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Fuzzy filters and fuzzy prime filters of bounded $R\ell$-monoids and pseudo BL-algebras

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Abstract

The logical foundations of processes handling uncertainty in information use some classes of algebras as algebraic semantics. The sets of provable formulas in corresponding inference systems from the point of view of uncertain information can be described by fuzzy filters of those algebraic semantics. Bounded residuated lattice ordered monoids ($R\ell$-monoids) are a common generalization of pseudo BL-algebras (and consequently of pseudo MV-algebras) and Heyting algebras, i.e., algebras behind fuzzy and intuitionistic reasoning. In the paper we introduce and investigate fuzzy filters of bounded $R\ell$-monoids and fuzzy prime filters of pseudo BL-algebras.

Keywords: $R\ell$-monoid, pseudo BL-algebra, pseudo MV-algebra, filter, fuzzy filter, prime filter, fuzzy prime filter

1 Introduction

As it is known, an important task of the artificial intelligence is to make the computers simulate human being in dealing with certainty and uncertainty in information. Logic gives a technique for laying the foundations of this task. While information processing

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*Corresponding author. Tel.: +420 585 634 650, fax: +420 585 634 663 dealing with certain information is based on the classical two-valued logic, non-classical logics including logics behind fuzzy reasoning handle information with various facets of uncertainty such as fuzziness, randomness, vagueness, etc. (For a generalized theory of uncertainty see [28].) So, non-classical logics have become as a formal and useful tool for computer science to deal with fuzzy and uncertain information. Furthermore, one can observe that human reasoning need not be strictly commutative and often can depend, e.g., on circumstances and on consecutive information in time. On the other side, there are logic concurrent programming languages based on non-commutative logics.

The classical two-valued logic has Boolean algebras as an algebraic semantics. Similarly, for important non-classical logics there are algebraic semantics in the form of classes of algebras. Using these classes, one can obtain an algebraization of inference systems that handle various kinds of uncertainty. The sets of provable formulas in inference systems are described by filters, and from the point of view of uncertain information, by fuzzy filters of corresponding algebras.

Bounded residuated lattice ordered monoids ($R\ell$-monoids) form a large class of algebras which contains, among others, certain classes of algebras behind fuzzy reasoning. Namely, pseudo BL-algebras [5], [6] and pseudo MV-algebras [9] ($=GMV$-algebras [23]), and consequently BL-algebras [10] and MV-algebras [3], [4], can be considered as bounded $R\ell$-monoids. Recall that BL-algebras and pseudo BL-algebras are algebraic semantics of Hájek's BL-logic [11] and pseudo BL-logic [12], respectively, as well as MV-algebras and pseudo MV-algebras are algebras of the Łukasiewicz infinite valued logic [3] and the non-commutative Łukasiewicz logic [20], respectively. Moreover, the class of bounded $R\ell$-monoids also contains the class of Heyting algebras [1], i.e. algebras of the intuitionistic logic.

Fuzzy ideals (or in the dual form, fuzzy filters) of MV-algebras were introduced and developed by Hoo in [13], [14] and their generalizations for pseudo MV-algebras by Jun and Walendziak in [16] and by Dymek in [8]. Certain classes of fuzzy filters or ideals were also studied in lattice implication algebras [17], in R_0-algebras [21] and in BCK/BCI-algebras [26], and further kinds of filters or ideals in [22] and in integral residuated ℓ-monoids [27].

In the paper we define and study fuzzy filters of bounded $R\ell$-monoids. For this general case we describe connections between filters and fuzzy filters and characterize fuzzy filters generated by fuzzy sets. Further, for the case of pseudo BL-algebras we introduce and study fuzzy prime filters and show their connections to prime filters. We characterize, by means of fuzzy filters, linearly ordered pseudo BL-algebras and give conditions under which a fuzzy filter is contained in a fuzzy prime filter.

2 Bounded $R\ell$-monoids

A *bounded $R\ell$-monoid* is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0, \rangle$ satisfying the following conditions:

(i) $(M;\, \odot, 1)$ is a monoid (need not be commutative).

(ii) $(M;\, \vee, \wedge, 0, 1)$ is a bounded lattice.

(iii) $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$ for any $x,\, y \in M$.

(iv) $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$.

Recall that the lattice $(M;\, \vee, \wedge)$ is distributive and that bounded $R\ell$-monoids form a variety of algebras of the indicated type. Moreover, the bounded $R\ell$-monoids can be recognized as bounded integral generalized BL-algebras in the sense of [2] and hence it is possible to prove that the operation "$\odot$" distributes over the lattice operations "$\vee$" and "$\wedge$".

In what follows, by an $R\ell$-monoid we will mean a *bounded* $R\ell$-monoid.

For any $R\ell$-monoid M we define two unary operations (negations) "$^-$" and "$^\sim$" on M such that $x^- := x \to 0$ and $x^\sim := x \rightsquigarrow 0$ for every $x \in M$.

Now we can characterize algebras of the above mentioned propositional logics in the class of $R\ell$-monoids.

An $R\ell$-monoid M is

a) a pseudo BL-algebra ([18]) if and only if M satisfies the identities of pre-linearity $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$;

b) a pseudo MV-algebra (GMV-algebra) ([23]) if and only if M fulfils the identities $x^{-\sim} = x = x^{\sim-}$;

c) a Heyting algebra ([25]) if and only if the operations "$\odot$" and "$\wedge$" coincide on M.

