

Finite size effects and the supersymmetric sine-Gordon models

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Abstract

We propose nonlinear integral equations to describe the groundstate energy of the fractional supersymmetric sine-Gordon models. The equations encompass the $N = 1$ supersymmetric sine-Gordon model as well as the $\phi_{\text{id,id,adj}}$ perturbation of the $SU(2)_L \times SU(2)_K / SU(2)_{L+K}$ models at rational level K . A second set of equations are proposed for the groundstate energy of the $N = 2$ supersymmetric sine-Gordon model.

1 Introduction

The sine-Gordon (SG) and supersymmetric sine-Gordon (SSG) [1, 2, 3, 4, 5] models belong to an interesting set of integrable quantum field theories that are said to have fractional supersymmetry [6, 7], thus attracting the name fractional supersymmetric sine-Gordon models (FSSG). The field content [8, 9] consists of a single boson interacting with a Z_L generating parafermion, the choices $L = 1$ and 2 yielding the SG and SSG models respectively. The models correspond to the fractional conformal field theory obtained from the Wess-Zumino-Witten model $SU(2)_L$ by compactifying the free boson on a circle of radius $2R = \beta/\sqrt{4\pi}$, perturbed by

$$\Phi(z, \bar{z}) = \frac{g\sqrt{4\pi}}{\beta} \Psi_1(z) \bar{\Psi}_1(\bar{z}) e^{-i\frac{\beta}{\sqrt{4\pi}}\varphi(z, \bar{z})} + \text{c.c.} . \quad (1)$$

The mass scale is set by the dimensionful parameter g , and the real dimensionless parameter β determines the particle content. Since the boson $\varphi(z, \bar{z})$ and generating parafermion $\Psi_1(z, \bar{z})$ have conformal dimensions 1 and $1-1/L$ respectively, the perturbing term has conformal dimension $\beta^2/8\pi + 1 - 1/L$. The ultraviolet effective central charge is $1+2(L-1)/(L+2) = 3L/L+2$.

In the limit $\beta^2/8\pi \rightarrow 1/L$ the perturbed model becomes the current-current perturbation of the WZW model $SU(2)_L$ [8]. Instead if we tune the parameter

$$\xi = \frac{L\beta^2/8\pi}{1/L - \beta^2/8\pi} \quad (2)$$

to $K+2$ we obtain the unitary coset $SU(2)_L \times SU(2)_K / SU(2)_{L+K}$ perturbed by the operator $\phi_{\text{id,id,adj}}$ [8]. The indices indicate which representation of $SU(2)_L$, $SU(2)_K$ and $SU(2)_{L+K}$ is associated to the corresponding weights. This choice of ξ when $L = 1$ corresponds to the well-known quantum group restriction [10, 11] of the sine-Gordon model to the ϕ_{13} perturbed minimal models. In fact, the FSSG models were first proposed as the models obtained by ‘unrestricting’ the S-matrices describing the $\phi_{\text{id,id,adj}}$ perturbation of the cosets $\mathcal{M}_{L,K} \equiv SU(2)_L \times SU(2)_K / SU(2)_{L+K}$ [8].

The existence of higher spin conserved charges in 1+1 dimensional integrable quantum field theories strongly constrain the allowed scattering processes. There is no particle production, the individual particle momenta are conserved, and all n -particle scattering processes

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decompose into products of two-particle scattering amplitudes [12]. The S-matrix of a supersymmetric model should also commute with the supersymmetry generators. This typically leads to a further factorisation of the two-particle S-matrix into a direct product in which one factor encodes the supersymmetry dependence [8, 13, 9]. The S-matrices of the FSSG models exactly fit this pattern [8, 9]:

$$S(\beta) = S_L \otimes S_{\text{SG}}(\beta) . \quad (3)$$

The supersymmetry-related factor S_L describes the scattering of the massive ϕ_{13} perturbation of the minimal model $\mathcal{M}_{1,L}$ [7, 11, 6], and the bosonic factor $S_{\text{SG}}(\beta)$ is the S-matrix of the sine-Gordon model at coupling β [12]. The particle content of the FSSG consists of a soliton of mass M and its antisoliton together with a number of breathers of mass

$$M_j = 2M \sin\left(\frac{\pi j \xi}{2}\right) , \quad j = 1 \dots < \frac{1}{\xi} . \quad (4)$$

There are no breathers in the repulsive regime $\xi < 1$. Once the spectrum and associated S-matrices have been conjectured the thermodynamic Bethe ansatz [14, 15] is usually employed to check the proposed scattering theory. However when the scattering is nondiagonal, as above, deriving the TBA equations from the S-matrices becomes rather difficult.

