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Factorizations of complete graphs into brooms

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Abstract

Let r and n be positive integers with $r < 2n$. A broom of order $2n$ is the union of the path on P_{2n-r-1} and the star $K_{1,r}$, plus one edge joining the center of the star to an endpoint of the path. It was shown by Kubesa [9] that the broom factorizes the complete graph K_{2n} for odd n and $r < \lfloor \frac{n}{2} \rfloor$. In this note we give a complete classification of brooms that factorize K_{2n} by giving a constructive proof for all $r \leq \frac{n+1}{2}$ (with one exceptional case) and by showing that the brooms for $r > \frac{n+1}{2}$ do not factorize the complete graph K_{2n}.

Key words: Graph factorization, graph labeling, spanning trees
2000 MSC: 05C70, 05C78

1 Introduction and definitions

Graph decomposition is a well established topic of graph theory. Various techniques were introduced for decomposing graphs into edge disjoint subgraphs.

Definition 1.1 *Let H be a graph with m vertices. A* decomposition *of the graph H is a set of pairwise edge disjoint subgraphs $G_1, G_2, \ldots, G_s$ of H such that every edge of H belongs to exactly one of the subgraphs G_r. If each subgraph G_r is isomorphic to a graph G we speak about a G-*decomposition of *H. If G is a factor (i.e., a spanning subgraph) of H, then we call the G-decomposition a G-*factorization.

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In this paper we always take the complete graph K_m for H and a certain spanning tree T for G. There are some obvious necessary conditions for a T-factorization of K_m to exist. First, since the number of edges $m-1$ of T must divide the number of edges $m(m-1)/2$ of K_m, obviously m has to be even and there will be $m/2$ copies of T in the factorization. Moreover, since every vertex has degree at least 1 in every factor, $\Delta(T) \leq m/2$. Further structure–based necessary conditions are examined in Section 2.

Unlike the famous Graceful Tree Conjecture (all n-vertex trees have graceful labelings, which enable them to decompose K_{2n}), not all $2n$-vertex trees factorize K_{2n}. There is no easy necessary and sufficient condition known for a T-factorization to exist, and we do not expect such condition to exist.

Sufficient conditions include several types of graph labelings. If a given graph G allows a certain type of labeling, then there exist a G-factorization of K_{2n}. One such labeling, the blended labeling, was introduced by Fronček [2]. A fundamental notion in further constructions is the length of an edge.

We adopt the common convention of denoting vertices by their labels. Moreover, an edge xy we denote by (x, y) if x or y are integer expressions.

Definition 1.2 *Let G be a graph with $V(G) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = m$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, \ldots, (m-1)_i\}$ for both $i = 0$ and $i = 1$.*

The pure length *of an edge (x_i, y_i) with $x_i, y_i \in V_i$, where $i \in \{0, 1\}$, for $\lambda(x_i) = p_i$ and $\lambda(y_i) = q_i$ is defined as*

$$\ell_{ii}(x_i, y_i) = \min\{|p - q|,\ m - |p - q|\}.$$

The mixed length *of an edge (x_0, y_1) with $x_0 \in V_0$, $y_1 \in V_1$, for $\lambda(x_0) = p_0$ and $\lambda(y_1) = q_1$, is defined as*

$$\ell_{01}(x_0, y_1) = \begin{cases} q - p & \text{for } q \geq p \\ m + q - p & \text{for } q < p, \end{cases}$$

where p and q are the vertex labels without subscripts and lie in $\{0, 1, \ldots, m-1\}$. The edges (x_i, y_i) for $i \in \{0, 1\}$ with the pure length ℓ_{ii} are pure edges *and the edges (x_0, y_1) with the mixed length ℓ_{01} are* mixed edges.

Definition 1.3 *Let G be a graph with $4n+1$ edges such that $V(G) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = 2n+1$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, \ldots, (2n)_i\}$ for both $i = 0$ and $i = 1$, and define lengths as in Definition 1.2.*

We say G has a blended labeling *(also called* blended ρ-labeling*) λ if*

(1) $\{\ell_{ii}(x_i, y_i): (x_i, y_i) \in E(G)\} = \{1, 2, \ldots, n\}$ *for* $i = 0, 1$,
(2) $\{\ell_{01}(x_0, y_1): (x_0, y_1) \in E(G)\} = \{0, 1, \ldots, 2n\}$.

Fronček [2] showed that there exists a G-factorization of K_{2n} for odd n if G has a blended labeling. Meszka [11] showed that having a blended labeling is not necessary for a G-factorization to exist when G is a tree. Kovářová [6] (see also [4]) introduced 'swapping labeling' and showed that a G-factorization of K_{2n} for even n exists when G has a swapping labeling.

Definition 1.4 *Let G be a graph with $V(G) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = 2n$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, \ldots, (2n-1)_i\}$ for both $i = 0$ and $i = 1$, and define lengths as in Definition 1.2.*

We say that G with $4n-1$ edges has a swapping blended labeling *(briefly* swapping labeling*) λ if*

(1) $\{\ell_{ii}(x_i, y_i): (x_i, y_i) \in E(G)\} = \{1, 2, \ldots, n\}$, *for* $i = 0, 1$,
(2) there exists an isomorphism φ such that G is isomorphic to G', where $V(G') = V(G)$ and $E(G') = E(G) \setminus \{(k_0, (k+n)_0), (l_1, (l+n)_1)\} \cup \{(k_0, (l+n)_1), ((k+n)_0, l_1)\}$ for certain k, l,
(3) $\{\ell_{01}(x_0, y_1): (x_0, y_1) \in E(G)\} = \{0, 1, \ldots, 2n-1\} \setminus \{\ell_{01}(k_0, (l+n)_1)\}$.

We summarize the results by Fronček [2] and Kovářová [6] in the following theorem.

Theorem 1.5 *If a graph G on m vertices allows a blended labeling or a swapping labeling, then there exists a G-factorization of K_m.*

Various other labelings such as the ρ-symmetric labeling, $2n$-cyclic labeling, fixing labeling, and recursive labeling were introduced by several authors as sufficient conditions for certain G-factorizations to exist. For every admissible $d \geq 3$, a spanning tree of diameter d that factorizes K_{4n+2} was found by Fronček in [2]; the case for K_{4n} was completed by Kovářová in [6]. Among the most general result there is the determination of spanning caterpillars of diameter 4 that factorize K_{2n} (in a series of papers by Fronček [3], Kubesa [9,10] and Kovářová [7,8]). Spanning caterpillars of diameter 5 that factorize K_{2n} were determined through the years in a series of papers and finally completed in [4] by Fronček et.al. A T-factorization of K_{2n} for every $\Delta(T)$ possible, $2 \leq \Delta(T) \leq n$, was given by Kovář and Kubesa [5].

In this paper we give a complete characterization (analogously to the papers [4] and [5]) in the case when a tree consisting of a path with many leaves attached to one of its endvertices factorizes the corresponding complete graph. We show that every such graph factorizes the corresponding complete graph unless the number of attached leaves exceeds $(n+1)/2$ or unless it is one exceptional case. The primary motivation for studying this class was to examine graphs that do not have labelings of the types mentioned and yet do factorize the complete graph. It turned out that there were only finitely many such graphs in this particular class of graphs.

Let S_r denote the star $K_{1,r}$, and let P_k denote the path with k vertices. For $1 \leq r \leq 2n-3$, let $B_{2n}(r)$ denote the graph formed from the disjoint union of P_{2n-r-1} and S_r by adding one edge joining the center of the star to an endvertex of the path. The graph $B_{2n}(r)$ is called a *broom*, the center of the star is called *centrum*, the leaves of the star are called *bristles*, the path is the *broomstick*, and its vertices are called *broomstick vertices*. See Fig. 1.