If the operation "$\odot$" is commutative then an $R\ell$-monoid is called *commutative*. Recall that in such a case the implications "$\to$" and "$\rightsquigarrow$", as well as the negations "$^-$" and "$^\sim$", respectively coincide. Then commutative pseudo BL-algebras are precisely BL-algebras and commutative pseudo MV-algebras coincide with MV-algebras.

Lemma 2.1 *[24, 7]*
In any bounded $R\ell$-monoid M we have for any $x,\, y \in M$:

(1) $x \leq y \iff x \to y = 1 \iff x \rightsquigarrow y = 1.$

(2) $x \leq y \implies z \to x \leq z \to y, \quad z \rightsquigarrow x \leq z \rightsquigarrow y.$

(3) $x \leq y \implies y \to z \leq x \to z, \quad y \rightsquigarrow z \leq x \rightsquigarrow z.$

(4) $x \to x = 1 = x \rightsquigarrow x, \quad 1 \to x = x = 1 \rightsquigarrow x, \quad x \to 1 = 1 = x \rightsquigarrow 1.$

(5) $(x \to y) \odot x \leq x \leq y \to (x \odot y), \quad x \odot (x \rightsquigarrow y) \leq y \leq x \rightsquigarrow (x \odot y).$

(6) $1^{-\sim} = 1 = 1^{\sim-}, \quad 0^{-\sim} = 0 = 0^{\sim-}.$

(7) $x \leq x^{-\sim}, \quad x \leq x^{\sim-}.$

(8) $x^{-\sim-} = x^-, \quad x^{\sim-\sim} = x^\sim.$

(9) $x^- \odot x = 0 = x \odot x^\sim.$

(10) $x \to (y \to z) = (x \odot y) \to z, \quad x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z.$

Let $M = (M;\, \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ be an $R\ell$-monoid and $\emptyset \neq F \subseteq M$. Then F is called a *filter* of M if

(i) $x,\, y \in F \implies x \odot y \in F$;

(ii) $x \in F,\; y \in M,\; x \leq y \implies y \in F$.

Further properties of filters follow from [2] and [15].

If M is an $R\ell$-monoid and $D \subseteq M$ then D is called a *deductive system* of M if for each $x,\, y \in M$

(1) $1 \in D$;

(2) $x \in D,\; x \rightarrow y \in D \implies y \in D$.

It is possible to show that $H \subseteq M$ is a filter of M iff H is a deductive system of M iff H satisfies (1) and for each $x,\, y \in M$

(2') $x \in D,\; x \rightsquigarrow y \in D \implies y \in D$.

Denote by $\mathcal{F}(M)$ the complete lattice (with respect to the order by set inclusion) of filters of M. Note that infima in $\mathcal{F}(M)$ coincide with intersections. It is known ([19]) that $\mathcal{F}(M)$ is a complete Heyting algebra and hence $G \cap \bigvee\limits_{i \in I} F_i = \bigvee\limits_{i \in I} (G \cap F_i)$, for any $G,\, F_i \in \mathcal{F}(M)$, $i \in I$.

If $X \subseteq M$, denote by $Fil(X)$ the filter of M generated by X. For $X = \emptyset$, we have $Fil(\emptyset) = \{1\}$. If $X \neq \emptyset$ then

$$Fil(X) = \{y \in M : \; y \geq x_1 \odot \cdots \odot x_n \;\text{ for some }\; x_1, \ldots, x_n \in X,\; n \geq 1\}.$$

3 Fuzzy filters of $R\ell$-monoids

Let $[0, 1]$ be the closed unit interval of reals and $M \neq \emptyset$ be a set. Recall that a *fuzzy set* in M is any function $\nu : \; M \longrightarrow [0, 1]$.

If ν and λ are fuzzy sets in M, define $\nu \leq \lambda$ iff $\nu(x) \leq \lambda(x)$ for all $x \in M$.

If $\Gamma \subseteq [0, 1]$, put $\bigwedge \Gamma := \inf \Gamma$ in $[0, 1]$ and $\bigvee \Gamma := \sup \Gamma$ in $[0, 1]$. In particular, if $\alpha, \beta \in [0, 1]$, then $\alpha \wedge \beta = \min\{\alpha, \beta\}$ and $\alpha \vee \beta = \max\{\alpha, \beta\}$. Recall that $[0, 1]$ is a complete Heyting algebra.

A fuzzy set ν in an $R\ell$-monoid M is called a *fuzzy filter* of M if for any $x,\, y \in M$ it is satisfied:

(f1) $\nu(x \odot y) \geq \nu(x) \wedge \nu(y)$,

(f2) $x \leq y \implies \nu(x) \leq \nu(y)$.

By (f2), it follows immediately that

(f3) $\nu(1) \geq \nu(x)$ for every $x \in M$.

Lemma 3.1 *Let ν be a fuzzy filter of an $R\ell$-monoid M. Then it holds for any $x, y \in M$:*

(i) $\nu(x \vee y) \geq \nu(x) \wedge \nu(y)$,

(ii) $\nu(x \wedge y) = \nu(x) \wedge \nu(y)$,

(iii) $\nu(x \odot y) = \nu(x) \wedge \nu(y)$.

Proof. For any $x, y \in M$ we have $x \odot y \leq x \wedge y \leq x \vee y$. Then by (f2) and (f1), $\nu(x \vee y) \geq \nu(x \odot y) \geq \nu(x) \wedge \nu(y)$. Since $x \odot y \leq x \wedge y \leq x, y$, by (f1) and (f2), it follows that $\nu(x) \wedge \nu(y) \leq \nu(x \odot y) \leq \nu(x \wedge y) \leq \nu(x) \wedge \nu(y)$. ☐

Theorem 3.2 *A fuzzy set ν in an $R\ell$-monoid M is a fuzzy filter of M if and only if it satisfies (f1) and*

(f4) $\nu(x \vee y) \geq \nu(x)$ *for any $x, y \in M$.*

Proof. If ν is a fuzzy filter of an $R\ell$-monoid M then $x \leq x \vee y$ implies $\nu(x) \leq \nu(x \vee y)$.

