USING SIGMOID FUNCTIONS FOR MODELLING SOUTH AFRICAN GOLD PRODUCTION

VYUŽITÍ SIGMOIDNÍCH FUNKCÍ K MODELOVÁNÍ PRODUKCE ZLATA JIŽNÍ AFRIKY

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Abstract

Sigmoid functions were used to approximate the cumulative gold production curve of Southern Africa and to extrapolate them into the future. Southern Africa was chosen because one third of the historic global world gold production comes from the Witwatersrand basin area, and also because the peak gold point was already reached in 1970. All models give a good agreement with reality, when the negative effects of World War II and the positive effects of the apartheid boom 1950 - 1970 are compensated for. From today's viewpoint the Gompertz function was found to give the best model, although this model implies the peak occurred 10 years earlier and due to the apartheid boom was about 300 tons higher. It also shows that the production could possibly rise again in the future to a level of 400 or even 500 metric tons a year. Hence, a biomodal production curve with at least two peak points could be observed in the future. Mathematical results are discussed in respect to the political and economic history of this country.

Abstrakt

K bližšímu představení celkové produkce zlata v Jižní Africe a její extrapolaci do budoucnosti byly použity Sigmoidní funkce. Jižní Afrika byla vybrána, protože na jedné straně pochází jedna třetina celosvětové historické produkce zlata z regionu Witwatersrand, na druhé straně byl vrchol produkce zlata dosažen už v roce 1970. Všechny modely se dobře shodují s realitou, pokud vypočítáme negativní vlivy druhé světové války a pozitivní vlivy rozmachu apartheidu v letech 1950-1970. Gompertzova funkce představuje z dnešního pohledu nejlepší model, i když podle tohoto modelu vychází vrchol o deset let dříve a, vlivem rozmachu apartheidu, o 300 t výše. Ukazuje taky, že by produkce mohla ještě jednou vzrůst na 400 až 500 tun za rok. Tak by mohla v budoucnosti vzniknout bimodální produkční křivka s nejméně dvěma vrcholy. Jsou diskutovány matematické modely s ohledem na politický a ekonomický vývoj země.

Key words: Southern Africa, gold production, peak gold, mathematical model, sigmoid functions.

1 INTRODUCTION

The production figures of some mineral commodity of a certain country can be seen as an empirical picture of the technical, geological, economical and political situation of that country or region. This is certainly true for the political metal gold and its most important country of origin - Southern Africa. From 150.000 metric tons of gold, which were mined during the last 7.000 years of history, around one third came from the Witwatersrand basin around Johannesburg [1]. A salient yearly production peak occurred in 1970 with 1000 tons of gold. Since then, the production has declined to a level around 250 tons, see Fig. 1 [2].

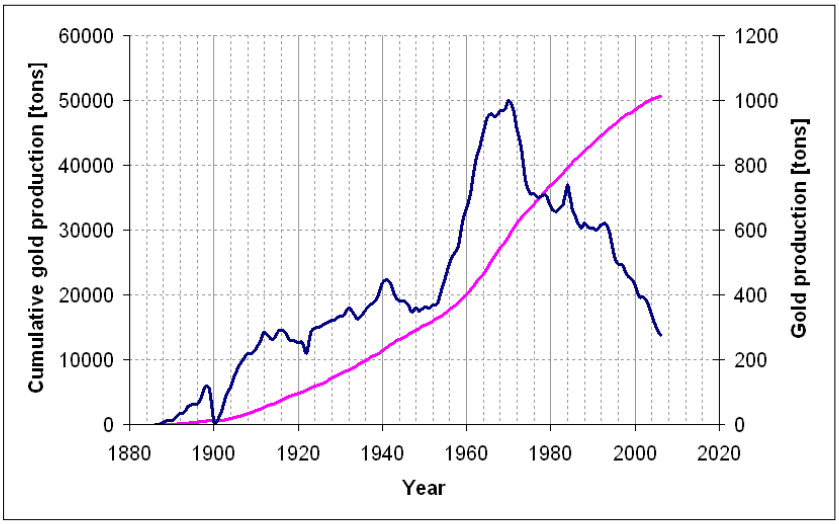

Fig. 1 South African gold production 1884 – 2007 (annual production in blue, right scale; summarized production in magenta, left scale) [2]

The U.S. Geological Survey enumerated in their last gold report the South African reserves to be 6.000 tons and the resources to be 36.000 tons. Globally, these figures represent approximately 14 % of the known reserves and 40 % of the known resources [3]. Obviously the frequently asked question is how the future production of these reserves and resources could qualitatively look like. In the field of mathematical approximations sigmoid functions offer a wide range of applications, such as modelling biological or economical growth processes. However, to the best of the authors' knowledge, there are no publications where cumulative gold production curves have been mathematically modelled by sigmoid functions.

In the following the historic gold production of Southern Africa in respect to the political and economical development will be discussed, as far as our models are concerned. Mainly, this is the Second World War and the so called "apartheid boom" from 1948 to 1970.

In the third section of this paper, the cumulative production curve of Fig. 1 will be approximated by 6 different sigmoid functions: the Logistic-, Gompertz-, Weibull-, Richards-, Johnson- and Gauss-Function (normal distribution). The mathematical approximations were calculated using the non-linear module of the JMP statistical software, version 8.0 [4]. In the fourth section the mathematical results and the possible future gold production of Southern Africa is qualitatively discussed.

2 HISTORICAL GOLD MINING IN SOUTHERN AFRICA

The large gold deposits of the Witwatersrand basin were discovered in the 80's of the 19th century. At the turn of the century the mining of gold was more than 100 tons per year and the interruption by the Second Boer War was only temporary.

Until the end of the 60's of the 20th century, the gold production laid within a rising trend channel with a rate of + 5,8 tons a year and a width of approximately 150 tons, see Fig. 2.

Within this trend channel the production fell only due to the First World War, marginally due to the great depression and longer continuing of the Second World War.

The fact that the Second World War had indeed a great impact on the global gold production can be seen by comparison of mining countries which took part in the war with those countries which were not involved, see Table 1.

