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Monadic GMV-algebras

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Abstract Monadic MV–algebras are an algebraic model of the predicate calculus of the Łukasiewicz infinite valued logic in which only a single individual variable occurs. GMV-algebras are a non-commutative generalization of MV-algebras and are an algebraic counterpart of the non-commutative Łukasiewicz infinite valued logic. We introduce monadic GMV-algebras and describe their connections to certain couples of GMV-algebras and to left adjoint mappings of canonical embeddings of GMV-algebras. Furthermore, functional $MGMV$-algebras are studied and polyadic GMV-algebras are introduced and discussed.

Keywords MV-algebra · GMV-algebra · monadic MV-algebra · monadic GMV-algebra · quantifier · left adjoint mapping · polyadic GMV-algebra

Mathematics Subject Classification (2000) 03B50 · 06D35 · 06F05 · 06F15

1 Introduction

MV-algebras have been introduced by C. C. Chang in [3] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic. The first author in [18] and, independently, G. Georgescu and A. Iorgulescu in [7], have introduced non-commutative generalization of MV-algebras (non-commutative MV-algebras in [18] and pseudo MV-algebras in [7]) which are equivalent. We will use for these algebras the name *generalized MV-algebras*, briefly *GMV-algebras*. Recently, I. Leuştean in [14] has introduced the non-commutative Łukasiewicz infinite valued logic and GMV-algebras can be taken as an algebraic semantics of this propositional logic.

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Recall that an intensive development of the theory of MV-algebras was made possible by the fundamental result of D. Mundici in [15] that gave a representability of MV-algebras by means of intervals of unital abelian lattice ordered groups (ℓ-groups). A. Dvurečenskij in [6] has generalized this result also for GMV-algebras, i.e., he has proved that every GMV-algebra is isomorphic to a GMV-algebra introduced by the standard method on the unit interval of a unital (non-abelian, in general) ℓ-group.

Monadic MV-algebras (MMV-algebras) were introduced and studied in [20] as an algebraic model of the predicate calculus of the Łukasiewicz infinite valued logic in which only a single individual variable occurs. MMV-algebras were also studied as polyadic MV-algebras in [21] and [22]. Recently, the theory of MMV-algebras has been developed in [1], [4] and [8]. Recall that monadic, polyadic and cylindric (Boolean) algebras, as algebraic structures corresponding to classical predicate logic, have been investigated in [11] and [12]. Similar algebraic structures have been considered for various logics in [16] and [17].

In this paper we extend the notion of an MMV-algebra to an arbitrary GMV-algebra which need not be commutative. We obtain monadic GMV-algebras ($MGMV$-algebras) and then we define the monadic non-commutative Łukasiewicz propositional calculus $\mathcal{MPL}$ using the non-commutative Łukasiewicz propositional calculus $\mathcal{PL}$ from [14].

Recall that the language of $\mathcal{PL}$ is based on unary connectives $\neg$ and $\sim$, and on binary connectives $\to$ and $\rightsquigarrow$. Denote the set of all formulas of $\mathcal{PL}$ by $Form(\mathcal{PL})$. For any $\varphi \in Form(\mathcal{PL})$ define $\varphi^\bullet$ as follows:

(1) if φ is a propositional variable then $\varphi^\bullet$ is φ;
(2) if φ is $\neg\psi$ then $\varphi^\bullet$ is $\sim(\psi^\bullet)$;
(3) if φ is $\sim\psi$ then $\varphi^\bullet$ is $\neg(\psi^\bullet)$;
(4) if φ is $\psi \to \chi$ then $\varphi^\bullet$ is $\psi^\bullet \rightsquigarrow \chi^\bullet$;
(5) if φ is $\psi \rightsquigarrow \chi$ then $\varphi^\bullet$ is $\psi^\bullet \to \chi^\bullet$.

