figure 8(a) the mean RSRG multi-phase thermal conductivity is plotted as a function of the mean box counting fractal dimension evaluated numerically. Also shown are the corresponding values using equation (7). Due to the confounding issue of poor $< dim_B A >$ recovery associated with high $\{\delta, n\}$ values we have restricted attention to the cases where $s(dim_B A) < 0.015$ (see Figure 6(b)). For this subset there is good agreement in almost all cases between the theoretical and numerical values. This subset of $\{\delta, n\}$ pairs also has relatively small variance (see Figure 8(b)).

4 Conclusions

The central aim of this paper was to derive an estimate for the thermal conductivity of particulate deposits from a knowledge of the internal pore structure and intrinsic thermal conductivity of the constituent particles. We have restricted our attention to two phase composites embedded in a two dimensional plane. In order to develop a fractal model of the pore structure it was necessary to introduce a new type of random fractal the shuffled Sierpinski carpet (SSC). One well established method of estimating the physical properties of random media is real space renormalisation group (RSRG) theory. It is particularly useful in deriving analytical results regarding the percolation properties of such media. This required a revision of the standard probability structure of cluster classes due to the *poorly mixed* nature of the SSC. Using this approach we were able to derive an equation which predicts the number of generation levels needed to achieve percolation threshold as a function of the underlying length generator of the SSC. We have utilised bounds on the thermal conductivity, in the limit of low porosity and high contrast in the conductivities of the two constituent phases, to provide an estimate of the deposit thermal conductivity which has an explicit dependency on the box counting dimension of the structure.

We have shown that the thermal conductivity of randomly accumulated deposits can be estimated from the geometry of the pore structure. We therefore suggest that non-invasive techniques which only recover geometrical information may be able to infer thermal properties of the plaque. The geometry is characterised by its box counting fractal dimension and an automated procedure for recovering the length scale generator and generation level is reported. The bulk flow and Monte Carlo simulations are currently being recast using a Lattice Boltzmann approach to provide a more realistic model of particle laden blood flow and the geometry of the artery.

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Figure 1.

A Shuffled Sierpinski Carpet (SSC) with phase A (white) conductivity $\sigma_A = 0.05$ and phase B (black) conductivity $\sigma_B = 0.8$ and $\delta = 0.3$, $n = 4$, $W = 2^\gamma$, $\gamma = 7$ and $p_B = 0.304504$. The phase B constituent particles are $N^i = \{1, 10, 102, 1034\}$ in number with respective sizes $W\delta^i = \{38, 12, 3, 1\}$ grid cells, and $dim_B A = 1.9304$.

Figure 2.

(a) The mean, RSRG multi-phase thermal conductivity $< \sigma_M^e >$ ($\bullet$) for mono-dispersed particles (phase B) in a continuum phase (phase A) versus the volume fraction of phase B, p_B. Also shown is the analytic, RSRG two-phase thermal conductivity σ_2^e ($-$).

(b) The standard deviation $s(\sigma_M^e)$, over the $r = 20$ realisations per volume fraction, versus p_B ($\sigma_A = 0.05$, $\sigma_B = 0.8$ and the grid size is $2^7 \times 2^7$).

Figure 3.

(a) The mean, RSRG multi-phase thermal conductivity $< \sigma_M^e >$ for a set of shuffled Sierpinski carpets. The *tremas* are the phase B particles in the continuum phase of the fractal set, phase A. The thermal conductivity values, averaged over $r = 20$ realisations, are shown as a function of the generating length scale δ, for various generation levels n.

(b) The standard deviation $s(\sigma_M^e)$ for each $\{\delta, n\}$ pair versus δ ($\sigma_A = 0.05$, $\sigma_B = 0.8$ and the grid size is $2^\gamma \times 2^\gamma$).

(c) The mean, RSRG multi-phase thermal conductivity $< \sigma_M^e >$ ($\bullet$) is shown as a function of phase B volume fraction p_B. Also shown is the analytic, RSRG two-phase thermal conductivity σ_2^e ($-$).

(d) The standard deviation $s(\sigma_M^e)$ for each $\{\delta, n\}$ pair as a function of phase B volume fraction p_B.

Figure 4.

The mean, RSRG multi-phase thermal conductivity $< \sigma_M^e >$ for a set of shuffled Sierpinski carpets (numerical) versus the corresponding analytic, RSRG two-phase thermal conductivity σ_2^e.

Figure 5.

(a) The relationship between generation level n and generation length δ for the Shuffled Sierpinski Carpet at percolation threshold given by equation (4) ($-$). Also shown are the $\{\delta, n\}$ pairs which gave rise to a percolation probability in the range $(0.3, 0.7)$ over $r = 20$ realisations ($\bullet$).

(b) The relationship between the percolation threshold, phase B volume fraction p_B and the generating length scale δ given by equation (5) ($-$). Also shown are the phase B volume fractions of the $\{\delta, n\}$ pairs shown in (a) ($\bullet$).

Figure 6.

(a) The mean box counting fractal dimension $< dim_B A >$ of a set of SSC is shown as a function of the length scale generator δ. The numbers indicate the particular generation level n. Also shown is the theoretical relationship given by equation (2) $(-)$.

(b) The standard deviation of the box counting fractal dimension $s(dim_B A)$ of a set of SSC is shown as a function of the length scale generator δ. The numbers indicate the particular generation level n.

Figure 7.

(a) The mean recovered length scale generator $< \delta^\star >$ as a function of the original length scale generator δ for a set of SSC. The numbers indicate the particular generation level. (The 45° line aids indentification of the locus of the points where an exact recovery is made.)

(b) The standard deviation of the recovered length scale generator $s(\delta^\star)$ as a function of the original length scale generator δ for a set of SSC.

(c) The mean recovered generation level $< n^\star >$ as a function of the original generation level n for a set of SSC.

(d) The standard deviation of the recovered generation level $s(n^\star)$ as a function of the original generation level n for a set of SSC.

Figure 8.

(a) The mean, RSRG multi-phase thermal conductivity $< \sigma_M^e >$ versus the mean box counting fractal dimension of continuum phase A $< dim_B A >$ for a subset of SSC where $s(dim_B A) < 0.015$ $(\square)$. Also shown are the theoretical values given by equation (7) $(\star)$ ($\sigma_A = 1$, $\sigma_B = 0$ and $g = 0.27$).

(b) The standard deviation of the RSRG multi-phase thermal conductivity $s(\sigma_M^e)$ versus the mean box counting fractal dimension of continuum phase A $< dim_B A >$ for a subset of SSC where $s(dim_B A) < 0.015$.