Conversely, if ν satisfies (f1) and (f4), and $x \leq y$, then $\nu(y) = \nu(x \vee y) \geq \nu(x)$. Hence ν is a fuzzy filter of M. ☐

Theorem 3.3 *Let ν be a fuzzy set in an $R\ell$-monoid M. Then the following conditions are equivalent.*

(1) *ν is a fuzzy filter of M.*

(2) *ν satisfies (f3) and for all $x, y \in M$,*

$$\nu(y) \geq \nu(x) \wedge \nu(x \to y). \tag{$*$}$$

(3) *ν satisfies (f3) and for all $x, y \in M$,*

$$\nu(y) \geq \nu(x) \wedge \nu(x \rightsquigarrow y). \tag{$**$}$$

Proof. (1) ⇒ (2): Let ν be a fuzzy filter of M and $x, y \in M$. Then, by Lemma 3.1(iii),

$\nu(y) \geq \nu(x \wedge y) = \nu((x \to y) \odot x) = \nu(x \to y) \wedge \nu(x)$.

Hence ν satisfies the condition (2).

(2) ⇒ (1): Let ν be a fuzzy set in M satisfying (f3) and ($*$). Let $x, y \in M$, $x \leq y$. Then $x \to y = 1$. Thus $\nu(y) \geq \nu(x) \wedge \nu(1) = \nu(x)$, hence (f2) holds.

Further, since $x \leq y \to (x \odot y)$, by ($*$) and (f2) we get $\nu(x \odot y) \geq \nu(y) \wedge \nu(y \to (x \odot y)) \geq \nu(y) \wedge \nu(x)$. Therefore (f1) is also satisfied and hence ν is a fuzzy filter of M.

(1) ⇔ (3): Analogously. ☐

Let F be a subset of M and $\alpha,\ \beta \in [0,1]$ such that $\alpha > \beta$. Define a fuzzy subset $\nu_F(\alpha,\beta)$ in M by

$$\nu_F(\alpha,\beta)(x) := \begin{cases} \alpha, & \text{if } x \in F, \\ \beta, & \text{otherwise.} \end{cases}$$

In particular, $\nu_F(1,0)$ is the characteristic function χ_F of F. We will use the denotation ν_F instead of $\nu_F(\alpha,\beta)$, for every $\alpha,\ \beta \in [0,1],\ \alpha > \beta$.

Theorem 3.4 *Let F be a non-empty subset of an $R\ell$-monoid M. Then the fuzzy set ν_F is a fuzzy filter of M if and only if F is a filter of M.*

Proof. Let M be an $R\ell$-monoid and $\emptyset \neq F \subseteq M$.

a) Let F be a filter of M and $x,\ y \in M$. If $x,\ y \in F$, then $x \odot y \in F$, hence $\nu_F(x \odot y) = \alpha = \nu_F(x) \wedge \nu_F(y)$. If $x \notin F$ or $y \notin F$, then $\nu_F(x) = \beta$ or $\nu_F(y) = \beta$, thus $\nu_F(x \odot y) \geq \beta = \nu_F(x) \wedge \nu_F(y)$. Therefore ν_F satisfies (f1). Further, let $x,\ y \in M,\ x \leq y$. If $y \in F$, then $\nu_F(y) = \alpha \geq \nu_F(x)$. If $y \notin F$, then also $x \notin F$, and hence $\nu_F(x) = \beta = \nu_F(y)$. Therefore ν_F satisfies (f2).

That means, ν_F is a fuzzy filter of M.

b) Let ν_F be a fuzzy filter of M. If $x,\ y \in F$, then $\nu_F(x) = \alpha = \nu_F(y)$, hence $\nu_F(x \odot y) \geq \nu_F(x) \wedge \nu_F(y) = \alpha$, thus $x \odot y \in F$. If $x \in F,\ y \in M$ and $x \leq y$, then $\alpha = \nu_F(x) \leq \nu_F(y)$, hence $\nu_F(y) = \alpha$, and so $y \in F$. Therefore F is a filter of M. ☐

Let ν be a fuzzy set in an $R\ell$-monoid M. Denote by M_ν the set

$$M_\nu := \{x \in M : \ \nu(x) = \nu(1)\}.$$

Theorem 3.5 *If ν is a fuzzy filter of an $R\ell$-monoid M, then M_ν is a filter of M.*

Proof. Let ν be a fuzzy filter of M. Let $x,\ y \in M_\nu$, i.e. $\nu(x) = \nu(1) = \nu(y)$. Then $\nu(x \odot y) \geq \nu(x) \wedge \nu(y) = \nu(1)$, hence $\nu(x \odot y) = \nu(1)$, thus $x \odot y \in M_\nu$.

Further, let $x \in M_\nu,\ y \in M$ and $x \leq y$. Then $\nu(1) = \nu(x) \leq \nu(y)$, hence $\nu(y) = \nu(1)$, and therefore $y \in M_\nu$.

That means M_ν is a filter of M. ☐

The converse implication to that from Theorem 3.5 is not true in general, even for pseudo MV-algebras, as it was shown in [8, Example 3.9].