The axioms of $\mathcal{PL}$ are as follows:

I. for any $\varphi, \psi, \chi \in Form(\mathcal{PL})$,

(P1) $\varphi \to (\psi \to \varphi)$;
(P2) $(\varphi \to \psi) \to ((\psi \to \chi) \rightsquigarrow (\varphi \to \chi))$;
(P3) $((\varphi \to \psi) \rightsquigarrow \psi) \to ((\psi \to \varphi) \rightsquigarrow \varphi)$;
(P4) $(\neg\psi \rightsquigarrow \neg\varphi) \to (\varphi \to \psi)$;
(P5) $\sim(\varphi \to \neg\psi) \to \neg(\psi \rightsquigarrow \sim\varphi)$;

II. if φ is an axiom then $\varphi^\bullet$ is an axiom too.

The deductive rules of $\mathcal{PL}$ are two modus ponens:

(MP$\to$) $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$

(MP$\rightsquigarrow$) $\dfrac{\varphi,\ \varphi \rightsquigarrow \psi}{\psi}$

Further connectives derived from $\{\neg, \sim, \to, \rightsquigarrow\}$ are defined as follows:
$\varphi \oplus \psi$ is $\sim\varphi \to \psi$ (equivalently $\neg\psi \to \varphi$);
$\varphi \odot \psi$ is $\sim(\neg\varphi \oplus \neg\psi)$ (equivalently $\neg(\sim\varphi \oplus \sim\psi)$);
$\varphi \vee \psi$ is $\varphi \oplus (\psi \odot \sim\varphi)$ (equivalently $\psi \oplus (\varphi \odot \sim\psi)$, $(\neg\psi \odot \varphi) \oplus \psi$, $(\neg\varphi \odot \psi) \oplus \varphi$).

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The monadic non-commutative Łukasiewicz propositional calculus $\mathcal{MPL}$ is now the logic containing $\mathcal{PL}$ in which the following formulas are axioms for arbitrary formulas φ and ψ:

(M1) $\varphi \to \exists\varphi,\ \ \varphi \rightsquigarrow \exists\varphi$;
(M2) $\exists(\varphi \vee \psi) \equiv \exists\varphi \vee \exists\psi$;
(M3) $\exists(\neg\exists\varphi) \equiv \neg\exists\varphi,\ \ \exists(\sim\exists\varphi) \equiv\ \sim\exists\varphi$;
(M4) $\exists(\exists\varphi \oplus \exists\psi) \equiv \exists\varphi \oplus \exists\psi$;
(M5) $\exists(\varphi \oplus \varphi) \equiv \exists\varphi \oplus \exists\varphi$;
(M6) $\exists(\varphi \odot \varphi) \equiv \exists\varphi \odot \exists\varphi$.

Let $\forall\varphi$ mean $\sim(\exists(\neg\varphi))$. Then the deductive rules in $\mathcal{MPL}$ are two modus ponens (MP$\to$) and (MP$\rightsquigarrow$), and the necessitation

(Nec) $\dfrac{\varphi}{\forall\varphi}$.

Now, analogously as in [1], we will consider a first-order language L based on $\{\oplus, \odot, \to, \rightsquigarrow, \neg, \sim, \exists\}$ and a monadic propositional logic L_m based on $\{\oplus, \odot, \to, \rightsquigarrow, \neg, \sim, \exists\}$. Let x be a fixed variable in L. For any propositional variable p in L_m choose a monadic predicate $F_p(x)$ in L. Then we introduce the mapping $\Delta : Form(L_m) \longrightarrow Form(L)$ such that

(1) $\Delta(p) = F_p(x)$, for any propositional variable p;
(2) $\Delta(\varphi \circ \psi) = \Delta(\varphi) \circ \Delta(\psi)$, for any $\circ \in \{\oplus, \odot, \to, \rightsquigarrow, \neg, \sim\}$;
(3) $\Delta(\exists\varphi) = \exists x \Delta(\varphi)$.

It is obvious that Δ makes it possible to identify formulas of L_m and monadic formulas of L containing x.

We show that monadic GMV-algebras ($MGMV$-algebras) can be characterized, analogously as MMV-algebras, by means of certain couples of GMV-algebras and by means of left adjoint mappings of canonical embeddings of GMV–algebras. We introduce the notion of a functional monadic GMV-algebra and show that every such an algebra is an $MGMV$-algebra. Furthermore, we study connections between congruences and ideals of $MGMV$-algebras. Moreover, we introduce polyadic GMV-algebras (a generalization of $MGMV$-algebras) as special cases of polyadic (Λ, I)-algebras which should be developed in the future.