Tab. 1 Gold production before and after the Second World War in different regions (* Europe without USSR, because only rough estimations are available for 1945) [5]

Region	**Production in 1938 [t]**	**Production in 1945 [t]**	**Change [%]**
Involved:			
North America	330,8	135,8	- 58,0
Europe*	18,9	6,2	- 67,1
Asia	114,4	13,9	- 87,9
Sum	**464,1**	**155,9**	**- 66,4**
Not involved:			
South America	50,2	39,7	- 21,0
Africa	454,1	434,8	- 4,3
Sum	**504,3**	**474,5**	**- 5,9**
World	**1178,1**	**811,8**	**- 31,1**

North America, Europe and Asia as the main participants of the war had to register a decrease in the gold production on the average of - 66,4 %, whereas the relatively not involved South American and African continents had a decrease just of - 5,9 %.

The second phase of political influence refers to the so called apartheid boom between 1948 and 1970. Due to the availability of the cheap black labour force, the South African economy grew in those years with a minimum rate of 5 % per year. Hence, due to the high net yield of up to 10 – 12 % a lot of capital from abroad was brought into Southern Africa in the form of direct investments or enterprise participations [6]. Additionally this boom was protected by tariffs, import quota and currency exchange restrictions. Especially the currency laws led to the situation, that earnings which were generated within Southern Africa had to stay and be reinvested in the country. For the mining industry this political and economical environment necessarily meant a large growth. Companies like Anglo American grew to huge conglomerates and used the mining profits to buy banks, breweries, farms, vineyards and so on. In the 1960‘s Anglo American concentrated in this manner more than half of the market capitalization of the Johannesburg Stock Exchange [7]. The cumulative gold production in the country nearly tripled from 350 tons to 1000 tons in 1970, see Figure 1. However, simultaneously the dependence on the industrial countries rose. At the end of the 1960‘s and the beginning of the 1970‘s, US President Nixon terminated the backing of the US dollar by gold and the Bretton Woods monetary system ended. The inflation rose and the first oil crises hit the world economy and so the South African boom came to an end. Also, strikes and rebellions of the poor black people added to the South African crisis, which culminated in 1976 in the Soweto Uprising. Our thesis that the production peak of 1000 tons in 1970, generated by the apartheid boom, was too early and too high compared to normal political circumstances, is supported by the following four arguments.

First, there is the already mentioned natural trend channel, which was built over several decades. Only the apartheid boom in 1958 could have caused the deviation from the trend channel.

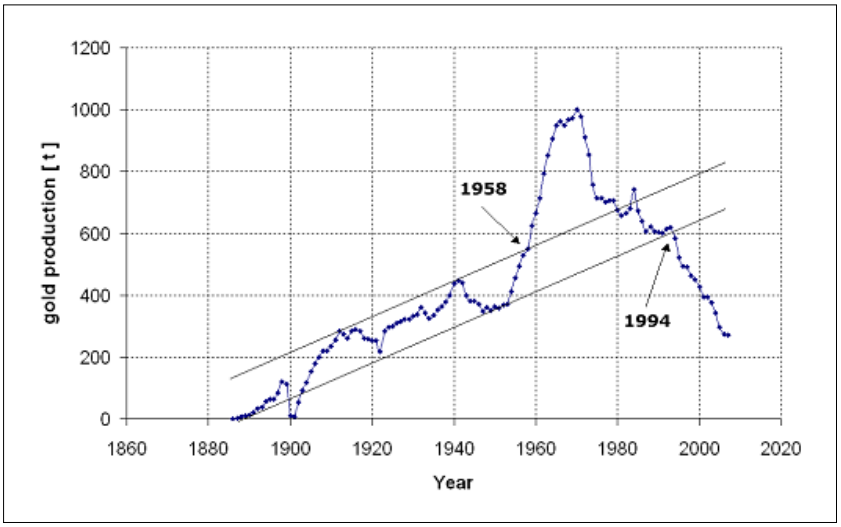

Fig. 2 Natural trend channel of the South African gold production. Outbreaks from this channel due to the Second Boer War in 1900, the Apartheid boom from 1958 onwards and after 1994

Surprisingly, if one extends those trend lines, the gold production just fell under this natural trend in 1994, when the ANC won the first general post-apartheid elections and took over political power.

The second argument for the peak having occurred too early is the development of the ore grades that had been mined in South Africa, see Figure 3. The data is available from 1893 onwards [8].

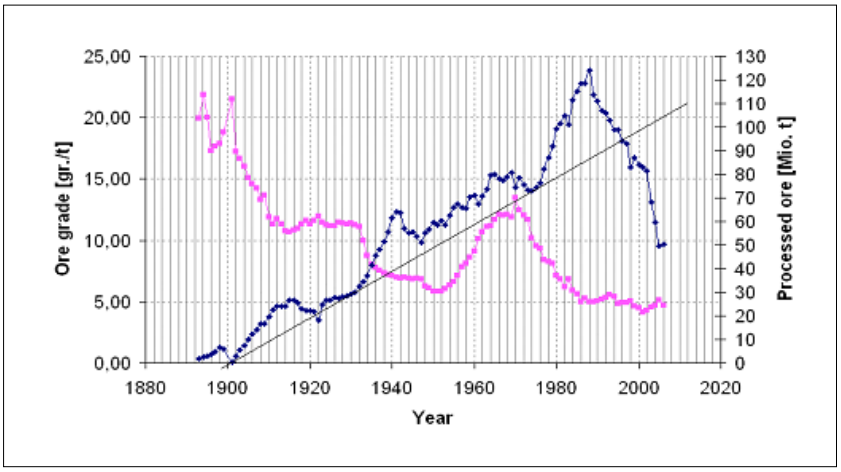

Fig. 3 Average ore grade and processed ore 1893 – 2006 [8]

Until the beginning of the apartheid era the ore grade declined on the average from around 20 grams per ton to around 6 grams per ton. From 1952 till 1970, however, this trend was reversed and the ore grade increased to 13,43 grams per ton in 1970. A lot of new high grade mines were brought into production during those years. The increase in ore grade was also accompanied by an increase in the processed ore, which rose from 50 million tons to about 80 million tons of ore (which of course was also a function of improved gold extracting technology). Thus, by expanding the mining activity both in the sense of mined ore and using the richest ore bodies available, it was possible to archive this gold peak point at 1000 tons in 1970. Also, from Fig. 3 it can be seen that after occurring the geological / technical peak, falling ore grades were tried to be compensated for by a further 50% increase to a clear peak of 120 million tons of ore, being processed in 1988.