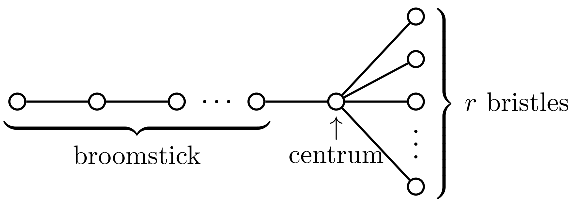

Fig. 1. A broom $B_{2n}(r)$.

We seek a decomposition of K_{2n} into n factors $T_1, T_2, \ldots, T_n$ that are isomorphic to a single spanning tree T, where $T \cong B_{2n}(r)$. Using the labeling of the vertices of K_{2n}, we designate the factors by isomorphisms $\phi_1, \phi_2, \ldots \phi_n$, writing $T_i = \phi_i(B_{2n}(r))$. This is an abuse of notation; actually, ϕ_i is the map of the vertex set.

2 Non-existence of a broom-factorization

Let $T_1, T_2, \ldots T_n$ be factors in K_{2n} that form a $B_{2n}(r)$-factorization of K_{2n}. There are at most three different vertex degrees in the broom $B_{2n}(r)$. Moreover, for $r > 1$ there are exactly three different degrees. The centrum has degree $r+1$, the bristles and one broomstick vertex have degree 1 and all remaining (broomstick) vertices have degree 2.

Lemma 2.1 *Let $B_{2n}(r)$ be a broom, and let the trees $T_1, T_2, \ldots, T_n$ form a $B_{2n}(r)$-factorization of K_{2n}. If $r > (n-1)/2$, then each vertex of the complete graph K_{2n} can be a centrum in at most one factor T_i.*

PROOF. We color the edges of K_{2n} so that all edges in one factor T_i are colored by the same color and we use a different color for each T_i. If some vertex u of K_{2n} is the image of the centrum v in two different factors T_i and T_j, then u is incident to $r+1$ edges in each of T_i and T_j and to at least one edge in each other factor. Hence $2(r+1)+(n-2) \leq 2n-1$, which requires $r \leq (n-1)/2$. $\square$

Theorem 2.2 *If $B_{2n}(r)$ factorizes the complete graph K_{2n}, then $r \leq (n+1)/2$.*

PROOF. By contradiction. Let $T = B_{2n}(r)$ be a broom with $r > (n+1)/2$ bristles and suppose it factorizes K_{2n}. By Lemma 2.1 each vertex of the complete graph K_{2n} is the map of the centrum in *at most one* factor F_i. Since there are only vertices of degree 1, 2, and $r+1$ in the tree T, we distinguish two types of vertices in K_{2n}. We say vertex u is Type A if it is the map of the centrum (of degree $r+1$) in one factor, the map of vertices of degree 2 in $n-r-1$ factors, and the map of leaves (not necessarily bristles) in r factors (each vertex in K_{2n} is of degree $2n-1$). Type B vertex is not the map of a centrum in any factor, but it is the map of vertices of degree 2 in $n-1$ factors and the map of a vertex of degree 1 in only one factor, since there are total $2n-1$ edges adjacent to it. There are n factors T_i which implies that there are n vertices of each type in K_{2n}.

Now we examine the edges between the centrum and the bristles. There is a total of nr bristles adjacent to the n centrums. Among these leaves at most n can mapped to some other Type B vertex. Since the centrum of each factor is mapped always to a Type A vertex, there have to be at least $nr-n = n(r-1) > n\left(\frac{n+1}{2}-1\right) = \binom{n}{2}$ edges among the Type A vertices. But there are only $\binom{n}{2}$ edges among the n Type A vertices which is the desired contradiction. □

3 Constructions of a broom-factorizations for odd n

Let $n = 2k+1$. The following lemma was proved in [9]. We give a simpler proof here.

Lemma 3.1 *The broom $B_{4k+2}(r)$ allows a blended labeling for every $k \geq 1$ and $1 \leq r \leq k$.*

PROOF. The proof is constructive. We split the broom $B_{4k+2}(r)$ into three subtrees T_0, T_{01}, and T_1. T_0 will contain only pure 00-edges, T_{01} only mixed edges, and T_1 only pure 11-edges; see Figs. 2 and 3.

The subtree T_0 is a broom $B_{k+1}(r)$ with r bristles and the broomstick of length $k-r$. The bristles are made by pure 00-edges $((k-\frac{k-r}{2})_0, x_0)$ for $x = \frac{k-r}{2}, \frac{k-r}{2}+1, \ldots, k-\frac{k-r}{2}-1$ when $k-r$ is even and $((\frac{k-r-1}{2})_0, x_0)$ for $x = \frac{k-r-1}{2}+1, \frac{k-r-1}{2}+2, \ldots, \frac{k+r-1}{2}-1$ when $k-r$ is odd. In the both cases the lengths of bristles are $1, 2, \ldots, r$.

The broomstick is the path $k_0, 0_0, (k-1)_0, 1_0, \ldots, (\frac{k-r}{2}-2)_0, (k-\frac{k-r}{2}+1)_0, (\frac{k-r}{2}-1)_0, (k-\frac{k-r}{2})_0$ for even $k-r$ and $k_0, 0_0, (k-1)_0, 1_0, \ldots, (k-\frac{k-r-1}{2}+$

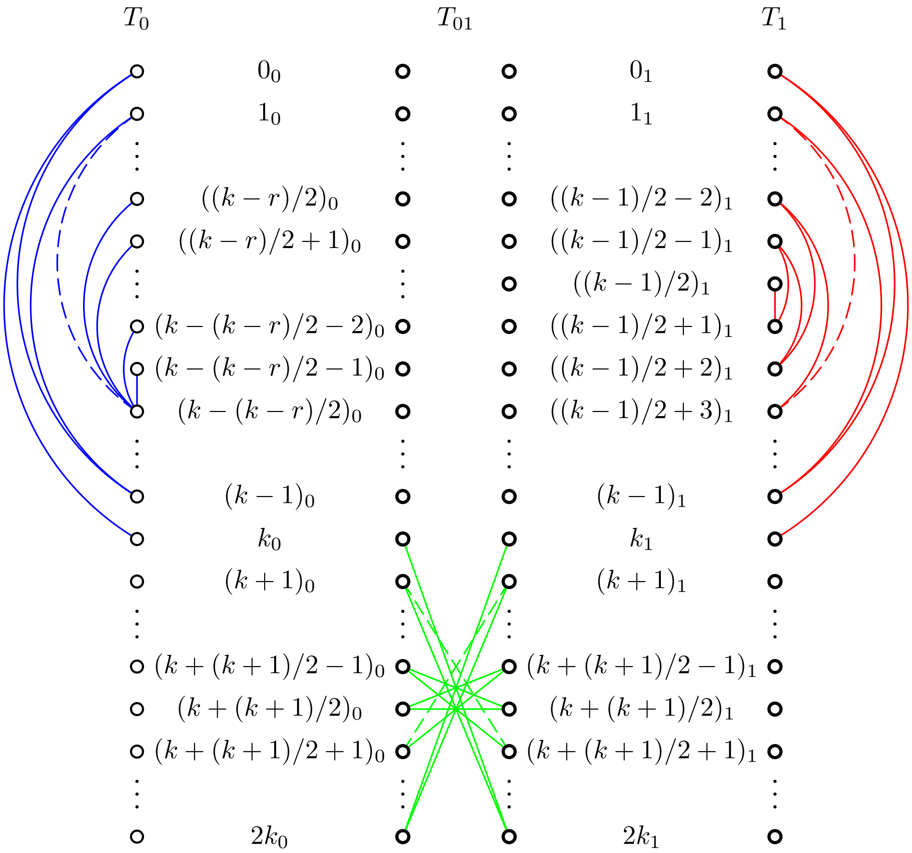

Fig. 2. Broom $B_{4k+2}(r)$ for odd k and odd r.