Fendley and Intriligator noted that one can immediately write down a TBA system for a model with a tensor product S-matrix of the form $S_G \otimes S_H$ if the TBA equations are already known for the models described by S-matrices S_G and S_H [16]. If the individual TBA equations are encoded on Dynkin-like diagrams of type G and H , each having a single massive node, then the TBA equations for the model with S-matrix $S_G \otimes S_H$ are found by gluing the individual TBA equations at the massive node [16]. In the compact notation of [17, 18] this corresponds to $(G \diamond H)_L$, where L indicates the massive node. It is easy to check that the TBA equations found for the FSSG model at the special values of the coupling [16, 19, 17, 20, 21]

$$\frac{\beta^2}{8\pi} = \frac{1}{L} - \frac{2}{m} , \quad m \in \mathbb{N}^+ , \quad L = 1, 2, \dots \leq \text{int}[m/2] - 1 , \quad (5)$$

and those for the $\phi_{\text{id,id,adj}}$ perturbation of the $\mathcal{M}_{L,K}$ models [22] all fit this pattern. TBA equations for the models arising as the Φ_{13} perturbation of the supersymmetric models $\mathcal{SM}_{2,4n+4}$ are known [23, 24, 25, 26], but currently lack an interpretation in terms of Dynkin-like diagrams.

A Y -system, from which the TBA equations can be obtained via Fourier transform [27], for the FSSG model at any coupling has been proposed using the gluing idea and the sine-Gordon Y -system [20, 21, 28]. The SG Y -system depends on writing the smaller of ξ or $1/\xi$ as a simple continued fraction. The number of TBA equations is found to be equal to the sum of the leading terms of the denominators in the continued fraction plus one further equation. Though there is a neat depiction in terms of Dynkin diagrams [20, 21] it is hard to deal with a large number of equations. Moreover those ξ for which the continued fraction does not terminate are associated to a TBA system with an infinite number of equations.

There is an alternative method of calculating the finite size effects of the sine-Gordon model, known as the nonlinear integral equation (NLIE) technique. The result is a single equation, first proposed for the groundstate of the sine-Gordon model by Destri and De Vega in 1992 [29]. A complete derivation to include the excited states of even topological charge followed in [30], and an equation for the excited states of odd topological charge was proposed in [31]. The conformal limit of the groundstate equation had been proposed in 1991 in an integrable lattice model context by Klümper, Batchelor and Pearce [33, 32]. Further motivation for this work came from the lattice model connection: the XXX spin chain at spin S is a model with underlying field theory $SU(2)_{2S}$. The spin- $\frac{1}{2}$ case corresponds to the lattice model

mentioned above; the spin-1 problem is discussed in [32,33]. Recently Suzuki has generalised the technique of [33,32] to obtain a finite set of nonlinear integral equations that describe the thermodynamics of the spin- S chain at finite temperature [34]. Given the relations between the spin chains and $SU(2)_{2S}$, and the FSSG models and perturbed $SU(2)_L$, we should expect the conformal limit of any integral equation describing the FSSG at $\beta^2 = 1/L$ to be closely related to the zero temperature version of Suzuki's NLIE.

Motivated by Suzuki's equations and the gluing idea we propose a set of nonlinear integral equations to describe the groundstate energy of the fractional supersymmetric sine-Gordon model at any value of the coupling constant β^2 . In §2 a short description of the component equations leads to the proposed NLIEs, and then a number of tests are carried out in §3.

The $N=2$ supersymmetric sine-Gordon model is not a member of the FSSG series. However the groundstate energy of this model can be studied by means of a further set of NLIEs, obtained by a simple generalisation of the FSSG equations. The details are in §4 along with a short discussion of related models: the sausage models and the parafermionic perturbation of the Z_M models. Directions for future work can be found in §5.

2 Nonlinear integral equations

There are two integrable quantum field theories associated with the ϕ_{13} perturbation of the minimal model $\mathcal{M}_{1,L}$. If the coupling λ is positive the action

$$\mathcal{A} = \mathcal{A}_{\text{CFT}} + \lambda \int d^2x \phi_{13} \quad (6)$$

describes a ‘massless flow’, or renormalisation group trajectory, from $\mathcal{M}_{1,L}$ in the ultraviolet to $\mathcal{M}_{1,L-1}$ in the infrared [35]. On the other hand, if the coupling is negative (6) describes a massive integrable quantum field theory with S-matrix S_L [36]. The TBA equations for the groundstate energy of the massive scattering theory were found by A.I.B. Zamolodchikov [37]. As we explain below one small change allows the same equations to unexpectedly describe the groundstate energy of the massless flows [38]. The equations are written in terms of the incidence matrix I_{ab} of the A_L Dynkin diagram [37,38]. The TBA system is*

$$\begin{aligned} f_a(\theta) &= g_a(\theta) + \sum_{b=1}^L \int_{-\infty}^{\infty} d\theta' I_{ab} \chi(\theta - \theta') \ln(1 + e^{f_b(\theta')}) \\ c_{\text{eff}}(r) &= \frac{6}{\pi} \sum_{a=1}^L \int_{-\infty}^{\infty} d\theta g_a(\theta) \ln(1 + e^{f_a(\theta)}) , \end{aligned} \quad (7)$$

with kernel $\chi(\theta) = 1/(2\pi \cosh \theta)$. Here $r = MR$ where M is the mass of the elementary kink, or in the massless theory the crossover scale, and R is the circumference of the cylinder on which the model is considered. The massive theory is found by setting the driving term to

$$g_a(\theta) = -\delta_{a,1} r \cosh \theta \quad (8)$$

while the massless theory appears if

$$g_a(\theta) = -\frac{r}{2} (\delta_{a,L} e^{\theta} + \delta_{a,1} e^{-\theta}) . \quad (9)$$

*Note that the TBA equations are usually written in terms of pseudoenergy $\varepsilon = -f$.