Let ν be a fuzzy set in M and $\alpha \in [0,1]$. The set

$$U(\nu;\alpha) := \{x \in M : \ \nu(x) \geq \alpha\}$$

is called the *level subset of ν determined by α.*

Note that from this point of view, $M_\nu = U(\nu;\ \nu(1))$, hence M_ν is a special case of a level subset of M.

Theorem 3.6 *Let ν be a fuzzy set in an $R\ell$-monoid M. Then ν is a fuzzy filter of M if and only if its level subset $U(\nu;\alpha)$ is a filter of M or $U(\nu;\alpha)=\emptyset$ for each $\alpha\in[0,1]$.*

Proof. Let ν be a fuzzy filter and $\alpha\in[0,1]$ such that $U(\nu;\alpha)\neq\emptyset$. Assume $x,\ y\in U(\nu;\alpha)$, then $\nu(x),\ \nu(y)\geq\alpha$, thus $\nu(x\odot y)\geq\nu(x)\wedge\nu(y)\geq\alpha$. Hence $x\odot y\in U(\nu;\alpha)$. Consider $x\in U(\nu;\alpha)$, $y\in M$ and $x\leq y$. Then $\alpha\leq\nu(x)\leq\nu(y)$, therefore $y\in U(\nu;\alpha)$. Consequently, $U(\nu;\alpha)$ is a filter of M.

Conversely, let us suppose that for every $\alpha\in[0,1]$ such that $U(\nu;\alpha)\neq\emptyset$, it is satisfied that $U(\nu;\alpha)$ is a filter of M. Let $x,\ y\in M$ and $\nu(x\odot y)<\nu(x)\wedge\nu(y)$. Writing $\beta=\frac{1}{2}(\nu(x\odot y)+(\nu(x)\wedge\nu(y)))$ yields $\nu(x\odot y)<\beta<\nu(x)\wedge\nu(y)$, thus $x,\ y\in U(\nu;\beta)$ and $x\odot y\notin U(\nu;\beta)$. That means $U(\nu;\beta)$ is not a filter of M, a contradiction. Therefore (f1) holds.

Finally, let $x,\ y\in M$ and $x\leq y$. Let us assume that $\nu(x)>\nu(y)$. Taking $\gamma=\frac{1}{2}(\nu(x)+\nu(y))$ we obtain $\nu(x)>\gamma>\nu(y)$, therefore $x\in U(\nu;\gamma)$ and $y\notin U(\nu;\gamma)$, a contradiction. Hence (f2) is fulfilled. From the above it follows that ν is a fuzzy filter of M. □

Theorem 3.7 *Let ν be a fuzzy subset in an $R\ell$-monoid M. Then the following conditions are equivalent.*

(1) ν *is a fuzzy filter of* M.

(2) $\forall x,\ y,\ z\in M;\ \ x\to(y\to z)=1\implies\nu(z)\geq\nu(x)\wedge\nu(y)$.

(3) $\forall x,\ y,\ z\in M;\ \ x\rightsquigarrow(y\rightsquigarrow z)=1\implies\nu(z)\geq\nu(x)\wedge\nu(y)$.

Proof. (1) ⇒ (2): Let ν be a fuzzy filter of M. Let $x,\ y,\ z\in M$ and $x\to(y\to z)=1$. Then by Theorem 3.3, $\nu(y\to z)\geq\nu(x)\wedge\nu(x\to(y\to z))=\nu(x)\wedge\nu(1)=\nu(x)$.

Moreover, also by Theorem 3.3, $\nu(z)\geq\nu(y)\wedge\nu(y\to z)$, hence we obtain $\nu(z)\geq\nu(y)\wedge\nu(x)$.

(2) ⇒ (1): Let a fuzzy set ν in M satisfy the condition (2). Let $x,\ y\in M$. Since $x\to(x\to 1)=1$, we have $\nu(1)\geq\nu(x)\wedge\nu(x)=\nu(x)$, hence (f3) is satisfied.

Further, since $(x\to y)\to(x\to y)=1$ we get $\nu(y)\geq\nu(x\to y)\wedge\nu(x)$, thus ν satisfies $(*)$, that means, by Theorem 3.3, ν is a fuzzy filter of M.

(1) ⇔ (3): Analogously. □

Corollary 3.8 *A fuzzy set ν in an $R\ell$-monoid M is a fuzzy filter of M if and only if for all $x,\ y,\ z\in M$, $x\odot y\leq z$ implies $\nu(z)\geq\nu(x)\wedge\nu(y)$.*

Corollary 3.9 *A fuzzy set ν in an $R\ell$-monoid M is a fuzzy filter of M if and only if for any $x,\ a_1,\ \dots,\ a_n\in M$, $a_1\odot\cdots\odot a_n\leq x$ implies $\nu(x)\geq\nu(a_1)\wedge\cdots\wedge\nu(a_n)$.*

For any fuzzy sets ν_i $(i\in I)$ in M we define the fuzzy set $\bigcap\limits_{i\in I}\nu_i$ in M as follows:

$$\left(\bigcap_{i\in I}\nu_i\right)(x)\ =\ \bigwedge_{i\in I}\nu_i(x).$$

Theorem 3.10 *Let* ν_i $(i \in I)$ *be fuzzy filters of an* $R\ell$*-monoid* M*. Then* $\bigcap\limits_{i\in I} \nu_i$ *is also a fuzzy filter of* M*.*