2 Preliminaries

Let $A = (A;\ \oplus, ^-, ^\sim, 0, 1)$ be an algebra of type $\langle 2, 1, 1, 0, 0\rangle$. Set $x \odot y := (x^- \oplus y^-)^\sim$ for any $x, y \in A$. Then A is called a *generalized MV-algebra* (briefly: *GMV-algebra*) if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = x = 0 \oplus x$;
(A3) $x \oplus 1 = 1 = 1 \oplus x$;
(A4) $1^- = 0 = 1^\sim$;
(A5) $(x^\sim \oplus y^\sim)^- = (x^- \oplus y^-)^\sim$;
(A6) $x \oplus (y \odot x^\sim) = y \oplus (x \odot y^\sim) = (y^- \odot x) \oplus y = (x^- \odot y) \oplus x$;
(A7) $(x^- \oplus y) \odot x = y \odot (x \oplus y^\sim)$;

Archive for mathematical logic. 2008, vol. 47, no. 3, p. 277-297. http://dx.doi.org/10.1007/s00153-008-0086-2

(A8) $x^{-\sim} = x$.

Proposition 1 *[7, Propositions 1.7 and 1.23] The following properties hold in any GMV-algebra:*

(1) $x \odot y = (x^{\sim} \oplus y^{\sim})^{-}$;
(2) $(x^{\sim})^{-} = x$;
(3) $0^{\sim} = 0^{-} = 1$;
(4) $x \odot 1 = 1 \odot x = x$,
(5) $(x \oplus y)^{-} = x^{-} \odot y^{-}$, $(x \oplus y)^{\sim} = x^{\sim} \odot y^{\sim}$;
(6) $(x \odot y)^{-} = x^{-} \oplus y^{-}$, $(x \odot y)^{\sim} = x^{\sim} \oplus y^{\sim}$;
(7) $x \oplus y = (x^{-} \odot y^{-})^{\sim} = (x^{\sim} \odot y^{\sim})^{-}$;
(8) $(x \wedge y)^{-} = x^{-} \vee y^{-}$, $(x \vee y)^{-} = x^{-} \wedge y^{-}$;
(9) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$, $(x \vee y)^{\sim} = x^{\sim} \wedge y^{\sim}$.

It is easily seen that the operations $\oplus$ and $\odot$ are mutually dual.

Proposition 2 *[7, Proposition 1.12] In every GMV-algebra the following properties hold:*

(1) $x \leq y \iff y^{-} \leq x^{-} \iff y^{\sim} \leq x^{\sim}$;
(2) $x \leq y \implies a \oplus x \leq a \oplus y, \quad x \oplus a \leq y \oplus a$;
(3) $x \leq y \implies a \odot x \leq a \odot y, \quad x \odot a \leq y \odot a$.

Proposition 3 *See [7] and [13].*

In any GMV-algebra, if the meets and joins on the left-hand side exist then so do those on the right-side and the following equalities hold:

$$a \oplus \left(\bigwedge_{i\in I} b_i\right) = \bigwedge_{i\in I}(a \oplus b_i), \quad \left(\bigwedge_{i\in I} b_i\right) \oplus a = \bigwedge_{i\in I}(b_i \oplus a),$$

$$a \odot \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a \odot b_i), \quad \left(\bigvee_{i\in I} b_i\right) \odot a = \bigvee_{i\in I}(b_i \odot a),$$

$$a \oplus \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a \oplus b_i), \quad \left(\bigvee_{i\in I} b_i\right) \oplus a = \bigvee_{i\in I}(b_i \oplus a),$$

$$a \odot \left(\bigwedge_{i\in I} b_i\right) = \bigwedge_{i\in I}(a \odot b_i), \quad \left(\bigwedge_{i\in I} b_i\right) \odot a = \bigwedge_{i\in I}(b_i \odot a),$$

$$a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \wedge b_i),$$

$$a \vee \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \vee b_i).$$

We define the operations $\oslash$ and $\odot$ as follows:

$$x \oslash y := x \odot y^{\sim}, \quad x \obslash y := y^{-} \odot x.$$

It is obvious that $x^{-} = 1 \obslash x$ and $x^{\sim} = 1 \oslash x$.

