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APPLICATION OF INCIDENCE MATRIX IN CONDITIONAL COMPOSITION OF LEVELLING NETS

POUŽITÍ MATICE SOUSEDNOSTI PŘI PODMÍNKOVÉM VYROVNÁNÍ NIVELAČNÍ SÍTĚ

Abstract

The main part of this article is devoted to a modified method for calculating levelling nets based on nullifying conditional equations. This modified method consists in ordering the measured data in the so-called incidence matrix, and the main acceleration of the computation process is achieved by an automated set-up of conditional equations and subsequent calculations. A part of this article comprises a comparison of this modified method with a method where conditional equations are set up in a standard way.

Abstrakt

Hlavní část článku je věnována modifikaci výpočtu nivelační sítě anulováním podmínkových rovnic. Tato modifikace spočívá v seřazení měřených dat do tzv. matice sousednosti, hlavní zrychlení výpočetního procesu pak spočívá v automatizovaném sestavení podmínkových rovnic a následných výpočtů. Součástí článku je porovnání této modifikované metody se způsobem, kdy jsou podmínkové rovnice sestavovány běžným postupem.

Key words: least square method, levelling net, incidence matrix

1 INTRODUCTION

Levelling nets can be composed by way of the method of the least square, where we consider the measured quantities to be intermediate, or they can be composed if we nullify conditional equation closures. The conditional method is very convenient for levelling nets; however, its disadvantage lies in the manual set-up of suitable conditional equations. It is possible to solve this problem by ordering the measured quantities in the so-called incidence matrix from which we can obtain a conditional equation closure vector and a matrix of transformed conditional equations coefficient by means of applying simple mathematical relations. The next step leading to correction vector achievement constitutes only a routine matrix calculation which can be easily algorithmized by suitable computer program.

2 MODEL TASK SET-UP

For the purpose of simplification a model task has been set up, based on which composition will be performed in a standard way and subsequently by using an incidence matrix.

Model task

By technical composition we surveyed a levelling net (see fig. 1), which is formed by 6 levelling sections. Spot A is a fixed bench mark with a known height; the task is to determine the heights of spots 1, 2, 3 in the levelling net. In fig. no. 1 arrows show the direction of the rise. 3 levelling sections are sufficient for us to determine the height of spots 1, 2, 3, the other 3 sections are therefore redundant and 3 conditional equations will be set up for composition of the net. Before the composition itself it is necessary that the linear independency of the set-up equations is checked. For the purpose of simplification, individual levelling sections will be considered to be of approximately the same length; therefore, the measured weights will not be included in the composition.

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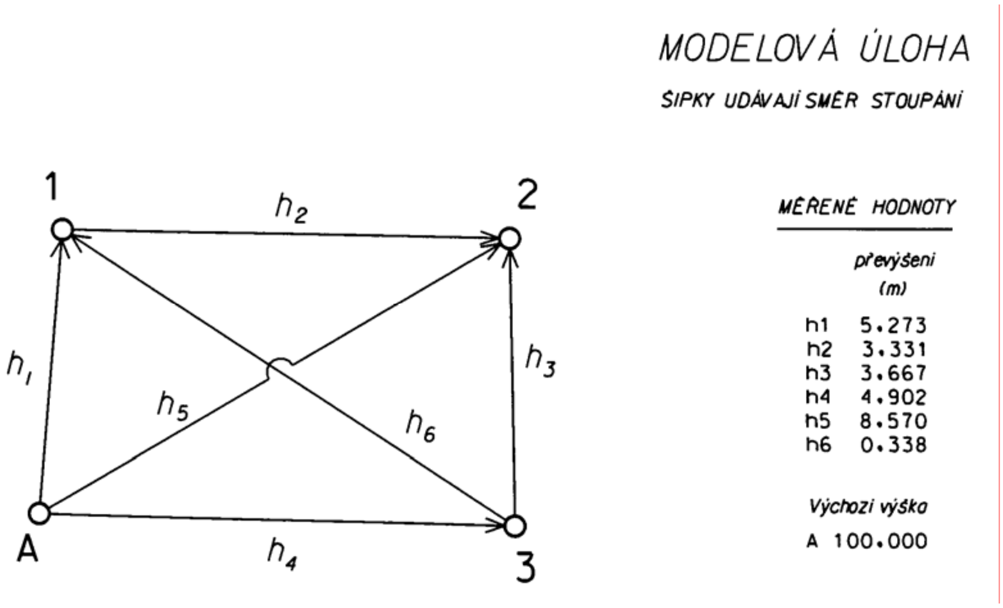