The third argument for the apartheid gold boom is the comparison with other mining countries during that time. Subtracting the South African gold production from the global gold production (see Figure 4) gives clear evidence that the global gold production between 1950 and 1980 was absolutely constant at 500 tons a year.

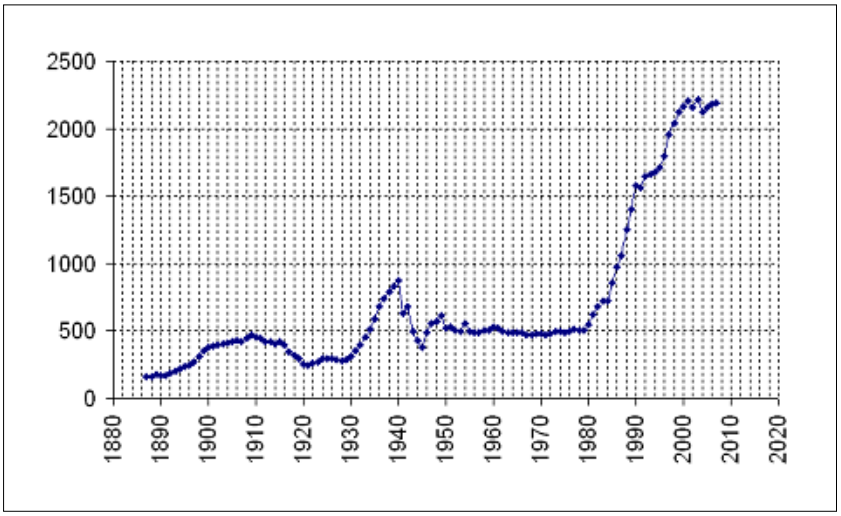

Fig. 4 Global gold production without Southern Africa

After the World War II, the gold price was still fixed at 35 US-$. The mining of other commodities was much more profitable and gold was mainly a by-product, being welcome to reduce costs. So while the rest of the world did not pay much attention to gold mining, Southern Africa was processing as much and as rich ore as possible. If we assume that South African production would have also been stable between 1950 and 1980 at the 1950-level of 350 tons, more than 11.000 tons would had not been mined and could have been "saved" for the future, i.e. for higher profit times.

Another interesting fact which can be seen from Fig. 3 is that for the last 20 years or so, only ore with an average concentration of 5 grams per ton is being mined. In fact, since 1989 the gold mining industry in Southern Africa has responded to lower gold prices by cutting costs and not replacing mines, which have exhausted their profitable life [9]. Hence, the processed ore decreased from 120 million tons to 50 millions tons nowadays and consequently the gold production decreased to around 270 tons.

The fourth argument supporting the thesis of the "too early and too high peak", is the number of employment in the South African mining industry. Between 1950 and 1980 the number of employees increased from 350.000 to 470.000 (+ 1/3) [10] and even the processed ore per employee increased from 170 tons to 211 tons (+ 1/4). Again, in times while the rest of the world had a constant gold output, the South African gold mining industry was "Apartheid-induced" growing in all respects.

Thus, compared to normal conditions (economically and politically) we can conclude that the gold production during the World War II was too low and during the subsequent apartheid boom was too high.

The yearly post-peak production so far can be divided into three periods, see the following table. The rates of change were obtained by fitting the respective time period with a linear function.

Tab. 2 Post-peak gold production in Southern Africa

Period	**decline [t]**	**rate of change [t/a]**
1970 – 1975	- 286,8	- 61,2
1975 – 1993	- 93,9	- 7,0
1993 – today	- 347,5	- 22,5

The mathematical models which we describe in the following section, show that the decline of the gold production after 1970 was geologically induced. However, after 1994 (break of the natural trend channel to the downside, see Figure 3) the decline was mainly due to economical and political reasons. This result is also confirmed by the Taylor formula, which calculates the optimum lifetime of the South African reserves and resources to be 93 years (fourth root of the reserves plus resources multiplied by 6,5). The optimum annual

production results in 451 tons a year (reserves plus resources divided by the optimum lifetime), i.e. around 2/3 higher than the current production level [11].

3 APPROXIMATING THE CUMULATIVE PRODUCTION CURVE

With the assumption that the currently given figures of the U.S. Geological Survey for the reserves and resources of together 42.000 tons will be extractable in the future, the cumulative historic gold production of Southern Africa will be 93.000 tons (production between 1884 and 2008 was approximately 51.000 tons). The cumulative production in the year 1970 was 29.135,4 tons, i.e. around one third of the assumed cumulative production. Hence, sigmoid functions with an inflection point at one third of the saturation value should be most suitable for modelling the South African gold production. For instance, the Gompertz-function [12], which has its inflection point at 1/e (i.e. 36,8 % of the saturation value) should be usable for our purpose.

In the following paragraphs, six sigmoid functions were used to approximate the production curve. In conclusion in section 4, we will summarize and discuss the mathematical results.

3.1 Logistic Function

This function was originally developed by the Belgian mathematician Pierre-François Verhulst in 1838 [13]. It was intended to be used for the description of population growth and today also finds its applications in investment calculations. The function has the form

$$P(t) = \frac{P_{max}}{1 + b \cdot e^{-ct}}$$

where P(t) is the production at time t, P_{max} the maximum cumulative production and b and c parameters which have to be approximated by the software to fit the historic data points. The inflection point of this function is exactly 50 % of the saturation value P_{max}. Simply because of that, the logistic function should not be suitable to model the South African gold production in a meaningful way. Figures 5 and 6 show the calculated fits for the cumulative and discrete curves.