Both theories have the same ultraviolet effective central charge: $c_{\text{eff}}(0) = 1 - 6/(L+2)(L+3)$, while the infrared value is zero or $1 - 6/(L+1)(L+2)$ for the massive/massless models respectively.

To avoid dealing with a different and possibly infinite set of equations for each choice of β^2 we consider the second type of nonlinear integral equation mentioned in the introduction. The equation involves a complex function $a(\theta)$ and its conjugate $\bar{a}(\theta)$ coupled via [33, 32, 29]

$$\ln a(\theta) = -i r \sinh \theta + \int_{\mathcal{C}_1} d\theta' \varphi(\theta - \theta') \ln(1 + a(\theta')) - \int_{\mathcal{C}_2} d\theta' \varphi(\theta - \theta') \ln(1 + a^{-1}(\theta')) . \quad (10)$$

The integration contours \mathcal{C}_1 and \mathcal{C}_2 run from $-\infty$ to $+\infty$, just below and just above the real θ -axis respectively. The kernel $\varphi(\theta)$ is proportional to the logarithmic derivative of the soliton-soliton scattering amplitude of the sine-Gordon model

$$\varphi(\theta) = \int \frac{dk}{2\pi} e^{ik\theta} \frac{\sinh(\xi - 1) \frac{\pi k}{2}}{2 \sinh \frac{\pi \xi k}{2} \cosh \frac{\pi k}{2}} . \quad (11)$$

The effective central charge found using

$$c_{\text{eff}}(r) = \frac{3ir}{\pi^2} \left(\int_{\mathcal{C}_1} d\theta \sinh \theta \ln(1 + a(\theta)) - \int_{\mathcal{C}_2} d\theta \sinh \theta \ln(1 + a^{-1}(\theta)) \right) \quad (12)$$

has ultraviolet value 1. With constant $i\pi\alpha$ [39] added to the RHS of (10) the effective central charge becomes $c_{\text{eff}}(0) = 1 - 6\alpha^2\xi/(\xi + 1)$. The choice $\xi = p/(q-p)$ and $\alpha = \pm 1/p$ [40, 41, 42] implements the restriction [10, 11] of the SG model to the ϕ_{13} perturbed minimal models.

The explanation for Fendley and Intriligator's gluing proposal lies in the derivation of the TBA equations [16]. The key is the factorisability of the S-matrix: it implies each factor can be diagonalised separately and each generates an individual set of pseudoparticles. These are seen to only couple through the massive particle, with the result being the 'factorised' or glued TBA system described in the introduction. There is no equivalent of the pseudoparticles in the derivation of the sine-Gordon NLIE. Moreover, in further contrast to the TBA, the S-matrix plays no role in the NLIE derivation. Instead one notes a posteriori that the kernel function of the resulting NLIE is related as described above to the sine-Gordon S-matrix. It therefore seemed unlikely that the gluing idea would be of any use in finding a NLIE for a model with an S-matrix of the form (3). Fortunately this turns out not to be the case.

We obtain the NLIEs for the FSSG model by gluing the minimal model TBA equations to the sine-Gordon NLIE at the massive node following the Fendley-Intriligator prescription [16]. However the resulting equations are not quite right and it is necessary to make a few simple modifications. Due to singularities in $\ln(1 + y_1(\theta))$ for real θ a TBA-like convolution $\int d\theta' \chi(\theta - \theta') \ln(1 + y_1(\theta'))$ along the real axis cannot be used. The solution is to replace this term with a convolution along the contours \mathcal{C}_1 and \mathcal{C}_2 encircling the real axis as in the SG NLIE. To ensure for real θ the reality of the $L-1$ TBA-like functions y_2, \dots, y_L , and the complex nature of the NLIE-like function y_1 we find we should couple y_1 and y_2 to each other with an extra shift of $i\pi/2$ in the kernel χ . The conjectured equations are

$$\begin{aligned} \ln y_1(\theta) &= \nu_1(\theta) + i\pi\alpha \\ &+ \int_{\mathcal{C}_1} d\theta' \varphi(\theta - \theta') \ln(1 + y_1(\theta')) - \int_{\mathcal{C}_2} d\theta' \varphi(\theta - \theta') \ln(1 + y_1^{-1}(\theta')) \\ &+ \int_{-\infty}^{\infty} d\theta' \chi(\theta - \theta' + \frac{i\pi}{2}) \ln(1 + y_2(\theta')) ; \end{aligned}$$

$$\begin{aligned}
\ln y_2(\theta) &= \nu_2(\theta) + \int_{\mathcal{C}_1} d\theta' \chi(\theta - \theta' - \frac{i\pi}{2}) \ln(1 + y_1(\theta')) - \int_{\mathcal{C}_2} d\theta' \chi(\theta - \theta' - \frac{i\pi}{2}) \ln(1 + y_1^{-1}(\theta')) \\
&\quad + \int_{-\infty}^{\infty} d\theta' \chi(\theta - \theta') \ln(1 + y_3(\theta')) ; \\
\ln y_a(\theta) &= \nu_a(\theta) + \sum_{b=1}^L I_{ab} \int_{-\infty}^{\infty} d\theta' \chi(\theta - \theta') \ln(1 + y_b(\theta')) \quad , \quad a = 3 \dots L , \quad (13)
\end{aligned}$$

where $\chi(\theta)$ remains as above. The driving terms are

$$\nu_a(\theta) = \begin{cases} -ir \sinh(\theta) & a = 1 \\ 0 & \text{otherwise} \end{cases} . \quad (14)$$