$1)_0, (\frac{k-r-1}{2}-1)_0, (k-\frac{k-r-1}{2})_0, (\frac{k-r-1}{2})_0$ for odd $k-r$. The lengths of pure 00-edges in the broomstick are $k, k-1, k-2, \dots, r+2, r+1$ in the both cases.

The subtree T_{01} is the path $k_0, (2k)_1, (k+1)_0, (2k-1)_1, \dots, (k+\frac{k}{2}+1)_1, (k+\frac{k}{2})_0, (k+\frac{k}{2})_1, (k+\frac{k}{2}+1)_0, \dots, (2k-1)_0, (k+1)_1, (2k)_0, k_1$ for even k and $k_0, (2k)_1, (k+1)_0, (2k-1)_1, \dots, (k+\frac{k+1}{2}-1)_0, (k+\frac{k+1}{2})_1, (k+\frac{k+1}{2})_0, (k+\frac{k+1}{2}-1)_1, \dots, (2k-1)_0, (k+1)_1, (2k)_0, k_1$ for odd k. In both cases the path has mixed edges of lengths $k, k-1, k-2, \dots, 1, 0, 2k, \dots, k+3, k+2, k+1$.

Finally, the subtree T_1 is the path $k_1, 0_1, (k-1)_1, 1_1, \dots, (\frac{k}{2}-2)_1, (\frac{k}{2}+1)_1, (\frac{k}{2}-1)_1, (\frac{k}{2})_1$ for even k and $k_1, 0_1, (k-1)_1, 1_1, \dots, (\frac{k-1}{2}+2)_1, (\frac{k-1}{2}-1)_1, (\frac{k-1}{2}+1)_1, (\frac{k-1}{2})_1$ for odd k. In both cases the path T_1 contains pure 11-edges of all lengths from 1 up to k.

Notice that the subtrees T_0 and T_{01} share only a single vertex, namely k_0, and T_{01}, T_1 also share a single vertex k_1. Therefore, we obtain a blended labeling of a broom $B_{4k+2}(r)$ for every $1 \le r \le k$. □

Lemma 3.2 *The broom $B_{4k+2}(k+1)$ allows a blended labeling for $k \ge 2$.*

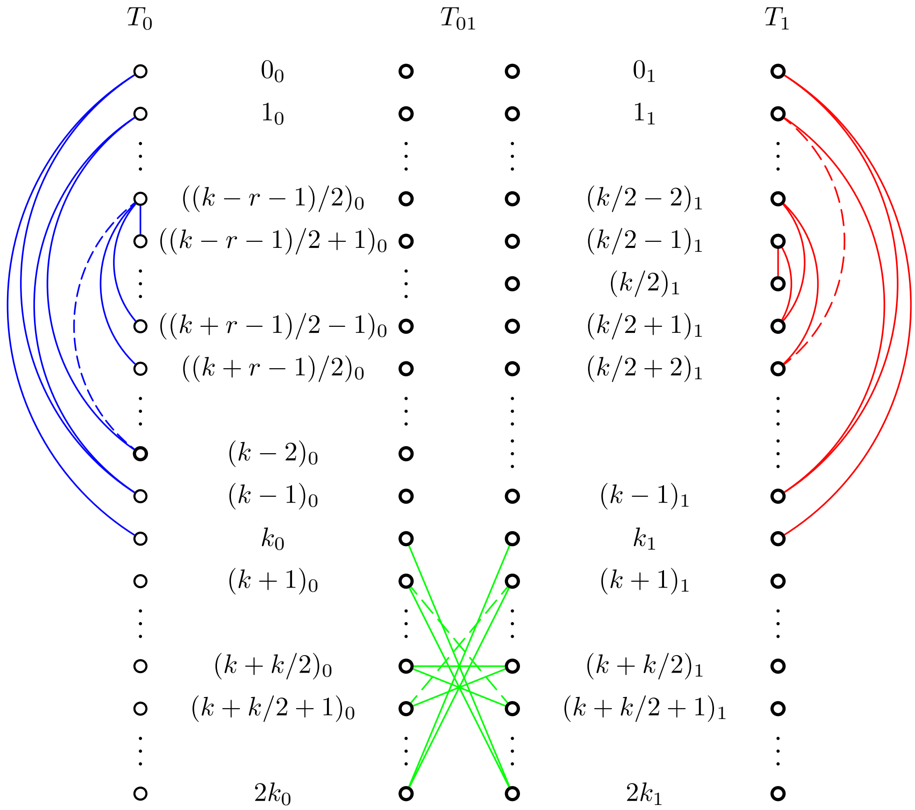

Fig. 3. Broom $B_{4k+2}(r)$ for even k and odd r.

PROOF. The proof is constructive. We divide the construction into two cases.

Case 1. Let k be even. We split the broom $B_{4k+2}(k+1)$ into four subtrees T_1, T_2, T_3, T_4, and a single edge. T_1 is a star $K_{1,k+1}$, T_2, T_3 are paths of length k and T_4 is a path of length $k-1$. Finally, we add one missing edge; see Fig 4.

The subtree T_1 contains pure 00-edges $(k_0, 0_0), (k_0, 1_0), (k_0, 2_0), \ldots, (k_0, (k-1)_0)$ of lengths $k, k-1, k-2, \ldots, 1$ and the mixed edge (k_0, k_1) of length 0. The subtree T_2 is the path $k_0, (2k)_1, (k+1)_0, (2k-1)_1, \ldots, (k+\frac{k}{2}+1)_1, (k+\frac{k}{2})_0$ with mixed edges of lengths $k, k-1, k-2, \ldots, 1$. The subtree T_3 is the path $(k+1)_1, 0_1, (k-1)_1, 1_1, (k-2)_1, 2_1, \ldots, (\frac{k}{2}-1)_1, (\frac{k}{2})_1$ with pure 11-edges of all lengths from 1 up to k. The subtree T_4 is the path $(k+1)_1, (2k)_0, (k+2)_1, (2k-1)_0, \ldots, (k+\frac{k}{2})_1, (k+\frac{k}{2}+1)_0$ with mixed edges of lengths $k+2, k+3, k+4, \ldots, 2k$. Notice that T_1, T_2 share only the vertex k_0 and T_3, T_4 also share only a single vertex $(k+1)_1$. Moreover, T_1 together with T_2 form one component and T_3 together with T_4 form another component. By adding the last mixed edge $((k+\frac{k}{2})_0, (\frac{k}{2})_1)$ of the so far unused length $k+1$ we obtain the broom $B_{4k+2}(k+1)$ with a blended labeling.