Fig. 1. Scheme of the surveyed levelling net (model task)

For composition of a levelling net the following symbols apply:

u	conditional equation closure vector
A	coefficient matrix of transformed conditional equations
s	incidence matrix (before composition)
S	incidence matrix (after composition)
v	correction vector
d_i	levelling section length
h_i	measured drop
H_i	height of the *i*-point

Standard solution of the task

E.g. these conditional equations can be set up for the levelling net (Fig. 1)

$$\begin{array}{cccccccc} +h_1 & +h_2 & -h_3 & -h_4 & & & = & 0 \\ +h_1 & & & -h_4 & & -h_6 & = & 0 \\ & & +h_3 & +h_4 & -h_5 & & = & 0 \end{array} \tag{1}$$

These conditions cannot be fulfilled if influenced by accidental errors. By substituting the measured quantities in the conditional equations it is we obtain the **u** closure vector, it is also possible for us to set up the **A** coefficient matrix of the transformed conditional equations.

$\mathbf{u} = (-0{,}001 \quad -0{,}003 \quad -0{,}001)^T$

$$\mathbf{A} = \begin{Vmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{Vmatrix}^T$$

We can calculate the **v** correction vector according to the relation [5]

$$\mathbf{v} = -\mathbf{A} \, .(\mathbf{A}^T . \, \mathbf{A})^{-1} . \, \mathbf{u} \qquad (2)$$

$\mathbf{v} = (0{,}0013 \;\; 0 \;\; 0{,}0008 \;\; -0{,}0005 \;\; -0{,}0008 \;\; -0{,}0013)^T$

Resulting heights of the spots to be determined

$H_1 = h_1 + v_1 = 105{,}238_2$ m

$H_2 = h_2 + v_2 = 108{,}569_2$ m

$H_3 = h_3 + v_3 = 104{,}901_5$ m

3 TASK SOLUTION OBTAINED BY ORDERING THE MEASURED QUANTITIES IN AN INCIDENCE MATRIX

Incidence matrix is a mathematical object used in many branches of science (graph theory, theory of electrical circuits [4]). In every branch work with this object is different; in geodesy the use of incidence matrix is completely new.

Incidence matrix **s** for a levelling net is build so that we write the individual bench marks in the headers of the rows and columns. Into the matrix **A**(*i,j*)we write the individual measured drops in the positions*(i,j)*, including their signs (according to rise/descent). Other positions are zero. For the calculation it is sufficient to fill out the upper part of the triangular matrix above the diagonal (marked blue in the matrix).

s =

i,j		*1*	*2*	*3*	*4*
		A	1	2	3
1	A	0	+5,237	+8,570	+4,902
2	1	0	0	+3,331	-0,338
3	2	0	0	0	-3,667
4	3	0	0	0	0

In matrix **s** we can always find a levelling line closed by the following relation:

$$-l_{i,j} + \sum_{i,i+1}^{j-1,j} l_{i,i+1}, \qquad l_{i,j} \neq 0, \qquad \min.(j) = i + 2 \qquad (3)$$

As closed levelling line needs to fulfil the condition of a zero drop, conditional equations can be simply obtained from the relation above (3)

$$-l_{i,j} + \sum_{i,i+1}^{j-1,j} l_{i,i+1} = 0 \qquad (4)$$

The principle of the relation (3) is the following: suitably defined algorithm starts its cycle from 1 do the no. of "dimension of matrix" position *i,j*, where *min.(j) = i+2*. In case, there is a number different from zero, its sign is changed and addition of elements defined by positions *i,i+1* and *j-1,j* in the row above the diagonal is added to this number.

