On Super $(a, 1)$-Edge-Antimagic Total Labelings of Regular Graphs

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Abstract

A *labeling* of a graph is a mapping that carries some set of graph elements into numbers (usually positive integers). An (a, d)-*edge-antimagic total labeling* of a graph with p vertices and q edges is a one-to-one mapping that takes the vertices and edges onto the integers $1, 2, \dots, p + q$, so that the sums of the label on the edges and the labels of their end vertices form an arithmetic progression starting at a and having difference d. Such a labeling is called *super* if the p smallest possible labels appear at the vertices.

In this paper we prove that every even regular graph and every odd regular graph with a 1-factor are super $(a, 1)$-edge-antimagic total. We also introduce some constructions of non-regular super $(a, 1)$-edge-antimagic total graphs.

Keywords: super edge-antimagic total labeling, regular graph

1 Introduction

We consider finite undirected graphs without loops and multiple edges. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. Let a, b, be two integers where $a < b$. By $[a, b]$ we denote the set of consecutive integers $\{a, a + 1, \dots, b\}$.

For a (p, q)-graph G with p vertices and q edges, a bijective mapping $f : V(G) \cup E(G) \to [1, p + q]$ is a *total labeling* of G and the associated *edge-weights* are $w_f(uv) = f(u) + f(uv) + f(v)$ for every $uv \in E(G)$.

An (a, d)-*edge-antimagic total labeling* ((a, d)-*EAT* for short) of G is the total labeling with the property that the edge-weights form an arithmetic progression starting from a and with difference d, where $a > 0$ and $d \geq 0$ are two given integers. Definition of an (a, d)-EAT labeling was introduced by Simanjuntak, Bertault and Miller in [11] as a natural extension of the *magic valuation*, which is also known as the *edge-magic labeling* defined by Kotzig and Rosa in [9]. Kotzig and Rosa [9] showed that all caterpillars have magic valuations and conjectured that all trees have magic valuations. An (a, d)-EAT labeling is called *super* if the smallest possible labels appear on the vertices. For more information on edge-magic and super edge-magic labelings, please see [8] and [13].

The (a, d)-EAT and super (a, d)-EAT labelings are two among several other "magic-type" labelings. Often results on one or a different type of magic-type labelings can be adapted or combined to obtain results on a different type. This idea has been studied by Figueroa-Centeno, Ichishima, and Muntaner-Batle [7]. In this paper, we will study a set of problems which are similar to the problems studied in [10] for vertex-magic total labelings. For an exhaustive survey on various magic-type labelings we again recommend [8].

A graph that admits an (a, d)-EAT labeling or a super (a, d)-EAT labeling is called an (a, d)-EAT graph or a super (a, d)-EAT graph, respectively. Sugeng et al. in [12] described how to construct super (a, d)-EAT labelings of all caterpillars for $d = 0, 1, 2$ and of certain caterpillars for $d = 3$. In [2] some constructions of super (a, d)-EAT labelings for disconnected graphs are presented using the notion of an α-labeling. Bača et al. [4] also studied super (a, d)-EAT labelings of path-like trees. Some other results on (a, d)-EAT graphs are presented in [1] and [6].

Let (p, q)-graph be a super (a, d)-EAT graph. It is easy to see that the minimum possible edge-weight is at least $p+4$ and the maximum possible edge-weight is not more than $3p + q - 1$. Thus

$$a + (q - 1)d \leq 3p + q - 1 \quad \text{and} \quad d \leq \frac{2p + q - 5}{q - 1} \ .$$

For any (p, q)-graph, where $p - 1 \leq q$, it follows that $d \leq 3$. In particular if G is connected then $d \leq 3$.

In this paper we deal with the existence of super $(a, 1)$-EAT labelings of regular graphs. We also give some constructions of non-regular super $(a, 1)$-EAT graphs.

2 Super $(a, 1)$-EAT labeling of regular graphs

Results in this and the following sections are based on the Petersen Theorem.

Proposition 2.1. (Petersen Theorem) *Let G be a $2r$-regular graph. Then there exists a 2-factor in G.*

Notice that after removing edges of the 2-factor guaranteed by the Petersen Theorem we have again an even regular graph. Thus, by induction, an even regular graph has a 2-factorization.

The construction in the following theorem allows to find a super $(a, 1)$-EAT labeling of any even regular graph. Notice that the construction does not require the graph to be connected.