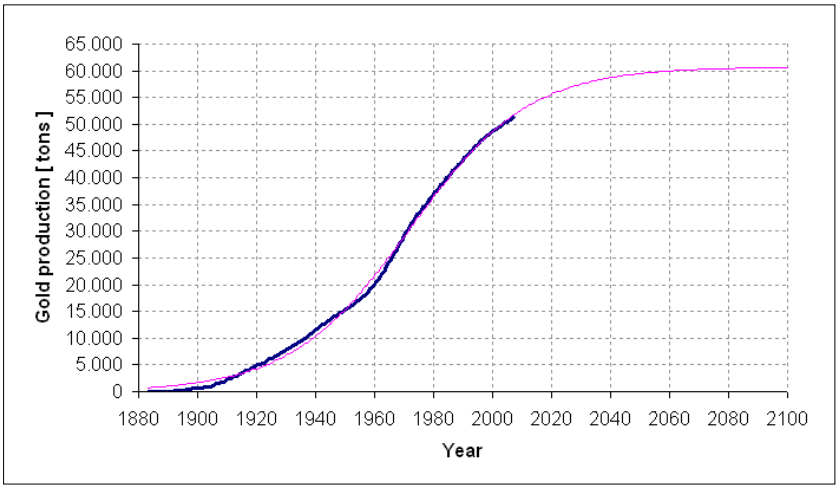

Fig. 5 Cumulative South African gold production (blue curve) with approximation by a logistic function (magenta curve)

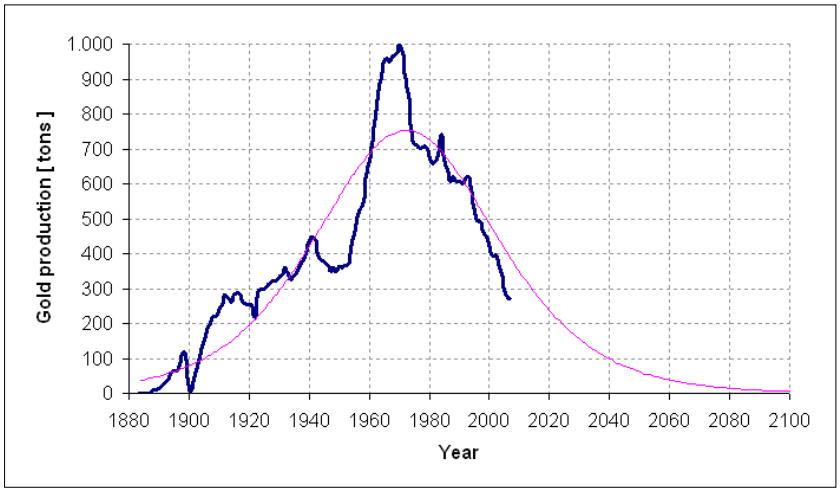

Fig. 6 South African gold production (blue curve) with approximation by a logistic function (magenta curve)

The model calculates the cumulative production in the amount of 60.717,9 tons (area under the magenta curve). The inflection point (maximum annual production) is in the year 1972 a value of 752,6 tons. In reality the point of peak production occurred in 1970 with a cumulative production of 29.135,4 tons. The logistic model calculates this value to be 30.455,4 tons. By the end of the 21th century, the gold production on an industrial scale should be nearly finished.

3.2 Gompertz function

The English mathematician Benjamin Gompertz formulated this function for the computation of population dynamics in 1825 [12].

$$P(t) = P_{max} \cdot e^{(-e^{(b-c\cdot t)})}$$

where P(t) is again the production at time t, P_{max} the maximum cumulative production and b and c parameters which have to be approximated. The inflection point is at 1/e of the saturation value (i.e. at approximately 36,8 %). As stated before, the Gompertz function should give a meaningful result because of the correct magnitude of the inflection point.

The following figures 7 and 8 show the approximations. The cumulative production is 95.256,1 tons, the inflection point is in the year 1980 at an annual production of 685,4 tons.

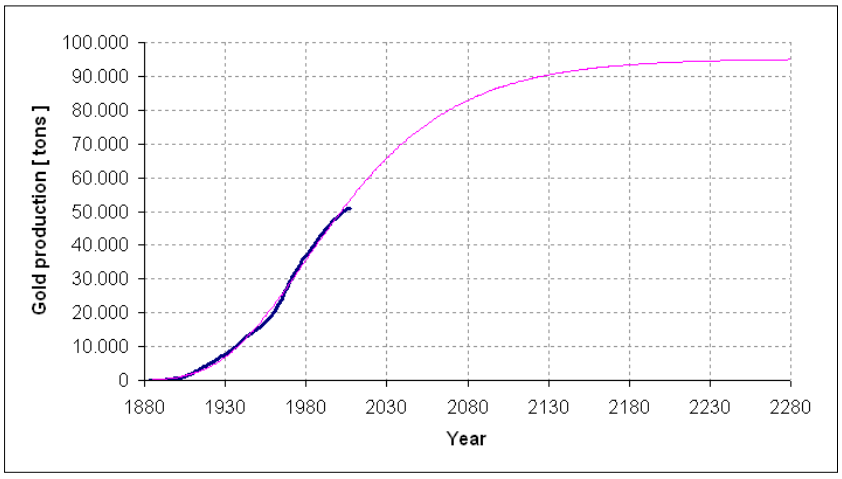

Fig. 7 Cumulative South African gold production (blue curve) with approximation by a Gompertz function (magenta curve)