For the purpose of our model task we consider the following positions

Position 1,3 = +8,570 –8,570 + (5,237 + 3,331)

Position 1,4 = +4,902 –4,902 + (5,237 + 3,331 – 3,667)

Position 2,4 = -0,338 +0,338 + (3,331 – 3,667)

It is obvious that the condition (4) need not be fulfilled owing to accidental errors. By applying the relation (4) we can extract individual conditional equations from the incidence matrix **S** together with the calculation of the closure vector **u** and the coefficient matrix of transformed conditional equations **A**.

If relation (2) is included in algorithm, we can obtain the correction vector **v** directly.

$$\mathbf{A} = \begin{Vmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{Vmatrix}^T$$

$\mathbf{u} = (-0{,}002\ \ -0{,}001\ \ 0{,}002)^T$

$\mathbf{v} = (\ 0{,}0013\ \ 0\ \ 0{,}0008\ \ -0{,}0007\ \ -0{,}0005\ \ -0{,}0013)^T$

Resulting heights of the spots to be determined

$H_1 = h_1 + v_1 = 105{,}238_3$ m

$H_2 = h_2 + v_2 = 108{,}569_3$ m

$H_3 = h_3 + v_3 = 104{,}901_5$ m

Incidence matrix after composition:

S =

i,j		*1*	*2*	*3*	*4*
		A	1	2	3
1	A	0	+5,2383	+8,5693	+4,9015
2	1	0	0	+3,3310	-0,3367
3	2	0	0	0	-3,6677
4	3	0	0	0	0

In matrix **S** conditions (4) have been fulfilled, which we can verify easily by a repeated launch of the program (after substitution **s** = **S**). If the closure vector is

$$\mathbf{v} = (0\ \ 0\ \ 0)^T \tag{5}$$

then composition of the net has been performed correctly.

Composition of a net by way of using an incidence matrix is convenient especially if the net forms a closed diagram in the circuit (Fig. 2). If we put a levelling net among several known bench marks, we would also be able to solve the task this way – by adding additional conditions (Fig. 2b).

It may happen that we will not be able to work with levelling nets this way at all or the solution will be very hard to find. This case will happen if some of the elements $\mathbf{s}_{i,i+2}$ of the incidence matrix is zero (Fig. 2c)