Theorem 2.2. *Let G be a graph on p vertices that can be decomposed into two factors G_1 and G_2. If G_1 is edge-empty or if G_1 is a super $(2p + 2, 1)$-EAT graph and G_2 is a $2r$-regular graph then G is super $(2p + 2, 1)$-EAT.*

Proof. First we start with the case when G_1 is not edge-empty. Since G_1 is a super $(2p+2,1)$-EAT graph with p vertices and q edges, there exists a total labeling $f: V(G_1) \cup E(G_1) \to [1, p+q]$ such that

$$\{f(v) + f(uv) + f(v) \colon uv \in E(G)\} = [2p+2, 2p+q+1] \,.$$

By the Petersen Theorem there exists a 2-factorization of G_2. We denote the 2-factors by F_j, $j = 1, 2, \ldots, r$. Let $V(G) = V(G_1) = V(F_j)$ for all j and $E(G) = \cup_{j=1}^{r} E(F_j) \cup E(G_1)$. Each factor F_j is a collection of cycles. We order and orient the cycles arbitrarily. Now by the symbol $e_j^{out}(v_i)$ we denote the unique outgoing arc from the vertex v_i in the factor F_j.

We define a total labeling g of G in the following way.

$$\begin{aligned} g(v) &= f(v) \quad \text{for } v \in V(G), \\ g(e) &= \begin{cases} f(e) & \text{for } e \in E(G_1), \\ q + (j+1)p + 1 - f(v_i) & \text{for } e = e_j^{out}(v_i). \end{cases} \end{aligned}$$

The vertices are labeled by the first p integers, the edges of G_1 by the next q labels and the edges of G_2 by consecutive integers starting at $p+q+1$. Thus g is a bijection $V(G) \cup E(G) \to [1, p+q+pr]$ since $|E(G)| = q + pr$.

It is not difficult to verify, that g is a super $(2p+2,1)$-EAT labeling of G. For the weights of the edges e in $E(G_1)$ is $w_g(e) = w_f(e)$. The weights form the progression $2p+2, 2p+3, \ldots, 2p+q+1$. For convenience we denote by v_k the unique vertex such that $v_i v_k = e_j^{out}(v_i)$ in F_j. The weights of the edges in F_j, $j = 1, 2, \ldots, r$ are

$$w_g(e_j^{out}(v_i)) = w_g(v_i v_k) = g(v_i) + (q + (j+1)p + 1 - f(v_i)) + g(v_k)$$

$$= f(v_i) + q + (j+1)p + 1 - f(v_i) + f(v_k) = q + (j+1)p + 1 + f(v_k)$$

for all $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r$. Since F_j is a factor, the set $\{f(v_k) \colon v_k \in F_j\} = [1, p]$. Hence we have that the set of the edge-weights in the factor F_j is $[q + (j+1)p + 2, q + (j+1)p + p + 1]$ and thus the set of all edge-weights in G is $[2p+2, q + (r+2)p + 1]$.

If G_1 is edge-empty it is enough to take $q = 0$ and proceed with the labeling of factors F_j. □

By taking an edge-empty graph G_1 we have the following theorem (we prefer call it a theorem though it is just a corollary of Theorem 2.2).

Theorem 2.3. *All even-regular graphs of order p with at least one edge are super $(2p+2,1)$-EAT.*

The construction from Theorem 2.2 can be extended also to the case when G_1 is not a factor. One can add isolated vertices to a graph and keep the property of being super $(a,1)$-EAT. A graph consisting of m isolated vertices is denoted by mK_1. We can obtain the following lemma.

Lemma 2.4. *If G is a super $(a,1)$-EAT graph then also $G \cup mK_1$ is a super $(a+m+2t,1)$-EAT graph for all $t \in [0,m]$.*

Proof. Since G is a super $(a, 1)$-EAT graph with p vertices and q edges, there exists such a total labeling $f : V(G) \cup E(G) \to [1, p + q]$ that

$$\{f(v) + f(uv) + f(v) \colon uv \in E(G)\} = [a, a + q - 1] .$$

Let t be any fixed integer from $[0, m]$. Let $(c_1, c_2, \ldots, c_m)$ be any permutation of the integers in $[1, p + m] \setminus [t + 1, t + p]$. We denote the vertices of mK_1 by $v_{c_1}, v_{c_2}, \ldots, v_{c_m}$ arbitrarily. Now we define a labeling g of the graph $H = G \cup mK_1$.

$$\begin{aligned} g(v) &= \begin{cases} f(v) + t & \text{for } v \in V(G), \\ i & \text{for } v = v_i, \text{ where } v_i \in mK_1, \end{cases} \\ g(e) &= f(e) + m \quad \text{for } e \in E(H). \end{aligned}$$

Obviously g is a bijection $V(H) \cup E(H) \to [1, p + q + m]$. The edges are labeled by the q highest labels and the vertices by the first $p + m$ integers. It is easy to verify that g is super $(a + m + 2t, 1)$-EAT labeling of H, since any edge $uv \in E(H)$ is also in $E(G)$.

$$\begin{aligned} w_g(uv) &= g(u) + g(uv) + g(v) \\ &= (f(u) + t) + (f(uv) + m) + (f(v) + t) = w_f(uv) + m + 2t \end{aligned}$$

and the claim follows. □

Notice that we can find $m+1$ different (up to isomorphism) super $(b, 1)$-EAT labelings of $G \cup mK_1$ but all with the same parity of the smallest edge-weight.