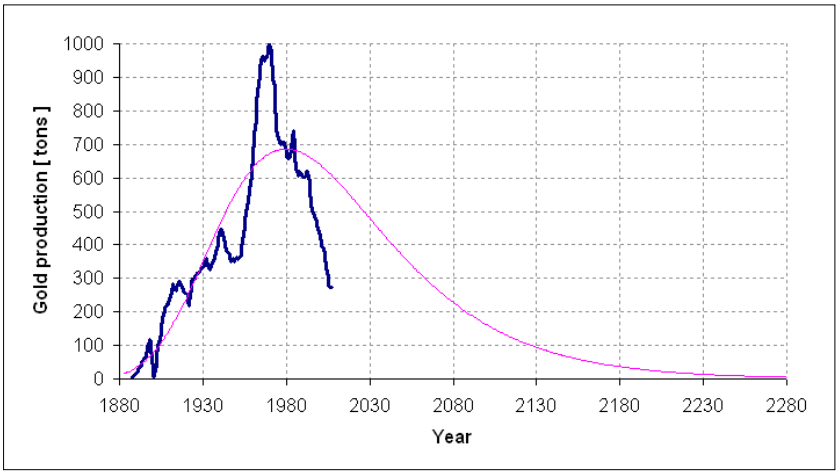

Fig. 8 South African gold production (blue curve) with approximation by a Gompertz function (magenta curve)

According to this model the gold production on an industrial scale should be possible even in the 22nd century. The real production peak according to the Gompertz model occurred 10 years earlier and was 314,7 tons higher.

3.3 Weibull-Function

The Swedish mathematician and engineer Waloddi Weibull formulated this statistical distribution to study quality management (e.g. life span of electronical devices or material fatigue) [14]. The 4-parameter Weibull-Function is

$$P(t) = P_{max} - a \cdot e^{(-e^{(b+c \cdot log(t))})}$$

where a, b and c are parameters which have to be approximated by the software.

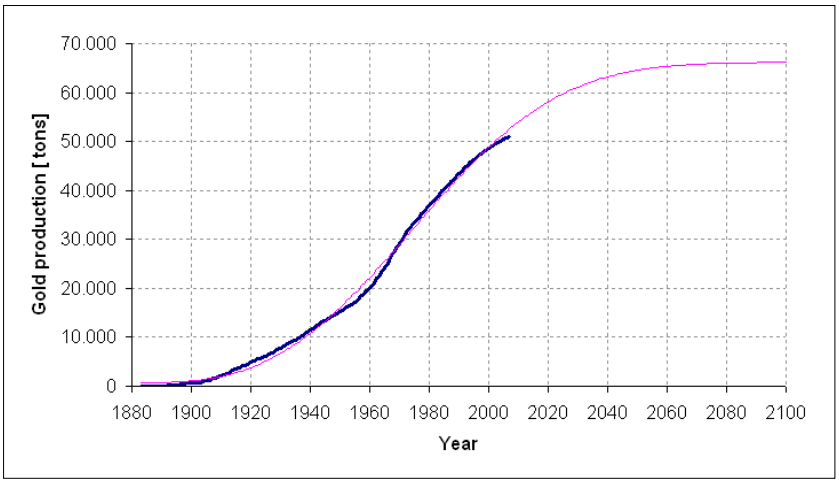

Fig. 9 Cumulative South African gold production (blue curve) with approximation by a Weibull function (magenta curve)

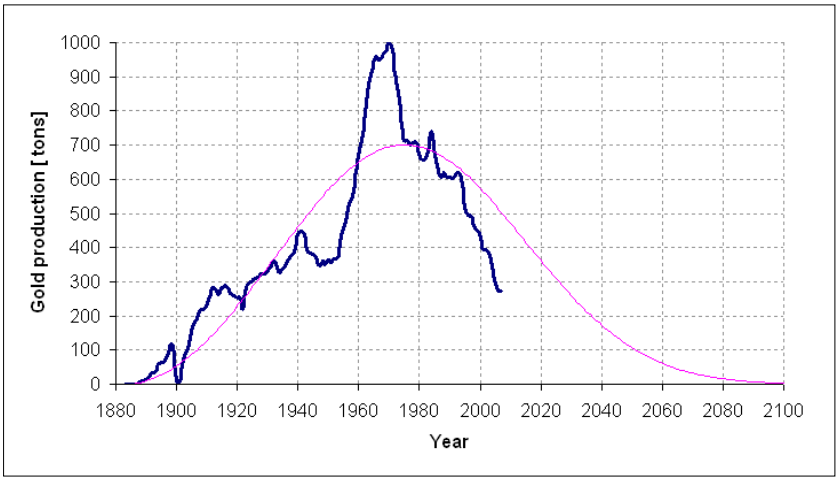

Fig. 10 South African gold production (blue curve) with approximation by a Weibull function (magenta curve)

The cumulative production is approximated by the Weibull function to 66.187,0 tons. The inflection point is at the level of the year 1975 with a discrete production of 700,5 tons in that year, which is 51,0 % of the overall production. Like with the logistic function the industrial gold production would come to an end by the end of this century.

3.4 Richards-Function

In 1959 the biologist F.J. Richards published an extension of the Bertalanffy-function, which was used for the empirical growth data of plants [15]. The 4-paramter formula is

$$P(t) = \frac{P_{max}}{[1 + a \cdot e^{(-b \cdot t)}]^{1/c}}$$

where a, b and c are again function parameters which have to be approximated by the software. The inflection point of this function is not constant, but depends on the parameter. With the Richards function it should therefore be possible to obtain a good mathematical fit of the real curve. However, due to the production decrease of the last couple of years since the end of the Apartheid, the cumulative production should be expected to be too low (similar to the logistic or Weibull function).