Case 2. Let k be odd. Again we split $B_{4k+2}(k+1)$ into four subtrees T_1, T_2, T_3, T_4, and a single edge; see Fig. 5. T_1 is the star $K_{1,k+1}$, T_2, T_3 are paths of

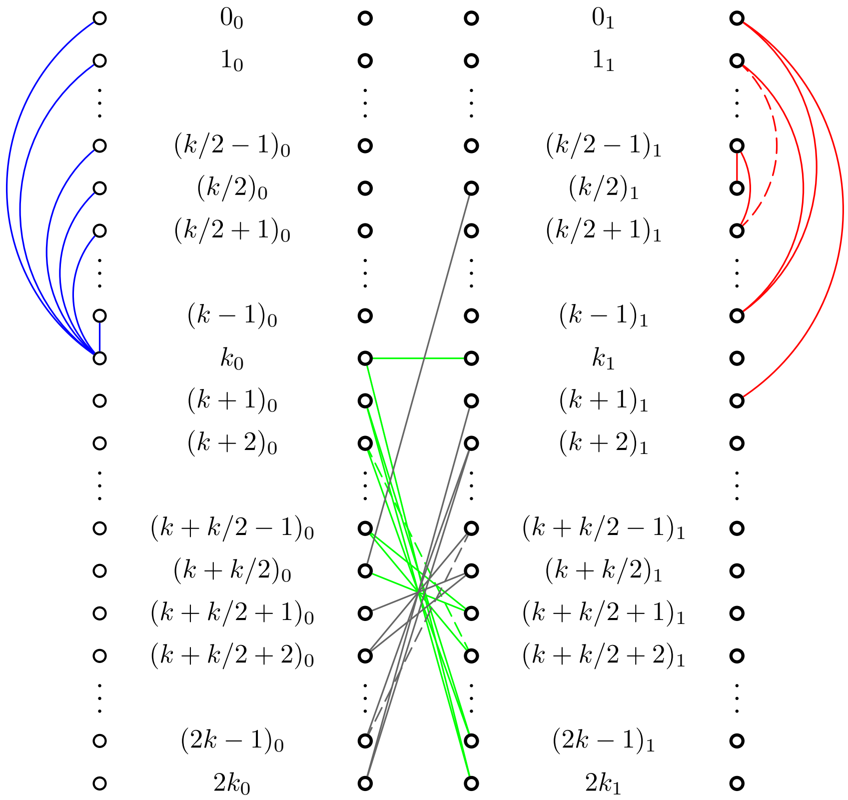

Fig. 4. Broom $B_{4k+2}(k+1)$ for even k.

lengths k and T_4 is a path of length $k-1$.

The subtree T_1 contains pure 00-edges $(k_0, 0_0), (k_0, 1_0), (k_0, 2_0), (k_0, (k-1)_0)$ of lengths $k, k-1, k-2, \ldots, 1$ and the mixed edge (k_0, k_1) of length 0. The subtree T_2 is the path $k_0, (2k)_1, (k+1)_0, (2k-1)_1, \ldots, (k+\frac{k+1}{2}-1)_0, (k+\frac{k+1}{2})_1$ with mixed edges of lengths $k, k-1, k-2, \ldots, 1$. The subtree T_4 is the path $(k+1)_1, (2k)_0, (k+2)_1, (2k-1)_0, \ldots, (k+\frac{k+1}{2}-1)_1, (k+\frac{k}{2}+1)_0, (k+\frac{k+1}{2})_1$ with mixed edges of lengths $k+2, k+3, k+4, \ldots, 2k$. Observe that the vertex $(k+\frac{k+1}{2})_0$ is isolated and T_2, T_4 share only the vertex $(k+\frac{k+1}{2})_1$ and T_1, T_2 share the vertex k_0.

Now we add the mixed edge $((k+\frac{k+1}{2})_0, (\frac{k+1}{2})_1)$ of the only missing length $k+1$ and show by induction that we can construct the path T_3 so that it has the endvertices $(k+1)_1$ and $(\frac{k+1}{2})_1$. Moreover, T_3 will have k pure 11-edges of all lengths from 1 up to k and $V(T_3)$ is the set $\{0_1, 1_1, 2_1, \ldots, (k-1)_1, (k+1)_1\}$ for every $k \geq 3$.

First, we show that above mentioned path T_3 exists for $k = 3, 5, 7$.

- $k = 3$: T_3 is the path $4_1, 1_1, 0_1, 2_1$.
- $k = 5$: T_3 is the path $6_1, 1_1, 4_1, 0_1, 2_1, 3_1$.
- $k = 7$: T_3 is the path $8_1, 1_1, 5_1, 0_1, 6_1, 3_1, 2_1, 4_1$.

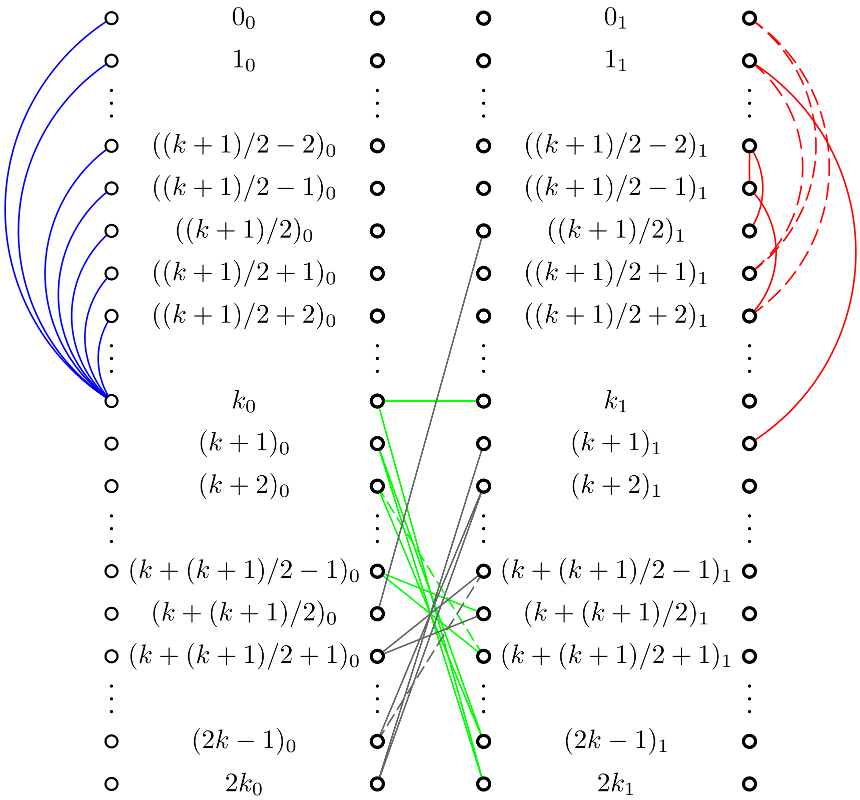

Fig. 5. Broom $B_{4k+2}(k+1)$ for odd k, $k \equiv 1 \pmod 6$.