Borrowing the TBA notation of [17, 18] our shorthand for equations of this type will be $(A_1 \diamond A_L)_a^{[i]}$. The superscript $[i]$ indicates the NLIE-like function and the subscript a the function on which the mass is placed. The equations above are therefore represented by $(A_1 \diamond A_L)_1^{[1]}$. The effective central charge defined in terms of the solution of (13) can be calculated using

$$c_{\text{eff}}(r) = -\frac{3}{\pi^2} \sum_{a=1}^L \int_{\mathcal{C}_1} d\theta \nu_a(\theta) \ln(1 + y_a(\theta)) - \int_{\mathcal{C}_2} d\theta \nu_a(\theta) \ln(1 + y_a^{-1}(\theta)) . \quad (15)$$

At $L = 1$ the system (13,14,15) trivially reduces to the NLIE describing the sine-Gordon equation. As expected these equations become equivalent to Suzuki's [34] for the spin- S XXX spin chain in the limit $\beta^2 \rightarrow 1/L$ ($\xi \rightarrow \infty$). The NLIE-like functions are approximately related via $y_1(\theta) \approx \ln \mathfrak{a}(\frac{2}{\pi}\theta + i)$, provided the spin chain driving term is replaced with the one relevant for the field theory.

Based on previous experience with NLIEs we expect to find the effective central charge for the FSSG model at level L and coupling β^2 by tuning ξ according to (2) and setting $\alpha = 0$. The $\phi_{\text{id,id,adj}}$ perturbation of the coset $\mathcal{M}_{L,K}$ should be found at

$$\xi = K+2 \quad , \quad \alpha = \pm 1/(K+2) . \quad (16)$$

In fact we hope the equations go a little further: even though the quantum group restriction of the S-matrices may not work it is thought that the FSSG models also describe the nonunitary cosets perturbed by $\phi_{\text{id,id,adj}}$. If we set

$$\xi = \frac{Lp}{q-p} \quad , \quad \alpha = \pm \frac{1}{p} , \quad (17)$$

then the NLIEs should yield the effective central charge of the $\phi_{\text{id,id,adj}}$ perturbation of $\mathcal{M}_{L,Lp/(q-p)-2}$. This is a coset at rational, and possibly negative, level. These models make sense provided the integers p, q are such that p and $(q-p)/L$ are coprime integers. At $L = 1$ this is the ϕ_{13} perturbation of the minimal models $\mathcal{M}(p, q)$, and at $L = 2$ the supersymmetry preserving perturbation Φ_{13} of the superminimal models $\mathcal{SM}(p, q)$.

In the same way that TBA equations for the massive perturbation of the minimal models can be converted into those describing the massless perturbations, we can describe *some* massless flows using the proposed NLIEs with new driving terms

$$\nu_a(\theta) = \begin{cases} -\frac{i}{2} r e^\theta & a = 1 \\ -\frac{1}{2} r e^{-\theta} \delta_{a, L-K+1} & \text{otherwise} \end{cases} . \quad (18)$$

	a	$1 + x_a^-$	$1 + x_a^0$	$1 + x_a^+$
Massive	1	1	$\frac{\sin \frac{\pi \xi \alpha (L+1)}{L+\xi}}{\sin \frac{\pi \xi \alpha}{L+\xi}} e^{i\pi \xi \alpha L / (L+\xi)}$	1
	$2 \dots L$	$\frac{\sin^2 \frac{\pi a}{L+2}}{\sin^2 \frac{\pi}{L+2}}$	$\frac{\sin^2 \frac{\pi \xi \alpha (L+2-a)}{L+\xi}}{\sin^2 \frac{\pi \xi \alpha}{L+\xi}}$	$\frac{\sin^2 \frac{\pi a}{L+2}}{\sin^2 \frac{\pi}{L+2}}$
Massless	1	$\frac{\sin \frac{\pi (K-L)}{L+2}}{\sin \frac{\pi}{L+2}} e^{i\pi (K+1) / (L+2)}$	$\frac{\sin \frac{\pi (L+1)}{L+K+2}}{\sin \frac{\pi}{L+K+2}} e^{i\pi L / (L+K+2)}$	1
	$2 \dots L-K$	$\frac{\sin^2 \frac{\pi (K+a)}{L+2}}{\sin^2 \frac{\pi}{L+2}}$	$\frac{\sin^2 \frac{\pi (L+2-a)}{L+K+2}}{\sin^2 \frac{\pi}{L+K+2}}$	$\frac{\sin^2 \frac{\pi a}{L+2}}{\sin^2 \frac{\pi}{L+2}}$
	$L-K+1 \dots L$	$\frac{\sin^2 \frac{\pi (L+2-a)}{K+2}}{\sin^2 \frac{\pi}{K+2}}$	$\frac{\sin^2 \frac{\pi (L+2-a)}{L+K+2}}{\sin^2 \frac{\pi}{L+K+2}}$	$\frac{\sin^2 \frac{\pi a}{L+2}}{\sin^2 \frac{\pi}{L+2}}$

Table 1: Stationary values for massive and massless perturbed models.