Proof. Let M be an $R\ell$-monoid, ν_i be a fuzzy filter of M for any $i \in I$ and $\nu = \bigcap\limits_{i\in I} \nu_i$. Suppose that x, y, $z \in M$ are such that $x \to (y \to z) = 1$. Then by Theorem 3.7, $\nu_i(z) \geq \nu_i(x) \wedge \nu_i(y)$, for every $i \in I$, hence

$$\begin{aligned}\nu(z) = \left(\bigcap_{i\in I} \nu_i\right)(z) &= \inf(\nu_i(z);\ i \in I) \geq \inf(\nu_i(x) \wedge \nu_i(y);\ i \in I)\\ &= \inf(\nu_i(x);\ i \in I) \wedge \inf(\nu_i(y);\ i \in I) = \left(\bigcap_{i\in I} \nu_i\right)(x) \wedge \left(\bigcap_{i\in I} \nu_i\right)(y) = \nu(x) \wedge \nu(y),\end{aligned}$$

therefore by Theorem 3.7, ν is a fuzzy filter of M. ☐

Corollary 3.11 *The set* $\mathcal{FF}(M)$ *of fuzzy filters of an* $R\ell$*-monoid* M *is a complete lattice in which infima coincide with intersections of fuzzy filters.*

Let ν be a fuzzy set in an $R\ell$-monoid M. Then the intersection of all fuzzy filters of M containing ν is called the *fuzzy filter of* M *generated by* ν, denoted by $FFil(\nu)$.

In two next theorems we give two descriptions of $FFil(\nu)$ which generalize analogous results from [16] and [8] concerning fuzzy ideals of pseudo MV-algebras.

Theorem 3.12 *Let* ν *be a fuzzy subset in an* $R\ell$*-monoid* M*. Put*

$$\nu^*(x) := \bigvee\{\alpha \in [0,1] :\ x \in Fil(U(\nu;\alpha))\} \ \text{ for any } \ x \in M.$$

Then $\nu^* = FFil(\nu)$*.*

Proof. If $\beta \in [0,1]$, put $\beta_n = \beta - \frac{1}{n}$ for every $n \in \mathbb{N}$. Let $\beta \in [0,1]$ be such that $U(\nu^*;\beta) \neq \emptyset$ and let $x \in U(\nu^*;\beta)$. Then

$$\nu^*(x) = \bigvee\{\alpha \in [0,1] :\ x \in Fil(U(\nu;\alpha))\} \geq \beta > \beta_n,$$

for each $n \in \mathbb{N}$. Thus for every $n \in \mathbb{N}$ there is $\gamma_n \in \{\alpha \in [0,1] :\ x \in Fil(U(\nu;\alpha))\}$ such that $\gamma_n > \beta_n$. Hence $x \in Fil(U(\nu;\gamma_n))$ for every $n \in \mathbb{N}$. That means $x \in \bigcap\limits_{n\in\mathbb{N}} Fil(U(\nu;\gamma_n))$. Moreover, let $x \in \bigcap\limits_{n\in\mathbb{N}} Fil(U(\nu;\gamma_n))$, i.e. $\gamma_n \in \{\alpha \in [0,1] :\ x \in Fil(U(\nu;\alpha))\}$ for each $n \in \mathbb{N}$. Then $\beta_n < \gamma_n \leq \bigvee\{\alpha \in [0,1] :\ x \in Fil(U(\nu;\alpha))\} = \nu^*(x)$, for every $n \in \mathbb{N}$. Thus $\beta \leq \nu^*(x)$, and hence $x \in U(\nu^*;\beta)$. Therefore we get $U(\nu^*;\beta) = \bigcap\limits_{n\in\mathbb{N}} (Fil(U(\nu;\gamma_n)) \in \mathcal{F}(M)$, and hence, by Theorem 3.6, ν^* is a fuzzy filter of M.

Let $x \in M$ and let $\beta \in \{\alpha \in [0,1] : x \in U(\nu;\alpha)\}$. Then $x \in U(\nu;\beta)$, and thus $x \in Fil(U(\nu;\beta))$. Hence $\beta \in \{\alpha \in [0,1] : x \in Fil(U(\nu;\alpha))\}$, therefore $\{\alpha \in [0,1] : x \in U(\nu;\alpha)\} \subseteq \{\alpha \in [0,1] : x \in Fil(U(\nu;\alpha))\}$. From this we get

$$\nu(x) \leq \bigvee\{\alpha \in [0,1] : x \in U(\nu;\alpha)\} \leq \bigvee\{\alpha \in [0,1] : x \in Fil(U(\nu;\alpha))\} = \nu^*(x),$$

thus $\nu \leq \nu^*$.

Now, let τ be a fuzzy filter of M containing ν. Let $x \in M$ and $\nu^*(x) = \beta$. Then $x \in U(\nu^*;\beta) = \bigcap\limits_{n\in\mathbb{N}} Fil(U(\nu;\gamma_n))$, thus $x \in Fil(U(\nu;\gamma_n))$ for every $n \in \mathbb{N}$. Hence $x \geq y_1 \odot \cdots \odot y_k$ for some $y_1, \ldots y_k \in U(\nu;\gamma_n)$, and so by Lemma 3.1, $\nu(x) \geq \nu(y_1)\wedge\cdots\wedge\nu(y_k) \geq \gamma_n$. Therefore $\tau(x) \geq \nu(x) \geq \gamma_n > \beta_n = \beta - \frac{1}{n}$ for every $n \in \mathbb{N}$, and since $n \in \mathbb{N}$ is arbitrary, we get $\tau(x) \geq \beta = \nu^*(x)$, that means $\nu^* \leq \tau$. □

Theorem 3.13 *Let ν be a fuzzy set in an $R\ell$-monoid M. Put*

$$\bar{\nu}(x) := \bigvee\{\nu(a_1) \wedge \cdots \wedge \nu(a_n) : a_1, \ldots, a_n \in M,\ x \geq a_1 \odot \cdots \odot a_n\},$$

for any $x \in M$. Then $\bar{\nu} = FFil(\nu)$.