Assume that T_3 exists for an arbitrary odd $k \geq 3$. We show that T_3 also exists for $\overline{k} = k+6$ The endvertices of T_3 are $(\frac{k+1}{2})_1$ and $(k+1)_1$, $V(T_3) = \{0_1, 1_1, 2_1, \ldots, (k-1)_1, (k+1)_1\}$. There are k 11-edges of all lengths from 1 up to k in T_3. Now we relabel the vertices in $V(T_3)$. If a vertex was labeled by i_1 then it gets the label $(i+3)_1$. Hence, $V(T_3) = \{3_1, 4_1, 5_1, \ldots, (k+2)_1, (k+4)_1\}$ and all lengths of edges remain the same. To construct a path $\overline{T}_3$, for $\overline{k}$, we add six vertices with labels 0_1, 1_1, 2_1, $(k+3)_1$, $(k+5)_1$, $(k+7)_1$ and the pure 11-edges $((k+7)_1, 1_1), (1_1, (k+5)_1), ((k+5)_1, 0_1)$ of lengths $k+6, k+4, k+5$ and the edges $(0_1, (k+3)_1), ((k+3)_1, 2_1), (2_1, (k+4)_1)$ of lengths $k+3, k+1, k+2$. Thus, $V(\overline{T}_3) = 0_1, 1_1, 2_1, \ldots, (k+5)_1, (k+7)_1$. Notice that the vertex $(k+3)_1$ is not incident with the mixed edge of length 0, while the vertex $(k+6)_1$ is. We obtain the path $\overline{T}_3$ and the proof is complete. □

Eldergill [1] proved that $(n, 2, n-1)$-caterpillar (the numbers $n, 2, n-1$ are the degrees of the vertices along the spine) of diameter 4 does not factorize K_{2n} for every $n \geq 3$. The broom $B_6(2)$ is a special case of such caterpillar for $n = 3$.

Lemma 3.3 *The broom $B_6(2)$ does not factorize K_6.*

Theorem 3.4 *If n is odd and at least 3, then $B_{2n}(r)$ factorizes K_{2n} if and*

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only if $r \leq \frac{n+1}{2}$ *and* $B_{2n}(r) \not\cong B_6(2)$.

The proof follows immediately from Lemmas 3.1–3.3 and Theorem 1.5.

4 Construction of a broom-factorization for even n

In this section we consider even n and let $k = n/2$.

Lemma 4.1 *The broom $B_{4k}(r)$ allows a swapping labeling when $k \geq 2$ and $1 \leq r \leq k-1$.*

PROOF. The proof is constructive. We split the broom $B_{4k}(r)$ into three subtrees T_0, T_{01}, and T_1; see Figs. 6 and 7. T_0 will contain only pure 00-edges, T_{01} will contain only mixed edges and T_1 will contain only pure 11-edges. Pure edges of length k in T_0 and T_1 will be swapped with mixed edges of length k in accordance with the swapping labeling.

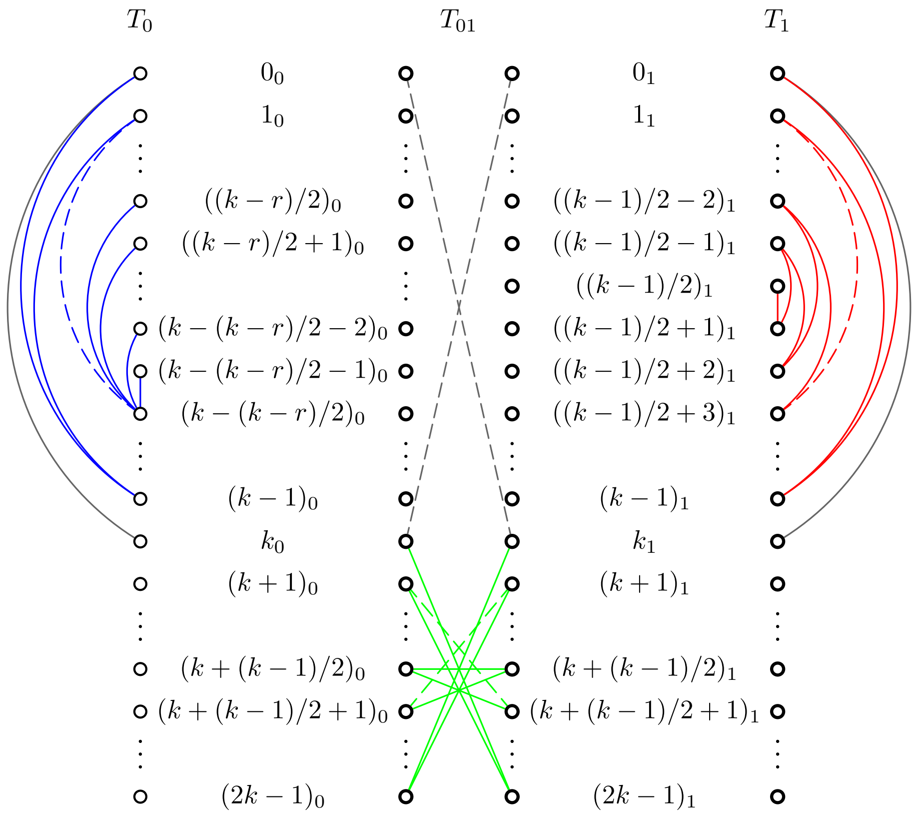

Fig. 6. Broom $B_{4k}(r)$ for odd k and odd r.

The subtree T_0 is a broom $B_{k+1}(r)$ with r bristles and a broomstick of length $k-r$. The bristles are made by pure 00-edges $((k-\frac{k-r}{2})_0, x_0)$ for $x = \frac{k-r}{2}, \frac{k-r}{2}+$

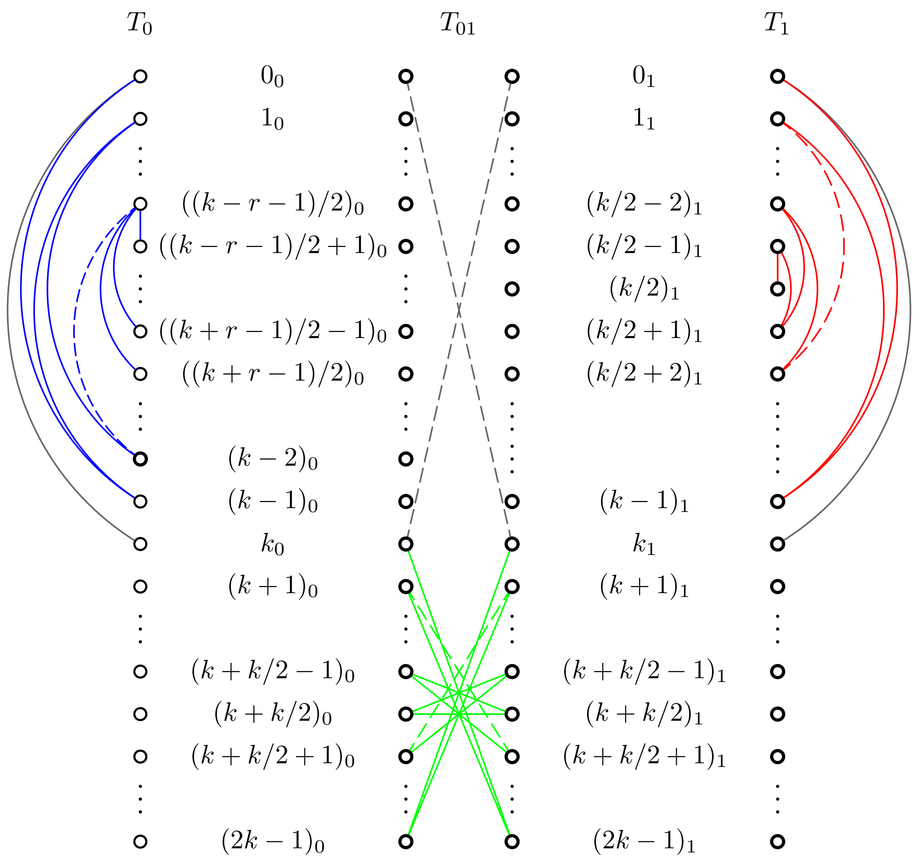

Fig. 7. Broom $B_{4k}(r)$ for even k and odd r.