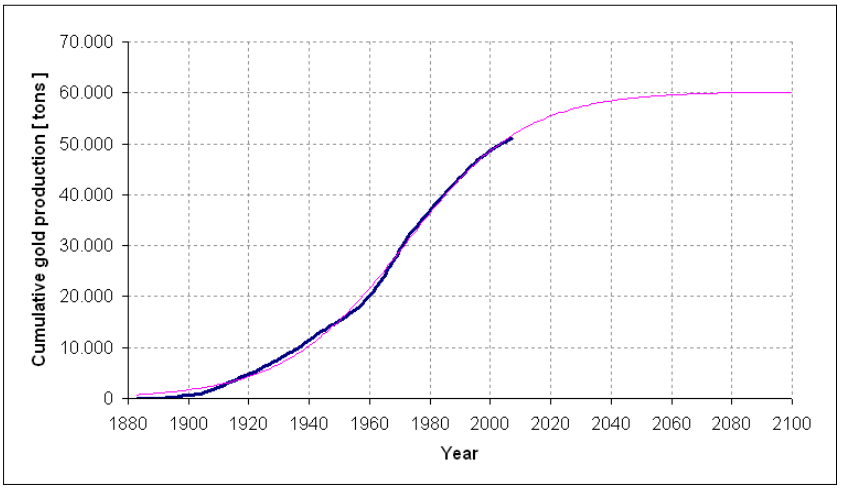

Fig. 11 Cumulative South African gold production (blue curve) with approximation by a Richards function (magenta curve)

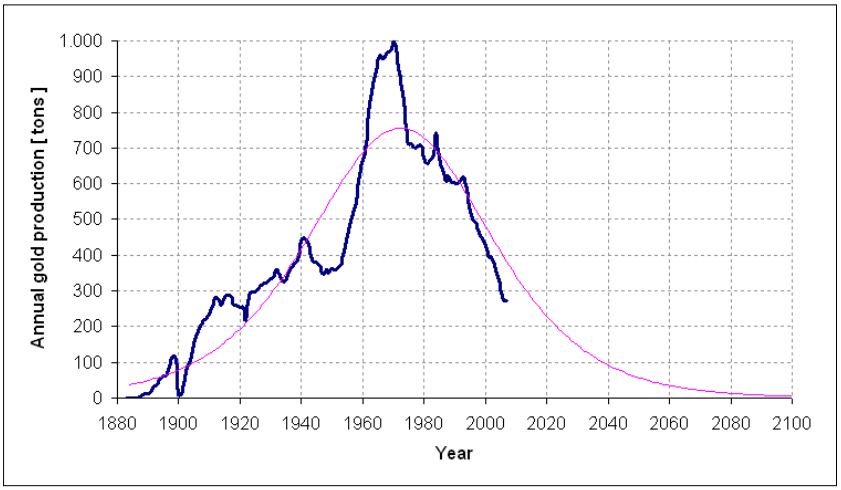

Fig. 12 South African gold production (blue curve) with approximation by a Richards function (magenta curve)

According to the Richards' model the production peak in the year 1972 is 755,5 tons. The cumulative production at that point in time is 30.459,8 tons. The maximum production is given to be 60.258,0 tons and hence in the same magnitude as in the Logistic and the Weibull model.

3.5 Johnson-Function

Contrary to the Logistic-, Weibull- or Richards functions, the Johnson function is approaching the saturation value very slowly. The inflection point occurs at a value of $P_{max} * e^{-2}$, i.e. already at 13,5 % of the saturation value. According to these mathematical reflections, the Johnson function should not yield a suitable model for the South African gold production. The function is given by

$$P(t) = P_{max} \cdot e^{-a/b+t}$$

where a and b are parameters of the function which have to be approximated by the software.

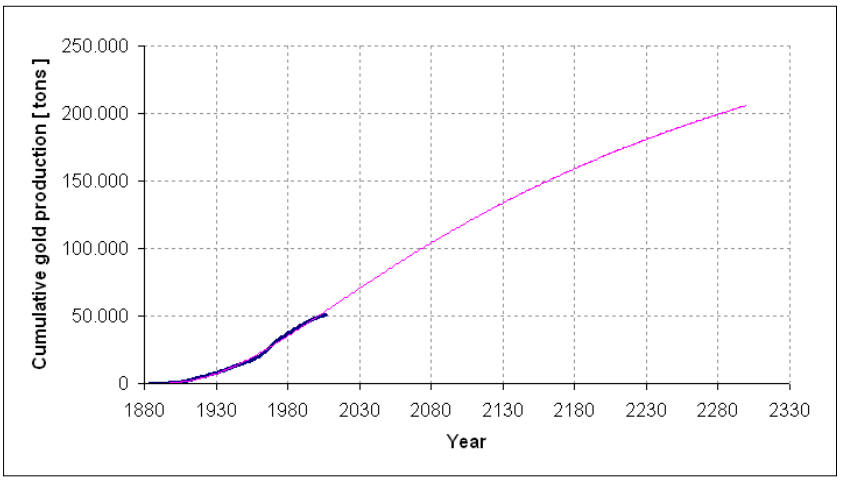

Fig. 13 Cumulative South African gold production (blue curve) with approximation by a Johnson function (magenta curve)

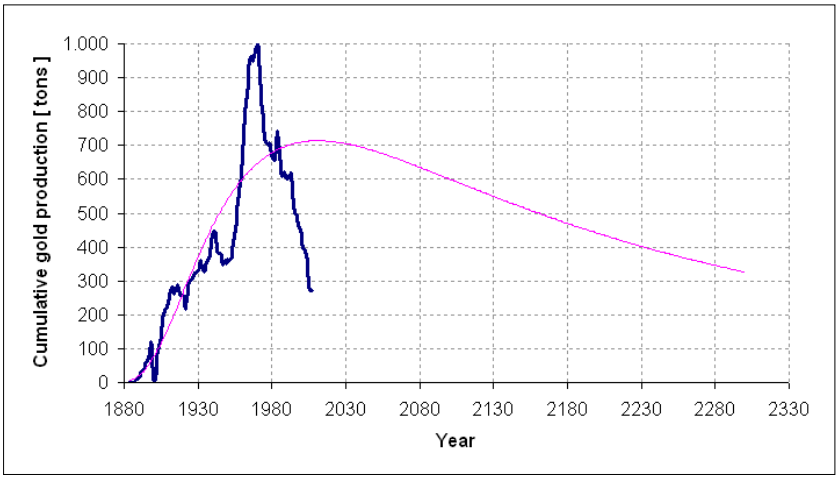

Fig. 14 South African gold production (blue curve) with approximation by a Richards function (magenta curve)

The Johnson model gives the peak production in the year 2011 in the amount of 713,8 tons. The saturation value is 418.299,6 tons. Hence, the Johnson function is a way out of reality and not suitable to give a meaningful model of the future production.