Proof. Obviously $\bar{\nu}(1) \geq \bar{\nu}(x)$ for any $x \in M$, hence $\bar{\nu}$ satisfies (f3).

Let $x, y \in M$ and let there exist $b_1, \ldots, b_n, c_1, \ldots, c_m \in M$ such that $x \geq b_1 \odot \cdots \odot b_n$, $x \rightsquigarrow y \geq c_1 \odot \cdots \odot c_m$. Then $y \geq x \wedge y = x \odot (x \rightsquigarrow y) \geq b_1 \odot \cdots \odot b_n \odot c_1 \odot \cdots \odot c_m$. Hence $\bar{\nu}(y) \geq \nu(b_1)\wedge\cdots\wedge\nu(b_n)\wedge\nu(c_1)\wedge\cdots\wedge\nu(c_m)$. Since $[0,1]$ is a Heyting algebra, we get $\bar{\nu}(x)\wedge\bar{\nu}(x \rightsquigarrow y) = \bigvee\{\nu(d_1)\wedge\cdots\wedge\nu(d_s) : d_1, \ldots, d_s \in M,\ x \geq d_1\odot\cdots\odot d_s\}\wedge\bigvee\{\nu(e_1)\wedge\cdots\wedge\nu(e_t) : x \rightsquigarrow y \geq e_1\odot\cdots\odot e_t\} = \bigvee\{\nu(d_1)\wedge\cdots\wedge\nu(d_s)\wedge\nu(e_1)\wedge\cdots\wedge\nu(e_t) : x \geq d_1\odot\cdots\odot d_s,\ x \rightsquigarrow y \geq e_1 \odot \cdots \odot e_t\}$. Hence we have $\bar{\nu}(y) \geq \bar{\nu}(x) \wedge \bar{\nu}(x \rightsquigarrow y)$, therefore by Theorem 3.3, $\bar{\nu}$ is a fuzzy filter of M. Furthermore, $x \geq x \odot x$, thus $\bar{\nu}(x) \geq \nu(x) \wedge \nu(x) = \nu(x)$ for every $x \in M$, hence $\nu \leq \bar{\nu}$.

Now, let τ be a fuzzy filter of M such that $\nu \leq \tau$. Then

$$\begin{aligned}\bar{\nu}(x) &= \bigvee\{\nu(a_1) \wedge \cdots \wedge \nu(a_n) : a_1, \ldots, a_n \in M,\ x \geq a_1 \odot \cdots \odot a_n\}\\ &\leq \{\tau(a_1) \wedge \cdots \wedge \tau(a_n) : a_1, \ldots, a_n \in M,\ x \geq a_1 \odot \cdots \odot a_n\} \leq \tau(x),\end{aligned}$$

by Corollary 3.9.

Therefore $\bar{\nu} \leq \tau$, that means $\bar{\nu} = FFil(\nu)$. □

4 Fuzzy prime filters of pseudo BL-algebras

Let ν be a non-constant fuzzy filter of an $R\ell$-monoid M. Then ν is called a *fuzzy prime filter* of M if for any $x, y \in M$,

$$\nu(x \vee y) = \nu(x) \vee \nu(y).$$

In this section we will focus on fuzzy prime filters of pseudo BL-algebras.

Theorem 4.1 *Let M be a pseudo BL-algebra and let ν be a non-constant fuzzy filter of M. Then the following conditions are equivalent.*

(1) ν *is a fuzzy prime filter of* M.

(2) $\forall x, y \in M;\ \nu(x \vee y) = \nu(1) \implies \nu(x) = \nu(1)$ *or* $\nu(y) = \nu(1)$.

(3) $\forall x, y \in M;\ \nu(x \to y) = \nu(1)$ *or* $\nu(y \to x) = \nu(1)$.

(4) $\forall x, y \in M;\ \nu(x \rightsquigarrow y) = \nu(1)$ *or* $\nu(y \rightsquigarrow x) = \nu(1)$.

Proof. (1) ⇒ (2): Let ν be a fuzzy prime filter of M, $x, y \in M$ and $\nu(x \vee y) = \nu(1)$. Then $\nu(x) \vee \nu(y) = \nu(x \vee y) = \nu(1)$, hence $\nu(x) = \nu(1)$ or $\nu(y) = \nu(1)$. (Note that this implication is true for an arbitrary $R\ell$-monoid.)

(2) ⇒ (3): Let ν be a non-constant fuzzy filter of M satisfying (2). Since M is a pseudo BL-algebra, $(x \to y) \vee (y \to x) = 1$ for any $x, y \in M$, thus $\nu((x \to y) \vee (y \to x)) = \nu(1)$, therefore $\nu(x \to y) = \nu(1)$ or $\nu(y \to x) = \nu(1)$ for every $x, y \in M$.

(3) ⇒ (1): Let a non-constant fuzzy filter ν of M satisfy (3). Let $x, y \in M$ and $\nu(x \to y) = \nu(1)$. We have

$$y \geq (x \wedge y) \vee ((x \to y) \odot y) = ((x \to y) \odot x) \vee ((x \to y) \odot y) = (x \to y) \odot (x \vee y),$$

hence

$$\nu(y) \geq \nu((x \to y) \odot (x \vee y)) \geq \nu(x \to y) \wedge \nu(x \vee y) = \nu(1) \wedge \nu(x \vee y) = \nu(x \vee y),$$

thus $\nu(x \vee y) = \nu(y)$.