$1, \dots, k - \frac{k-r}{2} - 1$ when $k-r$ is even and $((\frac{k-r-1}{2})_0, x_0)$ for $x = \frac{k-r-1}{2} + 1, \frac{k-r-1}{2} + 2, \dots, k - \frac{k-r-1}{2} - 1$ when $k-r$ is odd. In both cases the lengths of bristles are $1, 2, \dots, r$. The broomstick is the path $k_0, 0_0, (k-1)_0, 1_0 \dots, (\frac{k-r}{2} - 2)_0, (k - \frac{k-r}{2} + 1)_0, (\frac{k-r}{2} - 1)_0, (k - \frac{k-r}{2})_0$ for even $k-r$ and $k_0, 0_0, (k-1)_0, 1_0, \dots, (k - \frac{k-r-1}{2} + 1)_0, (\frac{k-r-1}{2} - 1)_0, (k - \frac{k-r-1}{2})_0, (\frac{k-r-1}{2})_0$ for odd $k-r$. In the both cases the lengths of the pure 00-edges along the broomstick are $k, k-1, k-2, \dots, r+2, r+1$.

The subtree T_{01} is the path $k_0, (2k-1)_1, (k+1)_0, (2k-2)_1, \dots, (k+\frac{k}{2}-1)_0, (k+\frac{k}{2})_1, (k+\frac{k}{2})_0, (k+\frac{k}{2}-1)_0, \dots, (2k-2)_0, (k+1)_1, (2k-1)_0, k_1$ for even k and $k_0, (2k-1)_1, (k+1)_0, (2k-2)_1, \dots, (k+\frac{k-1}{2}+1)_1, (k+\frac{k-1}{2})_0, (k+\frac{k-1}{2})_1, (k+\frac{k-1}{2}+1)_0, \dots, (2k-2)_0, (k+1)_1, (2k-1)_0, k_1$ for odd k. In both cases the path has mixed edges of lengths $k-1, k-2, k-3, \dots, 1, 0, 2k-1, \dots, k+3, k+2, k+1$. The mixed edge of length k is missing.

Finally, the subtree T_1 is the path $k_1, 0_1, (k-1)_1, 1_1, \dots, (\frac{k}{2}-2)_1, (\frac{k}{2}+1)_1, (\frac{k}{2}-1)_1, (\frac{k}{2})_1$ for even k and $k_1, 0_1, (k-1)_1, 1_1, \dots, (\frac{k-1}{2}+2)_1, (\frac{k-1}{2}-1)_1, (\frac{k-1}{2}+1)_1, (\frac{k-1}{2})_1$ for odd k. In the both cases the path T_1 contains pure 11-edges of all lengths from 1 up to k.

Now based on the labeling described above we obtain $2k$ pairwise edge disjoint factors of K_{4k} by adding $0, 1, \ldots, 2k-1$, respectively, to all vertex labels (addition is performed modulo $2k$) According the definition of swapping labeling the first k factors contain pure edges $(0_0, k_0)$ and $(0_1, k_1)$ (both are of length k). For the next k factors these will be replaced by mixed edges $(0_0, k_1)$ and $(0_1, k_0)$ of length k for each parity of k.

It is easy to observe (Figs. 6 and 7), that subtrees T_0 and T_{01} share only one vertex, namely k_0 in the first k factors and k_1 in the next k factors. Also T_{01}, T_1 share a single vertex k_1 in the first k factors and k_0 in the next k factors. Therefore, broom $B_{4k}(r)$ has a swapping labeling for every $1 \leq r \leq k-1$. □

Lemma 4.2 *The broom $B_{4k}(k)$ of order $4k$ with k bristles has a swapping labeling for all $k \geq 4$.*

PROOF. To construct the swapping labeling we consider three cases.

Case 1. Let $k \geq 7$ be odd. The following k edges make bristles of $B_{4k}(k)$: $k-1$ edges $((k+2)_0, x_0)$ for $x = 0, 1, k+3, k+4, \ldots, 2k-1$ and moreover the edge $((k+2)_0, (k+1)_1)$. In this way all pure 00-lengths except k and mixed length $2k-1$ are accommodated.

In the next step of the construction we divide the broomstick P of length $3k-1$ into three segments T_1, T_2, T_3, and one single edge. The first segment of P is a path T_1 of length 3: $(k+2)_0, 2_0, 2_1, (k+2)_1$.

The second segment, a path T_2 of length $2k-4$ has endvertices $(k+2)_1$ and 0_1. The edges of T_2 are: $((k+2-x)_0, (k+2+x)_1)$ for $x = 1, \ldots, \frac{k-1}{2}$, $((k+2-y)_0, (k+1+y)_1)$ for $y = 1, 2, \ldots, \frac{k-1}{2}, \frac{k+3}{2}, \frac{k+5}{2}, \ldots, k-1$, and $((k+2-z)_0, (k+z)_1)$ for $z = \frac{k+3}{2}, \frac{k+5}{2}, \ldots, k-1$.

The third segment, a path T_3, has endvertices $(\frac{k-1}{2})_1$, 0_1, internal vertices in the set $\{1_1, 3_1, 4_1, \ldots, (\frac{k-3}{2})_1, (\frac{k+1}{2})_1, \ldots, k_1\}$ and consists of $k-1$ edges. To construct T_3 we apply induction on k. T_3 exists for $k = 7, 9, 11$:

- $k = 7$: T_3 is the path $3_1, 4_1, 7_1, 5_1, 1_1, 6_1, 0_1$,
- $k = 9$: T_3 is the path $4_1, 9_1, 5_1, 3_1, 6_1, 7_1, 1_1, 8_1, 0_1$,
- $k = 11$: T_3 is the path $5_1, 11_1, 4_1, 8_1, 3_1, 6_1, 7_1, 9_1, 1_1, 10_1, 0_1$.

Now suppose that assertion is true when k is even and at least 10. We are going to construct a path $\overline{T}_3$ for $\overline{k} = k+6$. Apply the following relabeling for vertices of T_3: $i \to i+3$, for $i = 0, 1, 3, 4, \ldots, k$. Let $\overline{T}_3$ include all edges of T_3 and a path of length 6: $3_1, (\overline{k}-1)_1, 5_1, \overline{k}_1, 1_1, (\overline{k}-2)_1, 0_1$. Thus $\overline{T}_3$ has endvertices $(\frac{\overline{k}-1}{2})_1$, 0_1 and its internal vertices are $1_1, 3_1, 4_1, \ldots, (\frac{\overline{k}-3}{2})_1, (\frac{\overline{k}+1}{2})_1, \ldots, \overline{k}_1$; see Fig. 8.

The last part of P is the single edge $((\frac{k+3}{2})_0, (\frac{k-1}{2})_1)$. One can easily check that T_1 covers pure lengths k of both types and mixed length 0, T_2 covers all mixed lengths except 0, k, $2k-2$ and $2k-1$, T_3 covers all pure 11-lengths except k, and the single edge has mixed length $2k-2$.