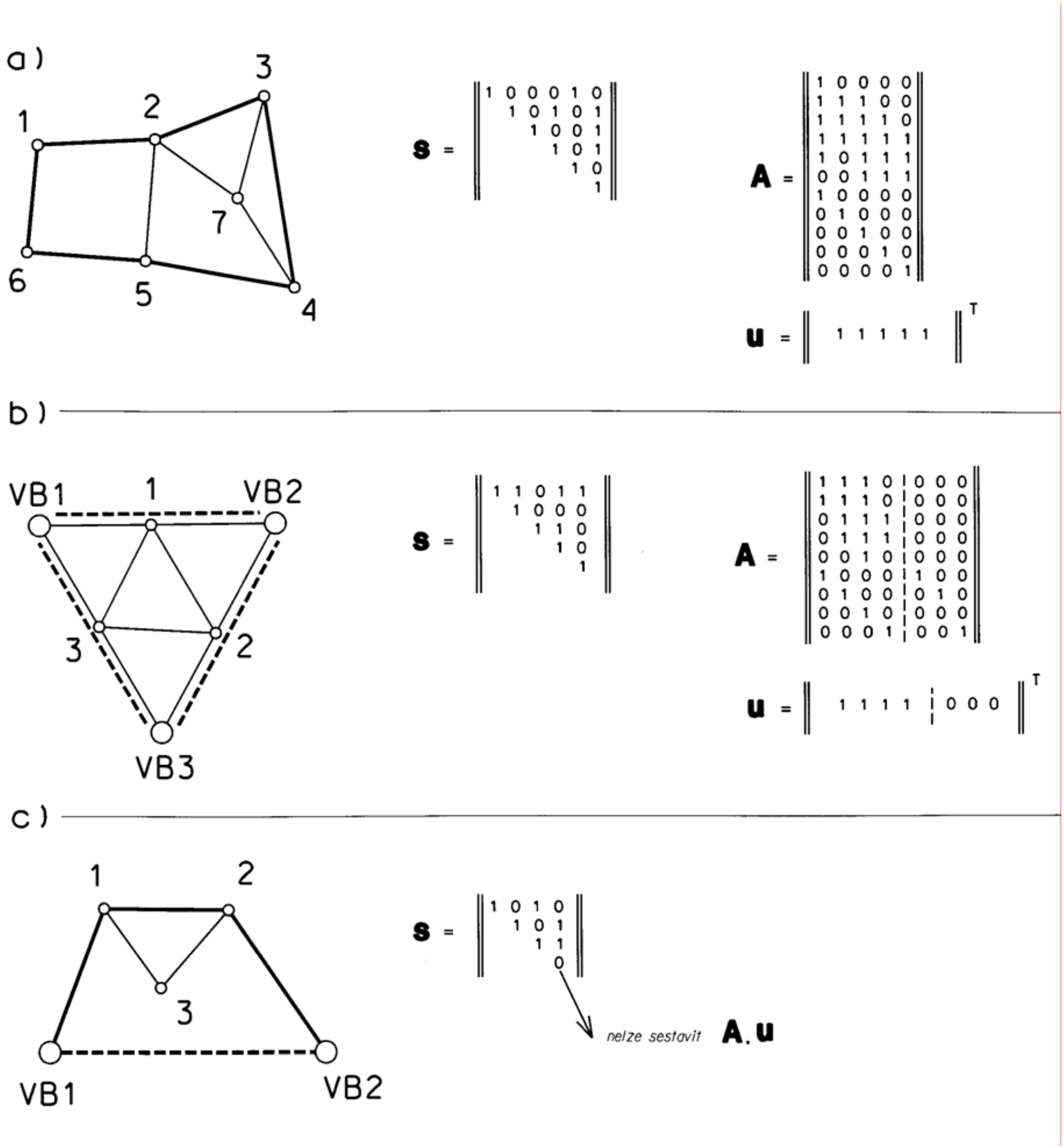

Fig. 2. Topology of levelling nets a) loose, b) inserted among 3 given bench marks, c) Unsuitable for the incidence matrix solution

4 EXAMPLE OF INCIDENCE MATRIX APPLIED TO A REAL LEVELLING NET

Subject matter which has been briefly explained by means of the trivial model task we can now apply to a real levelling net. As source data we used the accomplished altitude survey of an Observation Station for monitoring shifts and transformation of a mine tailings dam of Graphite Mines in Staré Město p/Sn. Phase 2001), elaborated in [3]. The surveyed altitudinal net is drawn in Fig. 3.