If (ξ, α) are fixed to the values prescribed in (16) the equations reproduce the flow $\mathcal{M}_{L,K} + \phi_{\text{id,id,adj}} \rightarrow \mathcal{M}_{L,K-L}$. Given the different nature of y_1 and y_{L-K+1} it is rather surprising that the resulting NLIEs do in fact describe massless flows. The massless flows within the sine-Gordon model ($L = 1$) can alternatively be studied via two coupled NLIE functions [39, 43].

Using the symmetry between L and K we can minimise the number of NLIEs for the coset models $\mathcal{M}_{L,K}$ by choosing $L < K$. However the massless flows only make sense if $L > K$. Both possibilities lead to a smaller number of equations than the relevant TBA equations. Therefore the new NLIEs are a computationally more efficient way of calculating the finite size effects of the FSSG models. In the next section we take a closer at the equations and obtain some results confirming the above conjectures.

3 Testing the NLIEs

We begin by examining the behaviour of the individual functions at $r = 0$ and $r = \infty$. By differentiating the first equation of (13) with respect to θ we see the function $i \ln(y_1)(\theta)$ becomes approximately constant in the region $|\theta| \ll \ln(2/r)$ as $r \rightarrow 0$. For $\theta \ll -\ln(2/r)$ it behaves as $-e^{r\theta}$, and for $\theta \gg \ln(2/r)$ as $e^{r\theta}$. Therefore y_1 is a complex constant, which we denote x_1^0 , inside the the central region $-\ln(2/r) \ll \theta \ll \ln(2/r)$. The real and imaginary parts of y_1 oscillate outside this region but because η is nonzero $y_1(\theta - i\eta)$ and $y_1^{-1}(\theta + i\eta)$ ultimately tend to zero. The functions y_2, y_3, \dots, y_L behave exactly as if they were TBA Y-functions[†]: they are symmetric in θ and real for real θ . From the behaviour of y_1 it follows that in the ultraviolet limit they become constants x_a^-, x_a^0, x_a^+ in the regions $\theta \ll -\ln(2/r)$, $-\ln(2/r) \ll \theta \ll \ln(2/r)$ and $\theta \gg \ln(2/r)$ respectively, while interpolating smoothly in between. A similar conclusion can be reached for the stationary solutions of the massless equations, apart from a small change in the behaviour of y_1 as it now becomes a nonzero constant in the region $\theta \ll -\ln(2/r)$. It is simple to check this picture by plotting some numerical solutions to the NLIEs. The exact values of the stationary solutions, derived from the NLIEs in the usual way [15, 44], are summarised for both massive and massless cases in table 1.

Using the standard methods [15, 44, 32] we find the ultraviolet effective central charge takes the form

$$c_{\text{eff}}(0) = -\frac{3}{\pi^2} \left[2\mathcal{L} \left(\frac{1}{1 + \bar{x}_1^0} \right) - \mathcal{L} \left(\frac{1}{1 + \bar{x}_1^+} \right) - \mathcal{L} \left(\frac{1}{1 + \bar{x}_1^-} \right) \right]$$

[†]in standard notation $Y_a(\theta) = \exp[\varepsilon_a(\theta)]$

$$\begin{aligned}
& + \sum_{a=1}^L 2\mathcal{L}\left(\frac{1}{1+x_a^0}\right) - \mathcal{L}\left(\frac{1}{1+x_a^+}\right) - \mathcal{L}\left(\frac{1}{1+x_a^-}\right) \\
& - \frac{i\pi\alpha}{2} \left(\ln \frac{(1+x_1^0)^2}{(1+x_1^-)(1+x_1^+)} - \ln \frac{(1+\bar{x}_1^0)^2}{(1+\bar{x}_1^-)(1+\bar{x}_1^+)} \right) \Big], \quad (19)
\end{aligned}$$

where \mathcal{L} is Rogers dilogarithm function

$$\mathcal{L}(x) = -\frac{1}{2} \int_0^x dt \frac{\ln 1-t}{t} + \frac{\ln t}{1-t}. \quad (20)$$

To evaluate (19) we use the standard properties and known sum rules of the dilogarithm (see for example [45])

$$\frac{6}{\pi^2} \sum_{a=2}^L \mathcal{L}\left(\frac{\sin^2[\pi/(L+2)]}{\sin^2[\pi a/(L+2)]}\right) = \frac{2(L-1)}{L+2}, \quad \frac{6}{\pi^2} \mathcal{L}(1) = 1, \quad (21)$$

and one further sum rule which involves both real and complex arguments

$$\frac{6}{\pi^2} \left(\mathcal{L}\left(\frac{1}{1+\bar{x}_1^0}\right) + \sum_{a=1}^L \mathcal{L}\left(\frac{1}{1+x_a^0}\right) \right) = 1. \quad (22)$$