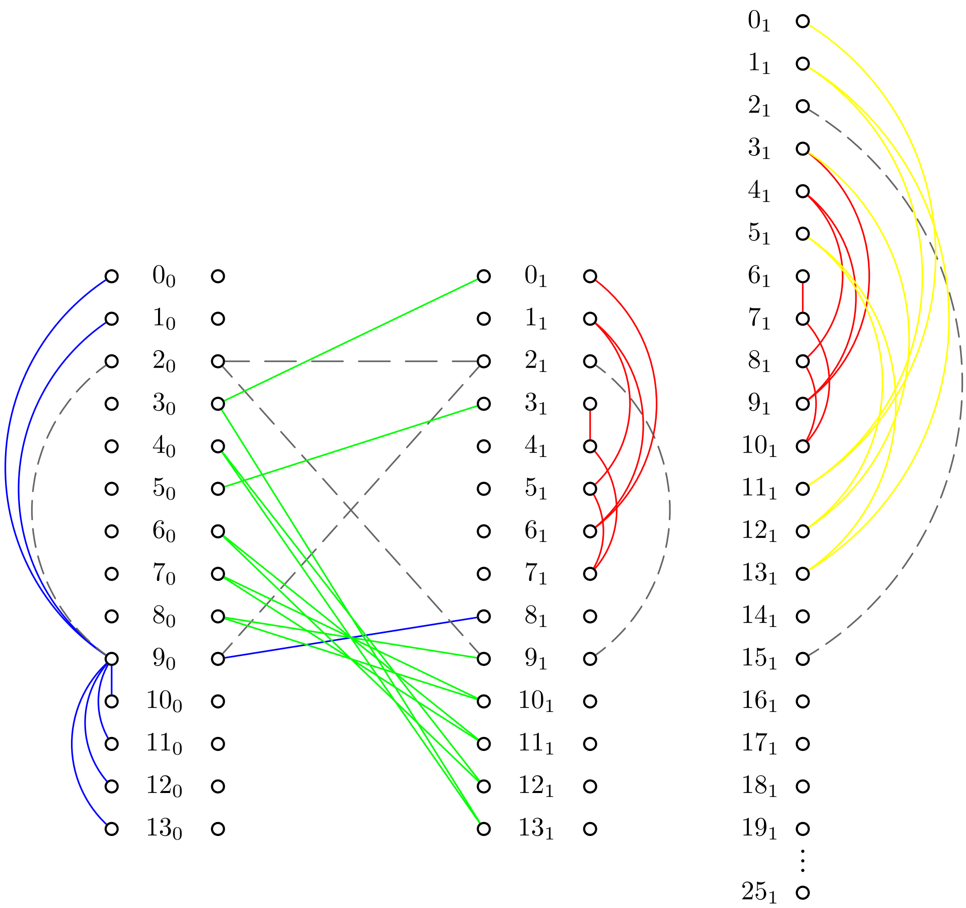

Fig. 8. Broom $B_{4k}(r)$ for odd $k \geq 7$ and $r = k$ along with the induction on k.

Case 2. Let $k \geq 10$ be even. The following k edges make bristles of $B_{4k}(k)$: $k-1$ edges $((k+2)_0, x_0)$ for $x = 0, 1, \frac{k}{2}, \frac{k}{2}+1, k+3, k+4, \ldots, \frac{3k}{2}+2, \frac{3k}{2}+5, \frac{3k}{2}+6, \ldots, 2k-1$ and moreover the edge $((k+2)_0, 4_1)$. In this way all pure 00-lengths except k and mixed length $k+2$ are accommodated.

Next we construct a path P of length $3k-1$ composed of four segments T_1, T_2, T_3, and T_4. The first segment of P is a path T_1 of length 3: $(k+2)_0, 2_0, 2_1, (k+2)_1$. The second segment, a path T_2 of length $2k-6$, has endvertices $(k+2)_1$ and 0_1. The edges of T_2 are: $((k+2-x)_0, (k+2+x)_1)$ for $x = 1, \ldots, \frac{k}{2}-1$, $((k+2-y)_0, (k+1+y)_1)$ for $y = 1, 2, \ldots, \frac{k}{2}, \frac{k}{2}+3, \frac{k}{2}+4, \ldots, k-1$, $((k+2-z)_0, (k+z)_1)$ for $z = \frac{k}{2}+3, \frac{k}{2}+4, \ldots, k-1$ and moreover the edge $((\frac{k+4}{2})_0, (\frac{3k+6}{2})_1)$. The addition is performed modulo $2k$, if necessary. The third segment, a path T_3, has endvertices $(\frac{k+12}{2})_1$, 0_1, internal vertices in the set $\{1_1, 3_1, 5_1, 6_1, \ldots, (\frac{k+10}{2})_1, (\frac{k+14}{2})_1, \ldots, (k+1)_1\}$ and consists of $k-1$ edges. To construct T_3 we apply induction on k. T_3 exists for $k = 10, 12, 14, 16$ and

18:

- $k = 10$: T_3 is the path $0_1, 9_1, 1_1, 7_1, 8_1, 5_1, 3_1, 10_1, 6_1, 11_1$,
- $k = 12$: T_3 is the path $0_1, 6_1, 11_1, 7_1, 9_1, 8_1, 5_1, 13_1, 3_1, 10_1, 1_1, 12_1$,
- $k = 14$: T_3 is the path $0_1, 7_1, 9_1, 10_1, 5_1, 11_1, 8_1, 12_1, 1_1, 14_1, 6_1, 15_1, 3_1, 13_1$,
- $k = 16$: T_3 is the path $0_1, 13_1, 1_1, 16_1, 5_1, 8_1, 12_1, 6_1, 15_1, 7_1, 17_1, 3_1, 10_1, 11_1, 9_1, 14_1$,
- $k = 18$: T_3 is the path $0_1, 14_1, 1_1, 18_1, 3_1, 19_1, 7_1, 9_1, 12_1, 5_1, 13_1, 8_1, 17_1, 6_1, 16_1, 10_1, 11_1, 15_1$.

Now suppose that assertion is true when k is even and at least 10. We are going to construct a path $\overline{T}_3$ for $\overline{k} = k + 10$. Apply the following relabeling for vertices of T_3: $i \to i + 5$, for $i = 0, 1, 3, 5, 6 \ldots, k + 1$. Let $\overline{T}_3$ include all edges of T_3 and a path of length 10: $5_1, (\overline{k} - 2)_1, 3_1, (\overline{k} + 1)_1, 7_1, (\overline{k} - 1)_1, 9_1, \overline{k}_1, 1_1, (\overline{k} - 3)_1, 0_1$. Thus $\overline{T}_3$ has endvertices $(\frac{\overline{k}+12}{2})_1$, 0_1 and its internal vertices are $1_1, 3_1, 5_1, 6_1, \ldots, (\frac{\overline{k}+10}{2})_1, (\frac{\overline{k}+14}{2})_1, \ldots, (\overline{k} + 1)_1$; see Fig. 9. The last segment of P is the path T_4: $(\frac{k+12}{2})_1, (\frac{3k+6}{2})_0, (\frac{3k+4}{2})_1, (\frac{3k+8}{2})_0$ of length 3.

One can easily check that T_1 covers pure lengths k of both types and mixed length 0, T_2 covers all mixed lengths except 0, k, $k + 2$, $k + 3$, $2k - 2$ and $2k - 1$, T_3 covers all pure 11-lengths except k, and T_4 covers mixed lengths $k + 3$, $2k - 1$ and $2k - 2$. Notice that if we replace in previous cases in the broom $B_{4k}(k)$ two pure edges $((k+2)_0, 2_0)$ and $((k+2)_1, 2_1)$ of the pure length k by two mixed edges $((k + 2)_0, 2_1)$ and $((k + 2)_1, 2_0)$ of the mixed length k then we obtain a broom isomorphic to $B_{4k}(k)$.

