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Finite Differences in a Small World

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1   Introduction

Many complex networks in nature exhibit two properties that are seemingly at odds. They are *clustered*—neighbors of neighbors are very likely to be neighbors—and they are *small worlds*—any two nodes can typically be connected by a relatively short path. Watts and Strogatz [17] referred to this as the *small world phenomenon* and proposed a network model that was shown through simulation to capture the two properties. The model incorporates a parameter that interpolates between fully local and fully global regimes. As the parameter is varied the small world property is roused before the clustering property is lost. These computations were later backed up by a semi-heuristic mean-field analysis in [12]. Motivated by [17] and the book [16], many authors have shown experimentally that the small world phenomenon can be found across a range of areas in science, medicine and technology [3, 8, 11, 10, 14, 15]. In this work we show that the small world phenomenon also emerges through a local-to-global cutoff in the context of Markov chains. Our model is based on a simple random walk and hence is relevant to many physical, sociological, epidemiological and computational applications [1, 2, 9, 13]. In particular, it may be regarded as a *teleporting random walk*, of the type used by the search engine Google to locate pages on the web [6, 7], restricted to a simple ring network.

An extremely attractive feature of our model is that it can be analyzed rigorously by using tools from numerical analysis.

Full details of the material summarized here can be found in [5]. Recent work that deals with the question of navigation in a small world setting appears in [4].

2   Model

Our model is a perturbed version of a period random walk on a ring. Imagine the numbers $1, 2, 3, \ldots, N$ arranged like the hours on a clock. Take a periodic random walk around this clock—at each time level flip a fair coin; if it lands heads, move clockwise, and if it lands tails, move anticlockwise. Now alter the process so that before taking each step you toss another, biased, coin that lands heads with small probability δ. In the event that the biased coins lands heads, then pick a number between 1 and N uniformly at random and move there. Otherwise, follow the fair coin procedure above. In this model, most of the time we follow a traditional periodic random walk, but occasionally (when the biased coin lands heads) we take a *teleporting jump* across the network. This teleporting idea is used by search engines such as Google to locate pages on the web. (Traverse the web by following links out of pages, but, to avoid reaching a cycle or a dead end, take

the occasional jump). Teleporting is very similar in spirit to the rewiring/shortcutting process that has been observed to produce small world networks of the type described in section 1, and we will show that an analogous cutoff effect arises.

3   Result

The small world phenomenon for networks involves the pathlength between nodes. Analogously, we may study the small world phenomenon for our random walk model by measuring the *mean hitting time*, that is, the average number of steps that it takes to reach B starting from A, where $A, B \in \{1, 2, \dots, N\}$ are chosen uniformly at random.

The problem of analysing the mean hitting time can be set up as a linear algebraic system of dimension $N-1$. We are interested in behaviour for large systems, that is, the $N \to \infty$ limit.

The key to the analysis is to identify the mean hitting time system as a finite difference approximation to an underlying continuum limit. The continuum limit itself depends upon N (i.e. the grid spacing), but careful analysis shows that convergence takes place in an appropriate sense, so that limiting behaviour may be captured.

The analysis shows that it is natural to let the teleporting jump probability, δ, scale with N according to

$$\delta = \frac{K}{N^2}, \qquad \text{for fixed } K.$$

In this regime, we find that, up to $O(N^{-1})$ terms, the mean hitting time, normalized by the mean hitting time when $\delta = 0$, has the form

$$y = \frac{6}{K}x, \tag{3.7}$$

where

$$x = \frac{\sqrt{2K}}{2\tanh\frac{\sqrt{2K}}{2}} - 1 \tag{3.8}$$

is an expression for the expected number of teleporting jumps taken per random walk.

We may now examine the small world property of the model by plotting y against x as K is varied. This shows how the introduction of shortcuts (teleporting jumps) makes it easier to get around the network. Figure 3.5 gives the picture. We see that there is an abrupt drop in the hitting time as x increases beyond $x \approx 10^{-1}$. This "order 1 shortcut cutoff effect" agrees with the fundamental observation of Watts and Strogatz [17] for their small world network model.

References

[1] H. C. BERG, *Random Walks in Biology*, Princeton University Press, 1983.