Numerical calculations confirm that (22) holds for *arbitrary* real numbers ξ and α , L is by definition an integer. The ultraviolet effective central charge predicted from the NLIEs takes the form

$$c_{\text{eff}}(0) = \frac{3L}{L+2} - \frac{6\alpha^2\xi L}{L+\xi}. \quad (23)$$

Does this reproduce the expected results? Yes, it does: with ξ related to β^2 as in (2) and $\alpha = 0$, the formula predicts $c_{\text{eff}}(0) = 3L/(L+2)$, whereas the choice (16) leads to the effective central charge of the unitary coset models $\mathcal{M}_{L,K}$:

$$c_{\text{eff}}(0) = \frac{3L}{L+2} - \frac{6L}{(L+K+2)(K+2)}, \quad (24)$$

and the possibility (17) yields the effective central charge for the nonunitary coset models

$$c_{\text{eff}}(r) = \frac{3L}{L+2} - \frac{6L}{pq}. \quad (25)$$

If we use the massless driving terms, the individual parts of (19) combine in a different way, but the final result is again (23). Since we have fixed the parameters $\xi = K+2$ and $\alpha = \pm 1/(K+2)$ we actually obtain (24). We need to check the infrared limit to be sure the NLIEs really do describe these massless flows. As $R \rightarrow \infty$ the contribution to $c_{\text{eff}}(\infty)$ from $y_{L+\xi+3}, \dots, y_L$ becomes vanishingly small. The remaining functions have stationary solutions given by the massless ones in table 1 but with L replaced by $L-K$. Plugging these values into (19) gives

$$c_{\text{eff}}(\infty) = \frac{3L}{L+2} - \frac{6L}{(K+2)(K-L+2)}, \quad (26)$$

which is exactly the ultraviolet effective central charge of $\mathcal{M}_{L,K-L}$.

Before obtaining the conformal dimension of the perturbing operator we comment on a surprising relation between the NLIEs at the coset points and the associated TBA equations. The massive $\phi_{\text{id,id,adj}}$ perturbation of $\mathcal{M}_{L,K}$ has a TBA system based on $(A_1 \diamond A_{L+K-1})_L$. The

symmetry between L and K means that we can either take a NLIE with L functions (below denoted y_a) and $\xi = K+2$, or one with K functions (denoted \tilde{y}_a) and $\xi = L+2$, in either case $c_{\text{eff}}(r)$ is exactly the same. Curiously the magnonic parts of both NLIEs are closely related to the TBA functions. Numerical evidence indicates

$$\exp(f_a) = \begin{cases} y_{L-a+1} & a = 1, \dots, L-1 \\ \tilde{y}_{a-K+1} & a = L+1, \dots, L+K-1 \end{cases} \quad (27)$$

The functions with nonzero driving term, $\exp f_L, y_1, \tilde{y}_1$, are not equal, but the integral equations imply they must be related. One such relation is

$$\int_{-\infty}^{\infty} d\theta' \chi(\theta - \theta') \ln(1 + e^{f_{L-2}(\theta')}) = \int_{\mathcal{C}_1} d\theta' \chi(\theta - \theta' - \frac{i\pi}{2}) \ln(1 + y_1(\theta')) - \int_{\mathcal{C}_2} d\theta' \chi(\theta - \theta' - \frac{i\pi}{2}) \ln(1 + y_1^{-1}(\theta')) \quad (28)$$

The question of the conformal dimension of the perturbing operator, predicted as usual from the periodicity of the functions entering the NLIEs [27], can be partially answered via the relation described above with the coset TBA systems. For $\xi = K+2$ and $\alpha = \pm 1/(K+2)$ the periodicity of the TBA-like functions y_2, \dots, y_L inferred from that of the TBA systems for the $\phi_{\text{id,id,adj}}$ perturbation of the coset $\mathcal{M}_{L,K}$ is $i\pi(\xi + L)$ [22]. It is not expected to depend on α and so one only needs to check what happens at other values of ξ . Doing so numerically leads us to believe all of the entire functions $y_a(\theta)$ have periodicity $i\pi(\xi + L)$, and therefore a Laurent expansion in powers of $\exp(2/(\xi + L)\theta)$. The quantities $\ln(1 + y_a(\theta))$ appearing in the kernels of the NLIEs have essentially the same shape as the quantities $\ln(1 + \exp(-\varepsilon_a(\theta)))$ that appear in a pure TBA system, that is a central plateau which quickly goes to zero for θ outside the region $|\theta| \ll \ln(2/r)$. Therefore we may employ Zamolodchikov's argument [27] to predict that $c_{\text{eff}}(r)$ expands in powers of $r^{4/(\xi+L)}$. Note that for some α , $x_1^0 = 0$ and Zamolodchikov's argument acquires a small modification, as explained in [46], with the result that $c_{\text{eff}}(r)$ instead expands in powers of $r^{2/(\xi+L)}$. However for the choices of α needed here x_1^0 is never identically zero. If the perturbing operator is odd we would expect an expansion in $r^{2(1-\Delta)}$ leading to the prediction $\Delta = 1 - 2/(\xi + L)$. Inserting the appropriate value of ξ (17) we find $\Delta = 1 - 2/(K+L+2)$, which exactly matches the conformal dimension of $\phi_{\text{id,id,adj}}$ for both unitary and nonunitary (with $K = Lp/(q-p)$) cosets. On the other hand if the perturbing operator is even we would expect an expansion in $r^{4(1-\Delta)}$ which implies $\Delta = 1 - 1/(\xi+L)$. Writing ξ in terms of β^2 using (2) we find the conformal dimension of the FSSG models: $\Delta = \beta^2/8\pi + 1 - 1/L$.