Moreover, $\nu(x \vee y) \geq \nu(x)$, hence $\nu(x \vee y) = \nu(x) \vee \nu(y)$.
Analogously, $\nu(y \to x) = \nu(1)$ implies $\nu(x \vee y) = \nu(x) \vee \nu(y)$.
Therefore ν is a fuzzy prime filter of M.

(1) ⇔ (4): Analogously. □

Let F be a proper filter of a pseudo BL-algebra M. Then F is called *prime* [5] if for all $x, y \in M$, $x \vee y \in F$ implies $x \in F$ or $y \in F$. By [5, Proposition 4.25], a proper filter F of M is prime iff $x \to y \in F$ or $y \to x \in F$, for all $x, y \in M$, iff $x \rightsquigarrow y \in F$ or $y \rightsquigarrow x \in F$, for all $x, y \in M$.

Theorem 4.2 *If M is a pseudo BL-algebra and F is a filter of M, then F is a prime filter of M if and only if ν_F is a fuzzy prime filter of M.*

Proof. Let F be a prime filter of M. Since F is a proper filter, ν_F is non-constant. Let $x, y \in M$. Then $(x \to y) \vee (y \to x) = 1 \in F$, thus $x \to y \in F$ or $y \to x \in F$, hence $\nu_F(x \to y) = \alpha = \nu_F(1)$ or $\nu_F(y \to x) = \alpha = \nu_F(1)$. Therefore by Theorem 4.1, ν_F is a fuzzy prime filter of M.

Conversely, let ν_F be a fuzzy prime filter of M. Then, also by Theorem 4.1, F is a prime filter of M. □

Theorem 4.3 *If M is a pseudo BL-algebra and ν is a fuzzy filter of M, then ν is a fuzzy prime filter if and only if M_ν is a prime filter of M.*

Proof. Let ν be a fuzzy prime filter of M. If $x, y \in M$ are such that $x \vee y \in M_\nu$, then $\nu(x) \vee \nu(y) = \nu(x \vee y) = \nu(1)$, hence $\nu(x) = \nu(1)$ or $\nu(y) = \nu(1)$, therefore $x \in M_\nu$ or $y \in M_\nu$ Consequently, M_ν is a prime filter of M.

Conversely, let ν be a fuzzy filter of M such that the filter M_ν is prime in M. (Then ν is non-constant.) If $x, y \in M$, then $(x \to y) \vee (y \to x) = 1 \in M_\nu$, thus $x \to y \in M_\nu$ or $y \to x \in M_\nu$, hence $\nu(x \to y) = \nu(1)$ or $\nu(y \to x) = \nu(1)$. That means, ν is a fuzzy prime filter of M. ☐

Theorem 4.4 *Let M be a pseudo BL-algebra and let ν be a non-constant fuzzy set in M. Then ν is a fuzzy prime filter of M if and only if for every $\alpha \in [0, 1]$, if $U(\nu; \alpha) \neq \emptyset$ and $U(\nu; \alpha) \neq M$, then $U(\nu; \alpha)$ is a prime filter of M.*

Proof. Let ν be a fuzzy prime filter of M. If $\alpha \in [0, 1]$ and $U(\nu; \alpha) \neq \emptyset$, then by Theorem 3.6, $U(\nu; \alpha)$ is a filter of M. Let $U(\nu; \alpha)$ be a proper filter of M and let $x, y \in M$. If $x \vee y \in U(\nu; \alpha)$, then $\nu(x \vee y) = \nu(x) \vee \nu(y) \geq \alpha$, hence $\nu(x) \geq \alpha$ or $\nu(y) \geq \alpha$, and so $x \in U(\nu; \alpha)$ or $y \in U(\nu; \alpha)$. That means, $U(\nu; \alpha)$ is a prime filter of M.

Conversely, let ν be a non-constant fuzzy set in M such that for every $\alpha \in [0, 1]$, if $U(\nu; \alpha) \neq \emptyset$, then $U(\nu; \alpha)$ is a prime filter of M. Let $x, y \in M$ and let $\nu(x \vee y) > \nu(x) \vee \nu(y)$, i.e. ν is not a fuzzy prime filter of M. Put $\beta = \frac{1}{2}(\nu(x \vee y) + (\nu(x) \vee \nu(y)))$. Then $\nu(x \vee y) > \beta > \nu(x) \vee \nu(y)$, thus $x \vee y \in U(\nu; \beta)$, $x \notin U(\nu; \beta)$, $y \notin U(\nu; \beta)$, hence $U(\nu; \alpha) \neq \emptyset$, but $U(\nu; \alpha)$ is not a prime filter of M, a contradiction. Therefore ν is a fuzzy prime filter of M. ☐

Theorem 4.5 *Let ν be a fuzzy prime filter of a pseudo BL-algebra M and let τ be a non-constant fuzzy filter of M such that $\nu \leq \tau$ and $\nu(1) = \tau(1)$. Then τ is a fuzzy prime filter of M.*

Proof. Let $x, y \in M$. Then by Theorem 4.1, $\nu(x \to y) = \nu(1)$ or $\nu(y \to x) = \nu(1)$. If $\nu(x \to y) = \nu(1)$, then $\tau(x \to y) = \tau(1)$. Similarly, $\nu(y \to x) = \nu(1)$ implies $\tau(y \to x) = \tau(1)$. Hence τ is a fuzzy prime filter of M. ☐

Theorem 4.6 *If M is a non-trivial pseudo BL-algebra, then the following conditions are equivalent.*

(1) M is linearly ordered.