3.6. Gauss-Function

The Gaussian normal distribution is widely used in probability calculations. The density function is given by

$$P(t) = P_{max} \cdot e^{\left(-(t-t_0)^2 \,/\, 2 \cdot w^2 \right)}$$

where t_0 is the ordinate shift and w the standard deviation. If one approximates the annual South African gold production with the statistical software, the graph displayed in Figure 15 is obtained.

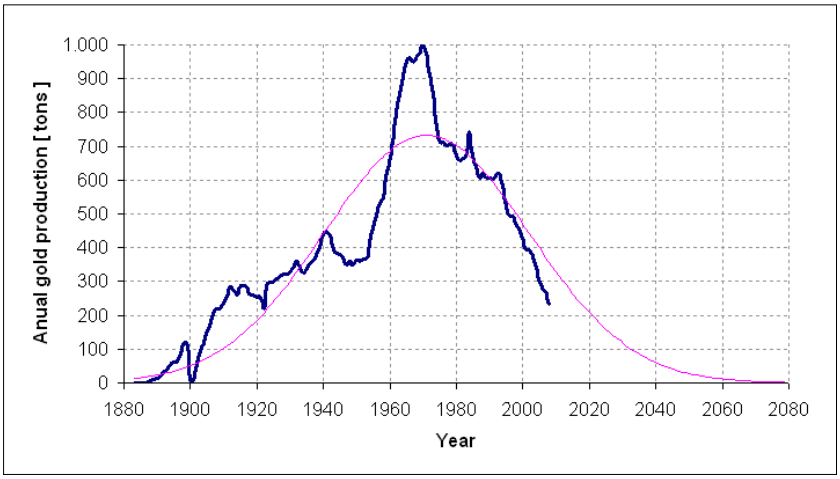

Fig. 15 South African gold production (blue curve) with approximation by a Gauss function (magenta curve)

The cumulative production according to the Gauss distribution is 56.437,5 tons, see Figure 16. The production peak is reached in 1971 in the amount of 731,7 tons.

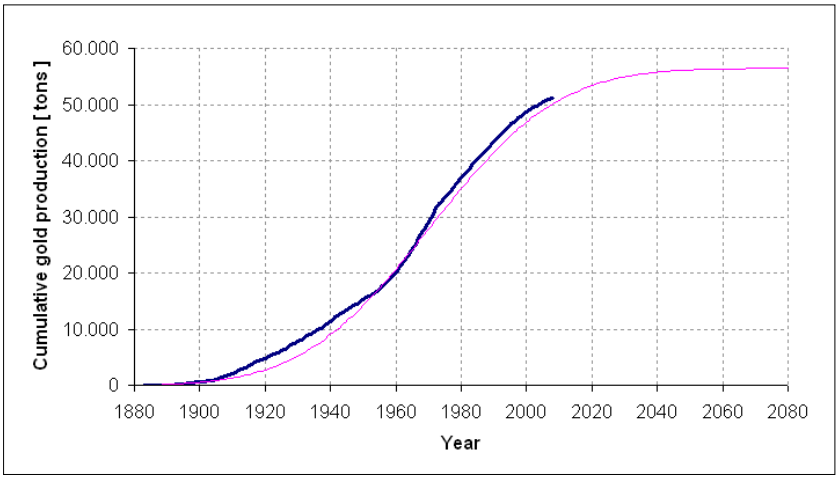

Fig. 16 Cumulative South African gold production (blue curve) with approximation by a Gauss function (magenta curve)

4 CONCLUSIONS

The following table lists the results which were approximated by these 6 functions, where:

$P_{cum,t->\infty}$	Maximum production at time to infinity
t_i	Year of the production peak (point of inflection)
P_i	Annual peak production at the point of inflection
$P_{cum,ti}$	Cumulative production up to t_i
R_a^2	Coefficient of determination of the approximated annual production curve
R_c^2	Coefficient of determination of the approximated cumulative production curve

The coefficients of determination were calculated using the free statistic software "R" version 2.6.0 [16].

Table 3: Results of the approximations

Function	**$P_{cum,t->\infty}$ [t]**	**t_i [a]**	**P_i [t]**	**$P_{cum,tw}$ [t]**	**R_a^2**	**R_c^2**
Logistic	60.718	1972	753	30.455	0,9234	0,9989
Gompertz	95.256	1980	685	35.576	0,8646	0,9984
Weibull	66.187	1975	701	32.427	0,8925	0,9986
Richards	60.258	1972	756	30.460	0,9261	0,9989
Johnson	418.300	2011	714	56.967	0,8247	0,9979
Gauss	56.438	1971	732	28.495	0,9146	0,9985

All functions which were used in this paper yielded in coefficients of determination greater than 0,997 when approximating the cumulative production curve. The annual production curves, however, showed some deviations in the value of R_a^2.

Those functions which give a lower overall cumulative production, also give a better mathematical correlation with the real production curve. The Gompertz function, which is the only model to give the cumulative production P_{cum} within a correct magnitude, indeed has got the lowest correlation (leaving the

Johnson model aside). However, this is only due to the large deviation of the model from 1994 onwards. This implies that the production in the last years fell more than it should have fallen due to geological reasons. The period 1883 – 1993 gives a correlation of the Gompertz model with reality of R_a^2 = 0,9101 which is in the magnitude of the other models.

Considering all 6 models, the arithmetically averaged peak annual production is 724 tons, which confirms our thesis that the real peak of 1000 tons in 1970 was a kind of artificial peak, being generated by the political and economical circumstances in Southern Africa of that time.

The Logistic-, Weibull-, Richards- and Gaus-Function which have the highest mathematical correlations with reality, likewise give the best correlation of the year of the annual peak production t_i. This high agreement, however, yields in a too low cumulative production $P_{cum,t->\infty}$. The named 4 functions give an arithmetical average for $P_{cum,t->\infty}$ of only 60.900 tons, which is in the magnitude of the historic production plus the currently known reserves (51.000 tons + 6.000 tons = 57.000 tons). The high South African resources, which have only low ore grades in hard granite rock and which lie in great depths, are not accounted for at all.