In the last part of this section we show that also all odd-regular graphs with a perfect matching are super $(a, 1)$-EAT. By P_n we denote the path on n vertices.

Lemma 2.5. *Let k, m be positive integers. Then the graph $kP_2 \cup mK_1$ is super $(2(2k + m) + 2, 1)$-EAT.*

Proof. We denote the vertices of the graph $G \cong kP_2 \cup mK_1$ by the symbols $v_1, v_2, \ldots, v_{2k+m}$ in such a way that $E(G) = \{v_i v_{k+m+i} \colon i = 1, 2, \ldots, k\}$ and the remaining vertices are denoted arbitrarily by the unused symbols.

We define the labeling $f : V(G) \cup E(G) \to [1, 3k + m]$ in the following way

$$\begin{aligned} f(v_j) &= j && \text{for } j = 1, 2, \ldots, 2k + m, \\ f(v_i v_{k+m+i}) &= 3k + m + 1 - i && \text{for } i = 1, 2, \ldots, k. \end{aligned}$$

It is easy to see that f is a bijection and that the vertices of G are labeled by the smallest possible numbers. For the edge-weights we get

$$\begin{aligned} w_f(v_i v_{k+m+i}) =& f(v_i) + f(v_i v_{k+m+i}) + f(v_{k+m+i}) \\ =& i + (3k + m + 1 - i) + (k + m + i) \\ =& 2(2k + m) + 1 + i \quad \text{for } i = 1, 2, \ldots, k. \end{aligned}$$

Thus f is a super $(2(2k + m) + 2, 1)$-EAT labeling of G. □

Now by taking $m = 0$ and observing that the number of vertices in kP_2 is $2k$, we immediately obtain the following theorem (we prefer to call it a theorem though it is just a corollary of Lemma 2.5 and Theorem 2.2).

Theorem 2.6. *If G is an odd regular graph on p vertices that has a 1-factor, then G is super $(2p+2,1)$-EAT.*

Unfortunately the construction does not solve the existence of $(a,1)$-EAT labelings for all odd-regular graphs, it only works for those that contain a 1-factor. We know that some graphs that arose by Cartesian products also satisfy this property, therefore, we can obtain the following corollary.

Corollary 2.7. *Let G be a regular graph. Then the Cartesian product $G \times K_2$ is a super $(a,1)$-EAT graph.*

Proof. If G is a $(2r+1)$-regular graph then the product $G \times K_2$ is $(2r+2)$-regular and by Theorem 2.3 it is super $(a,1)$-EAT. If G is $2r$-regular then $G \times K_2$ is a $(2r+1)$-regular graph with a 1-factor and thus according to Theorem 2.6 is super $(a,1)$-EAT. □

Let us point out that many results published on super $(a,1)$-EAT labelings (see [8]) follow from Theorems 2.3 and 2.6 as a corollary.

3 Some non-regular super $(a,1)$-EAT graphs

Theorem 2.2 is not restricted to regular graphs, it can be used also to obtain super $(a,1)$-EAT labelings of certain non-regular graphs. We illustrate the technique on a couple of examples. First we introduce the following lemmas.

Lemma 3.1. *Let k, m be positive integers, $k < 2m+3$. Then the graph $K_{1,k} \cup mK_1$ is super $(2(k+m+1)+2,1)$-EAT.*

Proof. We distinguish two subcases according to the parity of k.

Let k be an odd positive integer. We denote the vertices of the graph $G \cong K_{1,k} \cup mK_1$ by the symbols $v_1, v_2, \dots, v_{k+m+1}$ in such a way that $E(G) = \{v_i v_{m+2+\frac{k-1}{2}} : i = 1,2,\dots,k\}$ and the remaining vertices are denoted arbitrarily by the unused symbols. Notice that it is possible to use such notation as $k < 2m+3$.