(2) Every non-constant fuzzy filter of M is a fuzzy prime filter of M.

(3) Every non-constant fuzzy filter ν of M such that $\nu(1) = 1$ is a fuzzy prime filter of M.

(4) Fuzzy filter $\chi_{\{1\}}$ is a fuzzy prime filter of M.

Proof. (1) $\Rightarrow$ (2): Let M be a linearly ordered pseudo BL-algebra. Suppose that ν is a non-constant fuzzy filter of M. If $x, y \in M$, then $x \leq y$ or $y \leq x$, thus $x \to y = 1$ or

$y \to x = 1$. Hence $\nu(x \to y) = \nu(1)$ or $\nu(y \to x) = \nu(1)$, therefore ν is a fuzzy prime filter of M.

(2) ⇒ (3), (3) ⇒ (4): Obvious.

(4) ⇒ (1): Let $\chi_{\{1\}}$ be a fuzzy prime filter of M and let $x, y \in M$. Then $\chi_{\{1\}}(x \to y) = \chi_{\{1\}}(1) = 1$ or $\chi_{\{1\}}(y \to x) = 1$, thus $x \to y \in \{1\}$ or $y \to x \in \{1\}$, i.e. $x \to y = 1$ or $y \to x = 1$. Therefore $x \leq y$ or $y \leq x$. ☐

Theorem 4.7 *Let ν be a fuzzy prime filter of a pseudo BL-algebra M and $\alpha \in [0, \nu(1))$. Then the fuzzy set $\nu \vee \alpha$ in M such that $(\nu \vee \alpha)(x) = \nu(x) \vee \alpha$ is a fuzzy prime filter of M.*

Proof. Let ν be a fuzzy prime filter of M and $\alpha \in [0, \nu(1))$. Let $x, y, z \in M$. If $z \geq x \odot y$ (that means $x \to (y \to z) = 1$), then by Lemma 3.1, $\nu(z) \geq \nu(x \odot y) = \nu(x) \wedge \nu(y)$. Hence $(\nu \vee \alpha)(z) = \nu(z) \vee \alpha \geq (\nu(x) \wedge \nu(y)) \vee \alpha = (\nu(x) \vee \alpha) \wedge (\nu(y) \vee \alpha) = (\nu \vee \alpha)(x) \wedge (\nu \vee \alpha)(y)$, therefore by Theorem 3.7, $\nu \vee \alpha$ is a fuzzy filter of M. Since ν is not constant and $\alpha < \nu(1)$, we have $(\nu \vee \alpha)(1) = \nu(1) \vee \alpha = \nu(1) \neq (\nu \vee \alpha)(0)$, hence $\nu \vee \alpha$ is a non-constant fuzzy filter of M.

Moreover, $(\nu \vee \alpha)(1) = \nu(1)$ and $\nu \leq \nu \vee \alpha$, therefore by Theorem 4.5, $\nu \vee \alpha$ is a fuzzy prime filter of M. ☐

Theorem 4.8 *Let ν be a non-constant fuzzy filter of a pseudo BL-algebra M such that $\nu(1) \neq 1$. Then there is a fuzzy prime filter τ of M such that $\nu \leq \tau$.*

Proof. If ν is a non-constant fuzzy filter of M, then by Theorem 3.5, M_ν is a proper filter of M. Hence by [5, Theorem 4.28], there is a prime filter F of M such that $M_\nu \subseteq F$. Since F is a prime filter of M, χ_F is by Theorem 4.2, a fuzzy prime filter of M.

Denote $\alpha := \sup\{\nu(x) : x \in M \setminus F\}$ and suppose $\nu(1) \neq 1$. Then $\alpha \leq \nu(1) < 1$. Therefore by Theorem 4.7, $\tau = \chi_F \vee \alpha$ is a fuzzy prime filter of M satisfying $\nu \leq \tau$. ☐

5 Concluding remarks

The logical foundations of processes that handle various kinds of uncertainty in information use certain classes of algebras as algebraic semantics. Among others, the class of commutative bounded residuated lattice ordered monoids ($R\ell$-monoids) contains various classes of algebras behind the fuzzy logic, such as MV-algebras and BL-algebras, as well as the class of Heyting algebras, i.e. algebras of the intuitionistic logic. Hence commutative bounded $R\ell$-monoids can be taken as an algebraic counterpart of a logic which is a generalization of Hájek's basic fuzzy logic and the intuitionistic logic. Bounded $R\ell$-monoids, in which the multiplication (conjunction) need not be commutative, are generalizations of both commutative $R\ell$-monoids and pseudo BL-algebras, i.e. algebras of the non-commutative basic fuzzy logic (and consequently of pseudo MV-algebras = GMV-algebras, i.e. algebras of the non-commutative Łukasiewicz logic). Recall that non-commutative logics not only reflect the human reasoning which need not be strictly commutative, but they are also used in concurrent programming languages. The sets of provable formulas in corresponding inference systems are described by filters, and from the point of view of uncertain information, by fuzzy filters of algebraic semantics. Moreover, the properties of the sets of filters have a strong influence on the structure properties of bounded $R\ell$-monoids. In the paper we have described the fuzzy variants of filters and prime filters of bounded $R\ell$-monoids and, particularly, of pseudo BL-algebras.

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