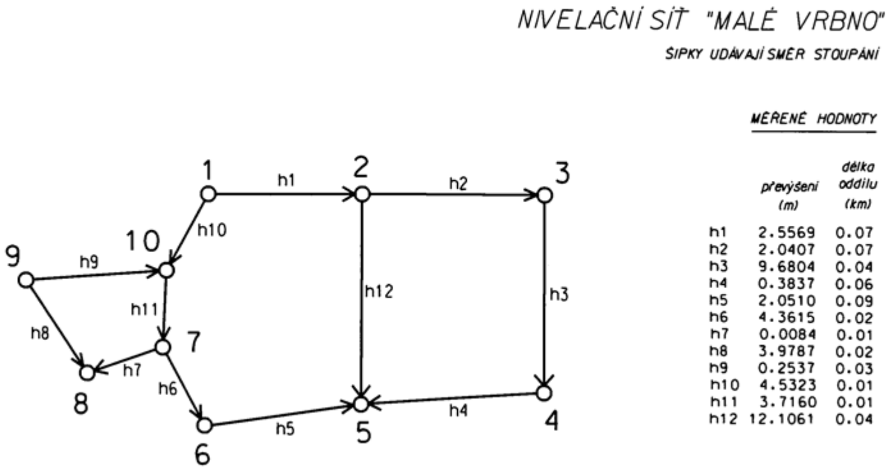

Fig. 3. Computation plan for the "Malé Vrbno" levelling net

For the levelling net (Fig. 3) we can build the following matrices in a standard way:

$$\mathbf{A} = \begin{Vmatrix} 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{Vmatrix}^{T}$$

$$\mathbf{u} = \begin{Vmatrix} 0,3 & -0,6 & -1,3 \end{Vmatrix}^{T}$$

$$\mathbf{P} = \text{diag.} \begin{Vmatrix} 0,1 & 0,1 & 0,3 & 0,2 & 0,1 & 0,5 & 1,0 & 1,0 & 0,3 & 0,3 & 0,5 & 1,0 \end{Vmatrix}$$

Calculation of corrections in case we build a weight matrix **P** is given by relation

$$\mathbf{v} = -\mathbf{P}^{-1} . \mathbf{A} . (\mathbf{A}^{T} . \mathbf{P}^{-1} . \mathbf{A})^{-1} . \mathbf{u} \tag{6}$$

or by means of a more transparent co-factor matrix **Q**

$$\mathbf{Q} = \mathbf{P}^{-1} \tag{7}$$

$$\mathbf{v} = -\mathbf{Q} . \mathbf{A} . (\mathbf{A}^{T} . \mathbf{Q} . \mathbf{A})^{-1} . \mathbf{u} \tag{8}$$

Based on the calculated corrections we can set inverse accuracy characteristics, i.e. unit standard deviation [5]

$$\sigma_0 = \pm\sqrt{[(\mathbf{v}^{T} . \mathbf{P} . \mathbf{v})/(n-1)]} \tag{9}$$

As we cannot consider the unit standard deviation σ_0 (9) to be characteristic of composition accuracy (it represents the accuracy of an auxiliary set of homogeneous quantities) [1], we set the accuracy of the individual drops h_i

$$\sigma_{hi} = \sigma_0.\sqrt{\mathbf{Q}_{ii}} \tag{10}$$

Diagonal elements of the matrix **P** form reverse quantities of levelling sections lengths in km.

Although weights of the measured homogeneous quantities have their own dimension, we can regard them as relative numbers and we can work with the dimension a bit more freely [1]. For the weights of levelling measuring we can write down [6]

$$p_1 : p_2 : \dots : p_n = c/d_1 : c/d_2 : \dots : c/dn \tag{11}$$

where c is a suitably selected constant.

In the example solved we set the weights with the modification

$$\mathbf{P}(i,i) = c\ .(d_i)^{-1}, c = 10^{-2} \tag{12}$$

Thus the weights remain in the following interval $\langle 0; 1 \rangle$.

By applying relation (8) we calculate the correction vector **v**

$$\mathbf{v} = \begin{Vmatrix} -0{,}72 & 0{,}49 & 0{,}16 & 0{,}24 & 0{,}72 & 0{,}14 & 0{,}07 & -0{,}14 & 0{,}24 & 0{,}07 & 0{,}14 & 0{,}40 \end{Vmatrix}^{T}$$

By substituting the composed drops in the conditional equations controls are met exactly.