In the infrared limit the groundstate energy of the massive model expands in powers of e^{-MR} [15]. Adapting the arguments of [15, 29] we obtain from the NLIEs

$$E(R) \sim -4 \cos \frac{\pi}{L+2} \cos \pi \alpha \int \frac{d\theta}{2\pi} \cosh \theta e^{-MR \cosh \theta} \quad (29)$$

With the choices of L and α as given in the text this agrees perfectly with TBA results obtained for the perturbed cosets $\mathcal{M}_{L,K} + \phi_{\text{id,id,adj}}$ [22], the Φ_{13} perturbation of the $N = 2$ supersymmetric minimal models [16] and the $SU(2)$ Gross-Neveu model [47].

We now test the NLIEs numerically, comparing against the effective central charge found using various TBA equations [22, 20, 16, 17]. These TBA equations correspond either to $\mathcal{M}_{L,K} + \phi_{\text{id,id,adj}}$ for integer K , or to the FSSG model at the specific coupling given in (5). We also compare against the magnonic TBA system based on $(A_1 \diamond T_n)_L$, identified in [17] with $\mathcal{M}_{L,n-3/2} + \phi_{\text{id,id,adj}}$. However this is only true at $L = 1$ and for general L these TBA

equations correspond to $\mathcal{M}_{L,n-L-1/2} + \phi_{\text{id,id,adj}}$. As no data for the chosen cases has previously been reported in the literature we solved both the TBA and NLIE equations numerically, using an iterative method. We checked stability of the solution by varying the number of iterations and the integration step size. The results are shown in table 2. The tables include a diagrammatic version of the TBA system with a solid circle representing the massive node. We also reproduce one massless flow whose TBA diagram has crosses to indicate the nodes on which the massless driving terms are placed. In all cases the agreement is very good.

Model	TBA	NLIE (L, ξ, α)	r	TBA $c_{\text{eff}}(r)$	NLIE $c_{\text{eff}}(r)$
$\mathcal{M}_{2,4} + \phi_{\text{id,id,adj}}$		$(4, 4, \pm \frac{1}{4})$	1E-05 100	1.2499968080510 1.0002668224816	1.2499968080550 1.0002668224816
$\mathcal{M}_{3,\frac{3}{2}} + \phi_{\text{id,id,adj}}$		$(3, \frac{7}{2}, \pm \frac{1}{7})$	0.01 0.3	1.5789577694079 1.3491110222344	1.5789577694082 1.3491110222443
$\mathcal{M}_{3,4} + \phi_{\text{id,id,adj}}$		$(3, 6, \pm \frac{1}{6})$	0.01 0.3	1.4590240568136 1.2821840541083	1.4590240568137 1.2821840541085
$S_2 \otimes S_{\text{SG}}(\frac{3}{10})$		$(2, 3, 0)$	0.01 0.3	1.4889948011771 1.2975198090220	1.4889948011793 1.2975198092226
$S_4 \otimes S_{\text{SG}}(\frac{1}{20})$		$(4, 1, 0)$	0.01 0.3	1.9854196180142 1.7223919114950	1.9854196180142 1.7223919114948

Table 2: Comparison of $c_{\text{eff}}(r)$ calculated using the TBA equations and the NLIE.

We can also enquire to which models the NLIEs based on $(A_1 \diamond A_L)_j^{[1]}$ correspond. If we set the j^{th} driving term to $-r \cosh \theta$ and all others to zero, the ultraviolet effective central charge becomes

$$c_{\text{eff}}(0) = \frac{3(L-j+1)}{L-j+3} - \frac{6\xi\alpha^2(L-j+1)}{(\xi+L)(\xi+j-1)}. \quad (30)$$

For $\xi=K+3-j$ and $\alpha=\pm 1/\xi$ we rediscover the massive $\phi_{\text{id,id,adj}}$ perturbation of a familiar model: $\mathcal{M}_{L-j+1,K}$.

4 $N=2$ supersymmetric sine-Gordon model

Recent interest in the $N=2$ supersymmetric sine-Gordon model [48] has arisen in two different contexts, both related to string theory. Though little is currently known about the excited state spectrum of the $N=2$ sine-Gordon model it is predicted to provide nontrivial information on the spectrum of states of closed superstrings in certain backgrounds [49]. Also, boundary versions of the $N=2$ SSG model (see for example [50,51]) are relevant for open string theory.