If we put $x \leq y$ if and only if $x^{-} \oplus y = 1$ then $L(A) = (A; \leq)$ is a bounded distributive lattice (0 is the least and 1 is the greatest element) with $x \vee y = x \oplus (y \odot x^{\sim})$ and $x \wedge y = x \odot (y \oplus x^{\sim})$.

GMV-algebras are in a close connection with unital ℓ-groups. (Recall that a unital ℓ-group is a pair (G, u) where G is an ℓ-group and u is a strong order unit of G.) If G is an ℓ-group, and $0 \leq u \in G$ then $\Gamma(G, u) = ([0, u]; \oplus, ^{-}, ^{\sim}, 0, u)$, where $[0, u] = \{x \in G : 0 \leq x \leq u\}$, and for any $x, y \in [0, u]$, $x \oplus y = (x + y) \wedge u$, $x^{-} = u - x$, $x^{\sim} = -x + u$, is a GMV-algebra.

Conversely, A. Dvurečenskij in [6] proved that every GMV-algebra is isomorphic to $\Gamma(G, u)$ for an appropriate unital ℓ-group (G, u). Moreover, the categories of GMV-algebras and unital ℓ-groups are by [6] equivalent.

3 Quantifiers on GMV–algebras

Let A be a GMV-algebra and $\exists : A \longrightarrow A$ be a mapping. Then $\exists$ is called an *existential quantifier* on A if the following identities are satisfied:

(E1) $x \leq \exists x$;
(E2) $\exists(x \vee y) = \exists x \vee \exists y$;
(E3) $\exists\left((\exists x)^{-}\right) = (\exists x)^{-}$, $\exists\left((\exists x)^{\sim}\right) = (\exists x)^{\sim}$;
(E4) $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$;
(E5) $\exists(x \odot x) = \exists x \odot \exists x$;
(E6) $\exists(x \oplus x) = \exists x \oplus \exists x$.

If A is a GMV-algebra and $\forall : A \longrightarrow A$ is a mapping then $\forall$ is called a *universal quantifier* on A if the following identities are satisfied:

(U1) $x \geq \forall x$;
(U2) $\forall(x \wedge y) = \forall x \wedge \forall y$;
(U3) $\forall((\forall x)^{-}) = (\forall x)^{-}$, $\forall((\forall x)^{\sim}) = (\forall x)^{\sim}$;
(U4) $\forall(\forall x \odot \forall y) = \forall x \odot \forall y$;
(U5) $\forall(x \odot x) = \forall x \odot \forall x$;
(U6) $\forall(x \oplus x) = \forall x \oplus \forall x$.

Lemma 1 *Let A be a GMV-algebra.*
(a) If $\exists$ is an existential quantifier on A then $\left(\exists x^{-}\right)^{\sim} = (\exists x^{\sim})^{-}$ for each $x \in A$.
(b) If $\forall$ is a universal quantifier on A then $\left(\forall x^{-}\right)^{\sim} = (\forall x^{\sim})^{-}$ for each $x \in A$.

Proof (a) Let $x \in A$. Then using (E1) and (E3) we obtain:

$$\begin{aligned} x^{-} \leq \exists x^{-} &\implies x^{-\sim} \geq (\exists x^{-})^{\sim} \implies x \geq (\exists x^{-})^{\sim} = \exists((\exists x^{-})^{\sim}) \\ &\implies x^{\sim} \leq (\exists((\exists x^{-})^{\sim}))^{\sim} \implies \exists x^{\sim} \leq \exists((\exists((\exists x^{-})^{\sim}))^{\sim}) \\ &\implies (\exists x^{\sim})^{-} \geq (\exists((\exists((\exists x^{-})^{\sim}))^{\sim}))^{-}. \end{aligned}$$

Put $y = (\exists x^{-})^{\sim}$. Then

$$(\exists x^{\sim})^{-} \geq (\exists((\exists y)^{\sim}))^{-} = ((\exists y)^{\sim})^{-} = \exists y = \exists((\exists x^{-})^{\sim}) = (\exists x^{-})^{\sim},$$

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therefore

$$(\exists x^{\sim})^{-} \geq (\exists x^{-})^{\sim}.$$

Analogously, we get $(\exists x^{-})^{\sim} \geq (\exists x^{\sim})^{-}$, that means $(\exists x^{-})^{\sim} = (\exists x^{\sim})^{-}$.