We can also solve the same task by ordering the surveyed data in an incidence matrix

$$\mathbf{S} = \begin{Vmatrix}
0 & 2{,}5569 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4{,}5323 \\
0 & 0 & 2{,}0407 & 0 & 12{,}1061 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 9{,}6804 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0{,}3837 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2{,}0510 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -4{,}3615 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0{,}0084 & 0 & -3{,}7160 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3{,}9787 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0{,}2537 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{Vmatrix}$$

An incidence matrix built this way is by applying relation (4) coming to the calculation of corrections through matrices (this step is done by algorithm).

$$\mathbf{A} = \begin{Vmatrix}
1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1
\end{Vmatrix}^{T}$$

$$\mathbf{u} = \begin{Vmatrix} 0{,}3 & -1{,}3 & -0{,}6 \end{Vmatrix}^{T}$$

P = diag. $\left\| 0{,}1 \quad 0{,}1 \quad 0{,}3 \quad 0{,}2 \quad 0{,}1 \quad 0{,}5 \quad 1{,}0 \quad 0{,}5 \quad 0{,}3 \quad 1{,}0 \quad 0{,}3 \quad 1{,}0 \right\|$

v = $\left\| -0{,}72 \quad 0{,}49 \quad 0{,}16 \quad 0{,}24 \quad 0{,}72 \quad 0{,}14 \quad 0{,}07 \quad -0{,}14 \quad 0{,}24 \quad 0{,}07 \quad -0{,}40 \quad 0{,}04 \right\|^T$

According to relation (9) and (10) it is then possible to define characteristics of inverse accuracy (mm)

σ_{hi} = $\left\| 0{,}5 \quad 0{,}5 \quad 0{,}3 \quad 0{,}3 \quad 0{,}5 \quad 0{,}2 \quad 0{,}2 \quad 0{,}2 \quad 0{,}3 \quad 0{,}2 \quad 0{,}2 \quad 0{,}3 \right\|^T$

5 COMPARISON OF BOTH THE METHODS FOR CORRECTION CALCULATION

Before we compare the results obtained by comparing both the methods of composition, we can carry out the following reasoning. When using the standard method for composition, mathematicians set up conditional equations according to their own opinion (providing there are enough redundant observations carried out and thus there are more possibilities how conditional equations could be set up). The result will then be, to a certain extent, dependent on how the coefficient matrix A of transformed conditional equations currently under construction will be conditioned (for more on the conditionality of matrices see [4]).

The task solved enables us to verify through an experiment which results we can achieve if we set up conditional sentences in various ways. For the individual variants qualitative evaluation of conditionality will be rated by a conditionality number – Cond(A) and by a matrix norm – Norm(A^T**.** A) – for more on this see [4]. The higher the values of the numbers Cond(A) and Norm(A^T**.** A) are, the better the task is conditioned. Other possible variants of matrix **A** are listed in matrices **A′** - **A′′′**.

Chart 1: Matrix **A** conditionality influence on values of the resulting elevations.

Cond	2,6002	2,6002	2,5543	2,0769	1,4353
Norm	12,1962	12,1962	12,3366	9,7417	6,7321
elevation	**s -> A***	**A**	**A′**	**A′′**	**A′′′**
	(m)	(m)	(m)	(m)	(m)
(1)	(2)	(3)	(4)	(5)	(6)
h1	2,5562	2,5562	2,5562	2,5562	2,5561
h2	2,0412	2,0412	2,0412	2,0412	2,0412
h3	9,6806	9,6806	9,6806	9,6806	9,6806
h4	0,3839	0,3839	0,3839	0,3839	0,3839
h5	2,0517	2,0517	2,0517	2,0517	2,0518
h6	4,3616	4,3616	4,3616	4,3616	4,3617
h7	0,0085	0,0085	0,0085	0,0085	0,0083
h8	3,9786	3,9786	3,9786	3,9786	3,9789
h9	0,2539	0,2539	0,2539	0,2539	0,2534
h10	4,5324	4,5324	4,5324	4,5324	4,5324
h11	3,7161	3,7161	3,7161	3,7161	3,7160
h12	12,1057	12,1057	12,1057	12,1057	12,1057