The S-matrix of the $N=2$ SSG model has the usual factorised form [52, 9, 53, 16]

$$S_{\text{SG}}^{N=2}(\tilde{\beta}) = S^{N=2} \otimes S_{\text{SG}}(\beta). \quad (31)$$

The supersymmetric piece $S^{N=2}$ is the sine-Gordon S-matrix at its $N=2$ supersymmetric point $\beta^2/8\pi = 2/3$. The bosonic piece is the sine-Gordon S-matrix at coupling β . The $N=2$ coupling constant $\tilde{\beta}$ is related to the sine-Gordon coupling via [52]

$$\tilde{\beta}^2 = \beta^2 / (1 - \beta^2/8\pi) \quad (32)$$

TBA equations derived from the $S^{N=2}$ S-matrix are encoded on $(A_1 \diamond A_3)_2$ [16]. As noted in [24], the TBA equations for the $N=2$ SSG model can, in theory, be obtained by gluing the $(A_1 \diamond A_3)_2$ TBA system to that of the sine-Gordon model at coupling β . In practice this has been done only for $\tilde{\beta}^2/8\pi = k + 1$, since at these points the SG TBA reduces to a simple Dynkin diagram: D_{k+2} . A second set of simple TBA equations exist for some related models: the minimal $N = 2$ conformal field theories perturbed by the superfield Φ_{13} [16].

To describe the groundstate energy of the $N=2$ SSG model at any coupling we replace the complicated sine-Gordon TBA equations with a single nonlinear integral equation, as we did above for the FSSG models. We propose equations similar to (13,14,15) but of type $(A_1 \diamond D_3)_1^{[1]}$ with $\alpha = 0$, rather than $(A_1 \diamond A_L)_1^{[1]}$. The nodes of D_n are labelled such that 1 is the first node of the tail, and $n, n-1$ label the fork nodes. The stationary values x_a^0 are all infinite, which suggests that numerical solutions for very small r will converge extremely slowly, if at all (this is equally true for any TBA equations with infinite stationary values). However combined with $x_2^\pm = x_3^\pm = 1$ and $x_1^\pm = 0$ we find the correct ultraviolet central charge of 3. The leading logarithmic corrections to this value have been proposed for some of the models in [54]. It is simple to check the proposed NLIEs have the expected infrared limit.

If the mass is moved onto either of the magnonic nodes so that the NLIEs are of type $(A_1 \diamond D_3)_2^{[1]}$, we find the effective central charge of the sausage models SST_λ^+ [55]. Again the possibly infinite set of TBA equations are in principle known. The simplest, occurring at $\lambda = 1/M$, are represented by an extended D_M Dynkin diagram [55]. A match is found with the NLIEs if we set $\xi = M-2$ and $\alpha = 0$. Alternatively if the twist α is set to $\pm 1/(M-2)$ we find the $\psi_1 \bar{\psi}_1 + \psi_1^\dagger \bar{\psi}_1^\dagger$ perturbation of the Z_{M-2} models, usually denoted $H_{M-2}^{(0)}$ [56, 19]. The models denoted $H_{M-2}^{(\pi)}$ refer to the massless perturbation of the Z_{M-2} models. The flow of the effective central charge from these models to the minimal models $\mathcal{M}(M-1, M)$ can be reproduced within the NLIEs provided the magnonic nodes are given driving terms of the form $-\frac{1}{2}re^\theta$ and $-\frac{1}{2}re^{-\theta}$. Upon resetting α to zero we find the effective central charge of the ‘massless’ sausage models $SST_{1/M}^-$.

Table 3 shows a numerical comparison of results found using this second set of NLIEs against some of the simplest TBA systems.

Model	TBA	NLIE (ξ, α)	r	TBA $c_{\text{eff}}(r)$	NLIE $c_{\text{eff}}(r)$
$S^{N=2} \otimes S_{SG}(\frac{3}{4})$		(3, 0)	0.01 0.3	2.5646432734143 1.9298961954447	2.5646432734187 1.9298961954452
$SST_{1/4}^+$		(2, 0)	0.01 0.3	1.7401548125701 1.3902395891608	1.7401548125703 1.3902395891609
$H_5^{(0)}$		(5, $\pm \frac{1}{5}$)	0.01 0.3	1.1373080354277 1.0190548066399	1.1373080354327 1.0190548066674
$H_4^{(\pi)}$		(4, $\pm \frac{1}{4}$)	1E-05 100	0.9999980062005 0.8003485312633	0.9999980062002 0.8003485312633

Table 3: Comparison of $c_{\text{eff}}(r)$ found using TBA equations and NLIEs of type $(A_1 \diamond D_3)_a^{[1]}$.

5 Summary

The SS model [54] is a two-parameter family of integrable models whose name follows from the S-matrix, which is a direct product of two sine-Gordon S-matrices. All of the models discussed above can be found as an SS model at specific values of the parameters [54, 57]. We leave for future investigation the possibility of gluing two sine-Gordon NLIEs together to describe the groundstate energy of this family of models.

Motivated by the existing results for the sine-Gordon model both sets of new NLIE should be generalised to encode the full finite size spectrum of the associated models. Furthermore, it is curious that the TBA driving term trick to find equations for the massless flows also works for the NLIEs; though only for the unitary coset models. A rather different method was used in [39] for the sine-Gordon model, also [43, 58], to obtain NLIEs describing massless flows at any value of β^2 . It remains to be seen whether a similar idea can be used on the new NLIEs. Finally it would be extremely interesting to develop the finite size effects of the fractional sine-Gordon models defined on a half-line.

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