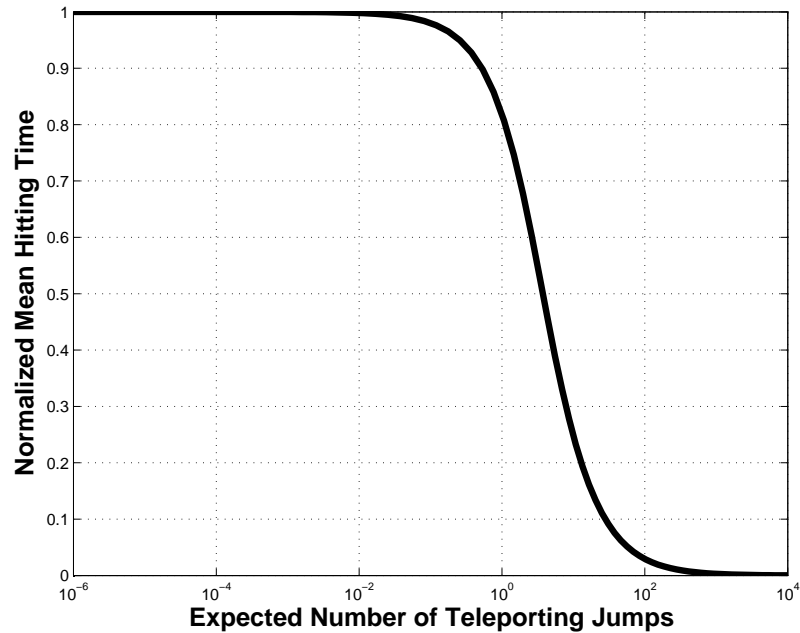[2] H. CASWELL, *Matrix Population Models*, Sinauer Associates, Inc., Sunderland, MA, second ed., 2001.

Figure 3.5: x-axis is the average number of teleporting jumps per excursion, y-axis is the mean hitting time (normalized by dividing by the mean hitting time with $\delta = 0$). The picture was produced from (3.7)–(3.8) by varying K.

[3] P. M. Gleiss, P. F. Stadler, A. Wagner, and D. A. Fell, *Relevant cycles in chemical reaction networks*, Advances in Complex Systems, 4 (2001), pp. 207–226.

[4] D. J. Higham, *Greedy pathlengths and small world graphs*, Tech. Rep. 8, University of Strathclyde, Glasgow, UK, 2002.

[5] ——, *A matrix perturbation view of the small world phenomenon*, SIAM J. Matrix Anal. Appl., (to appear (2003)).

[6] S. D. Kamvar, T. H. Haveliwala, C. D. Manning, and G. H. Golub, *Extrapolation methods for accelerating pagerank computations*, Proceedings of the Twelfth International World Wide Web Conference, (2003).

[7] C. Moler, *The world's largest matrix computation*, MATLAB News and Notes, October (2002).

[8] J. M. Montoya and R. V. Solé, *Small world patterns in food webs*, Journal of Theoretical Biology, 214 (2002), pp. 405–412.

[9] R. Motwani and P. Raghavan, *Randomized Algorithms*, Cambridge University Press, 1995.

[10] M. E. J. Newman, *The structure of scientific collaboration networks*, Proc. Natl. Acad. Sci., 98 (2001), pp. 404–409.

[11] ——, *The structure and function of complex networks*, SIAM Review, 45 (2003), pp. 167–256.

[12] M. E. J. Newman, C. Moore, and D. J. Watts, *Mean-field solution of the small-world network model*, Physical Review Letters, 84 (2000), pp. 3201–3204.

[13] J. R. NORRIS, *Markov Chains*, Cambridge University Press, 1997.

[14] S. H. STROGATZ, *Exploring complex networks*, Nature, 410 (2001), pp. 268–276.

[15] A. WAGNER AND D. A. FELL, *The small world inside large metabolic networks*, Proc. Roy. Soc. London, B., 268 (2001), pp. 1803–1810.

[16] D. J. WATTS, *Small Worlds: The Dynamics of Networks between Order and Randomness*, Princeton University Press, 1999.

[17] D. J. WATTS AND S. H. STROGATZ, *Collective dynamics of 'small-world' networks*, Nature, 393 (1998), pp. 440–442.