*) matrix A calculated from incidence matrix s

$$\mathbf{A}' = \left\| \begin{matrix} 1 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \end{matrix} \right\|^{T}$$

$$\mathbf{A}'' = \left\| \begin{matrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \end{matrix} \right\|^{T}$$

$$\mathbf{A}''' = \left\| \begin{matrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \end{matrix} \right\|^{T}$$

From chart 1 it is obvious that both the computation methods lead to the same results. (col. 2, 3).

After comparing both the computation methods it is clear that if we order measured quantities in an incidence matrix, several computation steps for composition take place at the same time, which significantly speeds up the whole computation process. Composition of a levelling net by this modified method is also easier from the point of the user as the only step which depends on the mathematician is the correct set-up of incidence matrix.

There is yet another advantage of this method. The set-up conditional equations are always linear-independent and the numerical stability of the matrix solution given mainly by the number of matrix conditionality **A** is the most suitable one among the possible matrix solutions (see Chart 1, heading marked in green colour).

The fact that the success of the whole composition process depends on the set-up of the matrix **A** is obvious from the comparison of results in columns (2)-(5) and in column (6). While the values of the resulting drops in columns (2)-(5) are in accordance, the resulting values in column (6) were calculated by not entirely correct set-up of conditional equations, which has shown in the stability of the solution of the whole set (the solution has the smallest number of conditionality from all the set-up calculation variants).

6 CONCLUSION

In this article we aimed to refer to other possibilities for setting up conditional equations, especially in case when composition is done by means of a matrix calculation, which is, owing to advanced computer technology, more transparent and faster than composition with the aid of correlative equations. The modified method of composition which is presented in the article can be applied in a more general way than just for composition of levelling nets, and it can be used to advantage for such surveying where there is no need for linearization of conditional equations, i.e. e.g. for depth surveying, base line measurements etc.

Application of incidence matrix for composition of land surveying using the conditional method represents an original idea e.g. for creation of a computer application that is based on a new, innovative method of entering input data achieved by surveying in the ground space and on simple communication of the operator and the computer program.

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RESUMÉ

Článek pojednává o výpočtu nivelačních sítí s uplatněním metody nejmenších čtverců způsobem, kdy výšky v uzavřených nivelačních obrazcích musí být rovny nule (podmínkové vyrovnání). V takovém případě se podmínkové rovnice sestavují ručně a další výpočet probíhá dosazením měřených hodnot do těchto podmínkových rovnic. U rozsáhlých nivelačních sítí tento postup může být zdlouhavý.

Měřené veličiny je ovšem možné sestavit do tzv. matice sousednosti (incidenční matice), čímž se docílí toho, že celý výpočetní postup je možné provést automatizovaně. Princip této metody spočívá v tom, že v matici sousednosti je možné odvodit jednoduchý matematický vztah, pomocí kterého lze vypočítat uzávěry podmínkových rovnic, které vlivem náhodných chyb obvykle nejsou rovny nule. V případě rozsáhlých nivelačních sítí může tato modifikace výpočtu znamenat velký přínos.

Součástí článku je sestavení experimentální úlohy – nivelační sítě, kterou je třeba vyrovnat. Vyrovnání je provedeno dosavadním výpočetním postupem a postupem modifikovaným – s využitím matice sousednosti.

V závěru článku je zřejmé, že oba výpočetní postupy dávají stejné výsledky vypočtených oprav pro jednotlivá převýšení. Správnost modifikovaného postupu je experimentální úlohou potvrzena, modifikovaný postup je úlohou podstatně rychlejší, neboť celou výpočetní část je možné sestavit do algoritmu.

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