We define the labeling $f : V(G) \cup E(G) \to [1, 2k+m+1]$ in the following way

$$f(v_j) = j \quad \text{for } j = 1,2,\dots,k+m+1,$$

$$f(v_i v_{m+2+\frac{k-1}{2}}) = \begin{cases} m + \frac{3k+1}{2} + i & \text{for } i = 1,2,\dots,\frac{k+1}{2}, \\ m + \frac{k+1}{2} + i & \text{for } i = \frac{k+3}{2}, \frac{k+5}{2}, \dots, k. \end{cases}$$

For the edge-weights we have

$$\begin{aligned} w_f(v_i v_{m+2+\frac{k-1}{2}}) &= f(v_i) + f(v_i v_{m+2+\frac{k-1}{2}}) + f(v_{m+2+\frac{k-1}{2}}) \\ &= \begin{cases} i + \left(m + \frac{3k+1}{2} + i\right) + \left(m + 2 + \frac{k-1}{2}\right) \\ \qquad = 2m + 2k + 2 + 2i \quad \text{for} \quad i = 1,2,\dots,\frac{k+1}{2}, \\ i + \left(m + \frac{k+1}{2} + i\right) + \left(m + 2 + \frac{k-1}{2}\right) \\ \qquad = 2m + k + 2 + 2i \quad \text{for} \quad i = \frac{k+3}{2}, \frac{k+5}{2}, \dots, k, \end{cases} \end{aligned}$$

i.e. the set of the edge-weights is $[2m+2k+4, 2m+3k+3]$. Thus for $2m+3 > k$, $k \equiv 1 \pmod 2$, f is a super $(2(k+m+1)+2,1)$-EAT labeling of G.

Notice that the edge $v_{\frac{k+1}{2}}v_{m+2+\frac{k-1}{2}}$ is labeled under the labeling f by the highest label $m+2k+1$ and has also the maximal edge-weight $2m+3k+3$. Thus it is possible to delete this edge from G and the obtained graph $K_{1,(k-1)} \cup (m+1)K_1$ will also be super $(2(k+m+1)+2,1)$-EAT. It means that it is possible to construct the required labeling also in the case when the star has even number of pending edges (for k even). □

Lemma 3.2. *Let k, m be positive integers, let m be even. Let H be an arbitrary 2-regular graph of order k. Then the graph $H \cup mK_1$ is super $(2(k+m)+2,1)$-EAT.*

Proof. According to Theorem 2.2 the graph H is super $(2k+2,1)$-EAT. Using Lemma 2.4 for $t = \frac{m}{2}$ we get that $H \cup mK_1$ is a super $(2(k+m)+2,1)$-EAT graph. □

Lemma 3.3. *Let k, m be positive integers, let m be even. Then the graph $P_k \cup mK_1$ is super $(2(k+m)+2,1)$-EAT.*

Proof. It is known that the path on k vertices is super $(2k+2,1)$-EAT, see [3]. According to Lemma 2.4 for $t = \frac{m}{2}$ we get that the graph $P_k \cup mK_1$ is super $(2(k+m)+2,1)$-EAT. □

Immediately from the previous lemmas and Theorem 2.2 we see that it is possible to "add" certain edges to an even-regular graph and obtain a super $(a,1)$-EAT graph. The edges are added in such a way that the graph induced by these edges is isomorphic to a collection of independent edges, to a star, to a 2-regular graph, or to a path.

Theorem 3.4. *Let k, m be positive integers. Let G be a graph on p vertices that can be decomposed into two factors G_1 and G_2. If G_2 is a $2r$-regular graph and either*

1) *G_1 is the graph $kP_2 \cup mK_1$, or*

2) *G_1 is the graph $K_{1,k} \cup mK_1$ for $k < 2m+3$, or*

3) *H is an arbitrary 2-regular graph of order k and $G_1 \cong H \cup mK_1$ for even m, or*

4) *G_1 is the graph $P_k \cup mK_1$ for even m,*

then the graph G is super $(2p+2,1)$-EAT.

Proof. Since the smallest edge-weight in G_1 in case 1) is $2(2k+m)+2 = 2p+2$ then the claim immediately follows by Lemma 2.5 and Theorem 2.2. By a similar argument one can prove cases 2), 3), and 4) using Theorem 2.2 and Lemmas 3.1, 3.2, and 3.3, respectively. □

Notice that in Lemmas 2.5, 3.1, 3.2 and 3.3 by taking $m = 0$ we obtain an $(2p'+2,1)$-EAT labeling of the corresponding graph on $p' = p-m$ vertices. Now adding m isolated vertices one can obtain by Lemma 2.4 not one, but $m+1$ different super $(a,1)$-EAT labelings of the graph G_1 in each of the cases of Theorem 3.4. This again implies several different super $(2p+2,1)$-EAT labelings of the graph G in Theorem 3.4. There can be significantly more than $m+1$

different labelings, since we may choose various orderings of an orientations of the 2-factors F_j of G_2 (as described in the proof of Theorem 2.2).

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