(b) Similarly, $(\forall x^{-})^{\sim} = (\forall x^{\sim})^{-}$ for every universal quantifier. □

Proposition 4 *If A is a GMV-algebra then there is a one-to-one correspondence between existential and universal quantifiers on A. Namely, if $\exists$ is an existential quantifier and $\forall$ is a universal one on A, then the mapping $\forall_{\exists} : A \longrightarrow A$ and $\exists_{\forall} : A \longrightarrow A$ such that for each $x \in A$,*

$$\forall_{\exists}\, x := (\exists x^{-})^{\sim} = (\exists x^{\sim})^{-}$$

and

$$\exists_{\forall}\, x := (\forall x^{-})^{\sim} = (\forall x^{\sim})^{-},$$

is a universal and an existential quantifier on A, respectively, and, moreover,

$$\exists_{(\forall_{\exists})} = \exists \quad \text{and} \quad \forall_{(\exists_{\forall})} = \forall.$$

Proof Let $\exists$ be an existential quantifier on A and let $\forall x = \forall_{\exists}\, x := (\exists x^{-})^{\sim} = (\exists x^{\sim})^{-}$. Let $x,\, y \in A$.

(U1) $x^{-} \leq \exists x^{-} \implies x^{-\sim} \geq (\exists x^{-})^{\sim} \implies x \geq \forall x$.

(U2) $\forall(x \wedge y) = (\exists(x \wedge y)^{-})^{\sim} = (\exists(x^{-} \vee y^{-}))^{\sim} = (\exists x^{-} \vee \exists y^{-})^{\sim} = (\exists x^{-})^{\sim} \wedge (\exists y^{-})^{\sim}$
$= \forall x \wedge \forall y$.

(U3) $\forall((\forall x)^{-}) = \forall((\exists x^{-})^{\sim -}) = \forall(\exists x^{-}) = (\exists(\exists x^{-})^{\sim}))^{-} = ((\exists x^{-})^{\sim})^{-} = (\exists x^{-})^{\sim -}$
$= (\forall x)^{-}$. The second identity analogously.

(U4) $\forall(\forall x \odot \forall y) = \forall((\exists x^{-})^{\sim} \odot (\exists y^{-})^{\sim}) = (\exists((\exists x^{-})^{\sim} \odot) \exists y^{-})^{\sim}))^{-})^{\sim}$
$= (\exists((\exists x^{-})^{\sim -} \oplus (\exists y^{-})^{\sim -}))^{\sim} = (\exists(\exists x^{-} \oplus \exists y^{-}))^{\sim} = (\exists x^{-} \oplus \exists y^{-})^{\sim}$
$= (\exists x^{-})^{\sim} \odot (\exists y^{-})^{\sim} = \forall x \odot \forall y$.

(U5) $\forall(x \odot x) = (\exists((x \odot x)^{-}))^{\sim} = (\exists(x^{-} \oplus x^{-}))^{\sim} = (\exists x^{-} \oplus \exists x^{-})^{\sim} = (\exists x^{-})^{\sim} \odot (\exists x^{-})^{\sim}$
$= \forall x \odot \forall x$.

(U6) $\forall(x \oplus x) = (\exists((x \oplus x)^{-}))^{\sim} = (\exists(x^{-} \odot x^{-}))^{\sim} = (\exists x^{-} \odot \exists x^{-})^{\sim} = (\exists x^{-})^{\sim} \oplus (\exists x^{-})^{\sim}$
$= \forall x \oplus \forall x$.

The proof of the remaining assertions is now obvious. □

As a consequence of Proposition 4, it will be sufficient to investigate only one from these kinds of quantifiers, e.g. the existential ones.

If A is a GMV-algebra and $\exists$ is an existential quantifier on A then the couple $(A, \exists)$ is called a *monadic GMV-algebra* (an *MGMV-algebra,* in brief).

In the following proposition we will prove some useful properties of $MGMV$-algebras.