Case 3. Let $k = 5, 6$ or 8. The following k edges make bristles of $B_{4k}(k)$: $k-1$ edges (k_0, x_0) for $x = 1, 2, \ldots k-1$ and moreover the edge $(k_0, (2k-2)_1)$. The remaining part of $B_{4k}(k)$ is a path of length $3k - 1$ and one endvertex k_0:

- $k = 5$: the path is $5_0, 0_0, 0_1, 5_1, 9_0, 6_1, 7_0, 9_1, 8_0, 2_1, 1_1, 3_1, 7_1, 4_1, 6_0$,
- $k = 6$: the path is $6_0, 0_0, 0_1, 6_1, 11_0, 1_1, 10_0, 3_1, 7_0, 5_1, 8_0, 9_1, 4_1, 2_1, 11_1, 7_1, 8_1, 9_0$,
- $k = 8$: the path is $8_0, 0_0, 0_1, 8_1, 15_0, 1_1, 14_0, 2_1, 13_0, 4_1, 10_0, 9_1, 11_0, 6_1, 9_0, 5_1, 3_1, 10_1, 11_1, 15_1, 12_1, 7_1, 13_1, 12_0$.

It is routine checking that all lengths except mixed length k are used in $B_{4k}(k)$.

Finally let $k = 4$. A swapping labeling of $B_{16}(4)$ is given in Fig. 10. □

Theorem 4.3 *If n is even, then $B_{2n}(r)$ factorizes K_{2n} if and only if $r \leq \frac{n}{2}$.*

PROOF. For $k \geq 4$ the assertion follows from the constructions given in the proofs of Lemmas 4.1 and 4.2 and from Theorem 1.5. A construction found

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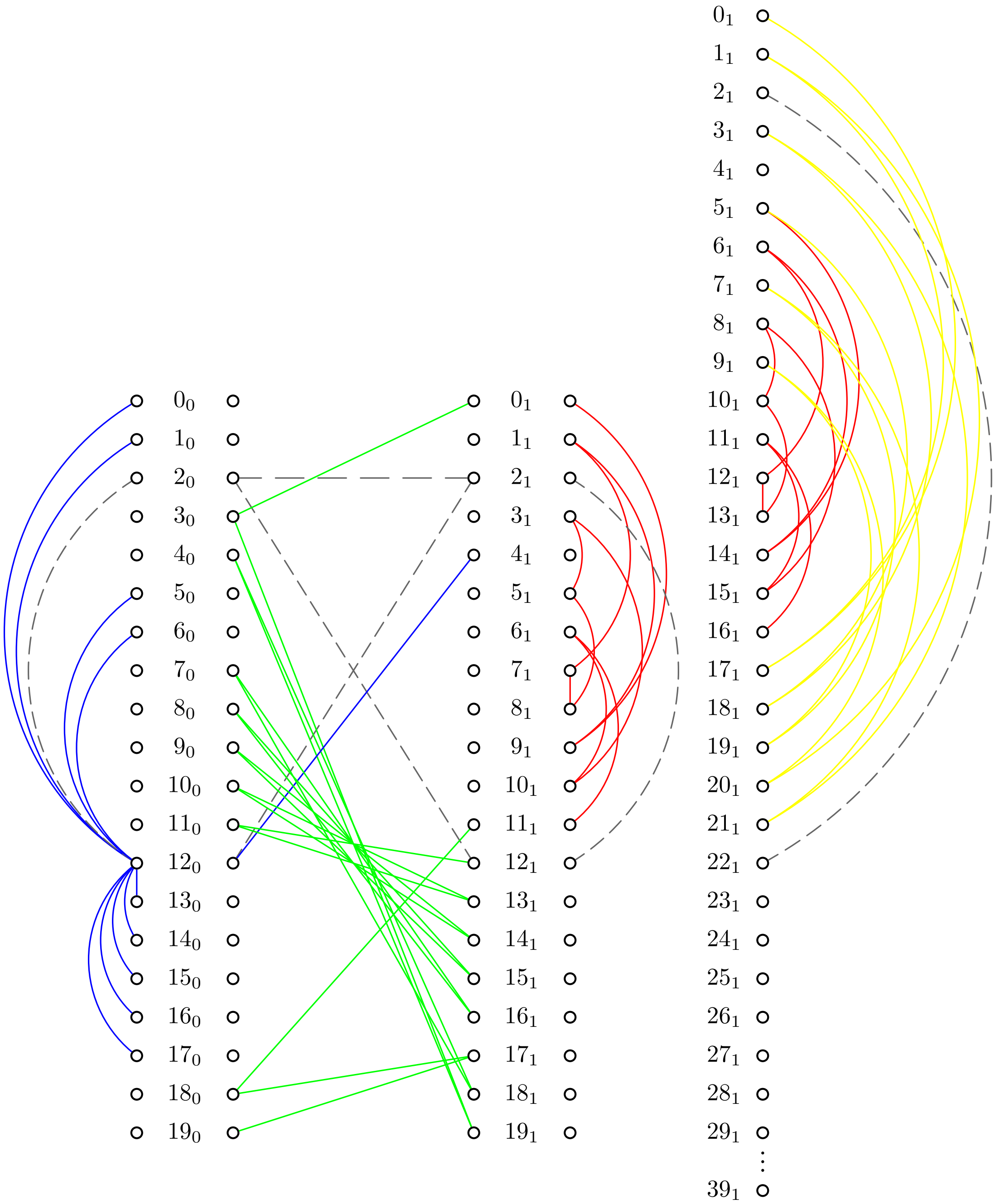

Fig. 9. Broom $B_{4k}(r)$ for even $k \geq 10$ and $r = k$ along with the induction on k.

by brute force for $k = 2$ is in Fig. 11 and for $k = 3$ in Fig. 12. $\square$

5 Conclusion

Theorems 3.4 and 4.3 yield our main result:

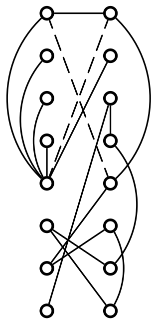

Fig. 10. A swapping labeling of $B_{16}(4)$.

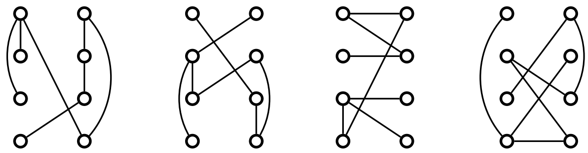

Fig. 11. $B_8(2)$-factorization of K_8.

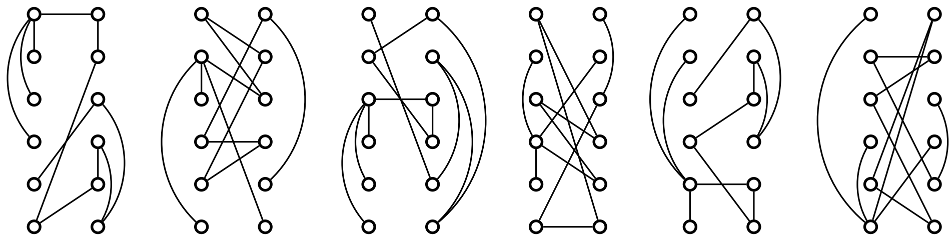

Fig. 12. $B_{12}(3)$-factorization of K_{12}.

Theorem 5.1 *The broom $B_{2n}(r)$ factorizes K_{2n} if and only if $r \leq \frac{n+1}{2}$ and $B_{2n}(r) \ncong B_6(2)$.*

The above theorem gives a complete classification of brooms that factorize the complete graph K_{2n}. In this way another step into solving the wide open general problem of tree factorizations of complete graphs is made.

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