As already described in section 2, all mathematical models show the highest deviation from reality during the World War II and during the Apartheid boom. Furthermore, it is noticeable that the production since the end of the Apartheid regime has dropped more markedly than all models do indicate. Figure 17 shows the arithmetically averaged deviation of all 6 models compared to the real production curve.

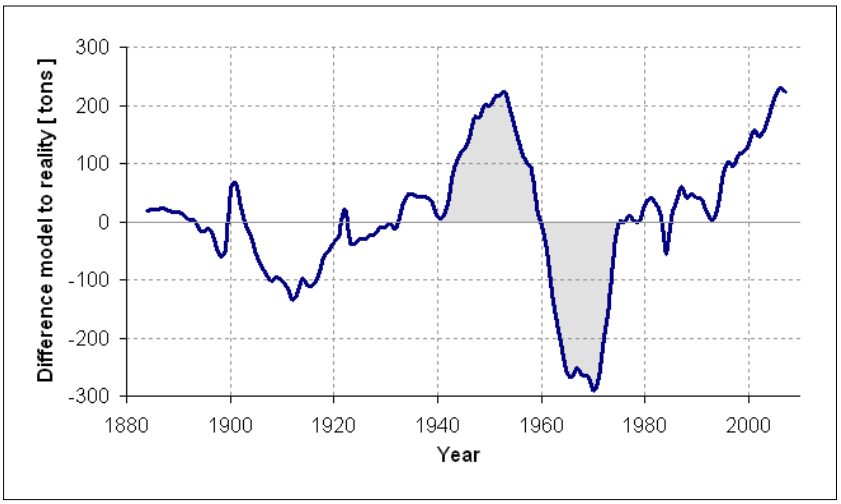

Fig. 17 Annual deviation of the models compared to the real production curve (arithmetically averaged over all 6 functions)

The summed up deviation of the years 1940 to 1959 (World War II and following years) gives a value of + 2.607 tons, i.e. our models gave a higher value than reality did. In the years of the Apartheid boom from 1960 to 1975, the real production was 2.833 tons higher than our models predicted (see grey areas in Fig. 17, respectively). Both values are within the same magnitude, which implies that these two historic effects should be mathematically compensated for. Hence, if one replaces the annual production of the years from 1940 to 1975 with the arithmetical average of those years of 621,6 tons, the correlation of this "historically corrected production curve" with e.g. our Gompertz model becomes clearer.

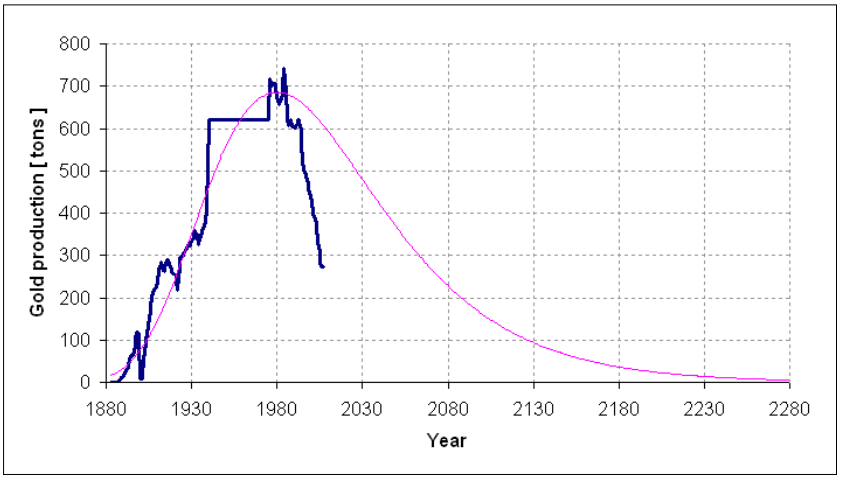

Fig. 18 Comparison of annual South African gold production withaveraged years 1940 – 1975 by the Gompertz model

The coefficients of determination R_a^2 between this historically compensated production curve and the Gompertz model rises from 0,8646 to 0,9251. If one accounts only the years up to 1993, R_a^2 rises further to a value of 0,9755. For the overall historic gold production up to the end of the Apartheid in 1994, the Gompertz model gives methodically and content-wise a very good approximation and/or a mathematical description.

The deviation since 1994 between this model and reality has two political motivated reasons beside the geological fact of decreasing ore grades. First the easing of currency exchange laws and secondly the taxation of South African mining companies. As described above, mining companies couldn't transfer their earnings out of the country. These laws were eased since the end of the Apartheid system so that e.g. the bank and brewery owner Anglo American again became a true mining company which left its roots in the South African gold mining industry more and more [7]. In other words: South African gold resources nowadays are in competition with Chilean copper ores or with coal projects in Australia. Where the largest profit can be expected, the money goes. The labour and capital intensive gold production in Southern Africa is currently in the sense of profit maximization of the invested capital not competitive, so that the very high resources with low ore grades currently are not mined at all.

The taxation of earnings is the other main factor for the declining gold production in Southern Africa in recent years. Mines which have a return on sales lower than 5%, are exempted from tax (known as the "tax tunnel") [17]. Due to this reason, South African gold mines are run near or only marginally above the "zero profit grade", simply to avoid taxation which increases superlinearly with rising profits [9].

Thus, the high South African gold resources will probably only be mined, when globally no other mining projects will be more profitable anymore. This can be the case either when all other more profitable projects on this planet will be finished or when the price of gold will substantially rise. Probably it will be a mixture of both.

Should the taxation of the gold mines remain the same as it is today, it is to be counted that the production of gold in Southern Africa will be at a constant level over many decades. However, this level of industrial scale will still be fixed near the limit of profitability.

From today's viewpoint a second production peak like the one in 1970 could only arise, if the basic economical conditions for mining companies are changed. The Gompertz model, which we favour, gave an annual production of 600 tons for 2007 and of around 500 tons for 2025. Such a second production peak could therefore lie within this range.

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