Proposition 5 *Let $(A, \exists)$ be an MGMV-algebra and $x,\, y \in A$. Then the following conditions are satisfied.*

(1) $\exists 1 = 1$;
(2) $\exists 0 = 0$;
(3) $\exists(\exists x) = \exists x$;
(4) $\exists(\exists x \odot \exists y) = \exists x \odot \exists y$;

Archive for mathematical logic. 2008, vol. 47, no. 3, p. 277-297. http://dx.doi.org/10.1007/s00153-008-0086-2

(5) $x \leq \exists y \iff \exists x \leq \exists y$;
(6) $x \leq y \implies \exists x \leq \exists y$.
(7) $\exists(x \oplus y) \leq \exists x \oplus \exists y$;
(8) $\exists(x \odot y) \leq \exists x \odot \exists y$;
(9) $(\exists x)^- \odot \exists y \leq \exists(x^- \odot \exists y)$, $(\exists x)^\sim \odot \exists y \leq \exists(x^\sim \odot \exists y)$,
$\exists y \odot (\exists x)^- \leq \exists(\exists y \odot x^-)$, $\exists y \odot (\exists x)^\sim \leq \exists(\exists y \odot x^\sim)$;
(10) $\exists(x \oslash y) \geq \exists x \oslash \exists y$, $\exists(x \oslash y) \geq \exists x \oslash \exists y$;
(11) $\exists x^- \geq (\exists x)^-$, $\exists x^\sim \geq (\exists x)^\sim$.

Proof (1) It is obvious.
(2) $0 = 1^- = (\exists 1)^- = \exists((\exists 1)^-) = \exists(1^-) = \exists 0$.
(3) $\exists(\exists x) = \exists(0 \oplus \exists x) = \exists(\exists 0 \oplus \exists x) = \exists 0 \oplus \exists x = 0 \oplus \exists x = \exists x$.
(4) $\exists x \odot \exists y = ((\exists x)^- \oplus (\exists y)^-)^\sim = (\exists((\exists x)^-) \oplus \exists((\exists y)^-))^\sim = (\exists(\exists((\exists x)^- \oplus \exists((\exists y)^-)))^\sim = \exists((\exists(\exists((\exists x)^-) \oplus \exists((\exists y)^-))^\sim) = \exists((\exists((\exists x)^-) \oplus \exists((\exists y)^-))^\sim) = \exists(((\exists x)^- \oplus (\exists y)^-)^\sim)$
$= \exists(\exists x \odot \exists y)$.
(5) $x \leq \exists y \iff x \vee \exists y = \exists y \implies \exists y = \exists(x \vee \exists y) = \exists x \vee \exists y \implies \exists x \leq \exists y$,
$\exists x \leq \exists y \implies x \leq \exists x \leq \exists y$.
(6) $x \leq y \implies y = x \vee y \implies \exists y = \exists(x \vee y) = \exists x \vee \exists y \implies \exists x \leq \exists y$.
(7) $x \oplus y \leq \exists x \oplus \exists y \implies \exists(x \oplus y) \leq \exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$.
(8) $x \odot y \leq \exists x \odot \exists y \implies \exists(x \odot y) \leq \exists(\exists x \odot \exists y) = \exists x \odot \exists y$.
(9) $(\exists x)^- \odot \exists y \leq x^- \odot \exists y \leq \exists(x^- \odot \exists y)$.
(10) $(x \oslash y) \oplus y = x \vee y \geq x \implies \exists x \leq \exists((x \oslash y) \oplus y) \leq \exists(x \oslash y) \oplus \exists y \implies \exists(x \oslash y) \geq \exists x \oslash \exists y$.
(11) $\exists x^- = \exists(1 \oslash x) \geq \exists 1 \oslash \exists x = 1 \oslash \exists x = (\exists x)^-$.

□

Remark 1 By (E1) and properties (3) and (6) of Proposition 5, every existential quantifier on an $MGMV$-algebra A is a closure operator on the lattice $L(A)$. Dually, every universal quantifier on A is an interior operator on $L(A)$.

4 Functional monadic GMV-algebras

Functional monadic Boolean algebras have been introduced and investigated in [11]. Since their elements are functions (mappings) from their domains to their value Boolean algebras and existential and universal quantifiers are suprema and infima, respectively, of their ranges, functional monadic Boolean algebras give a visualization of the general notion of monadic Boolean algebras.

In this section we introduce analogously functional monadic GMV-algebras and show that they are special instances of monadic GMV-algebras.

Let M be a GMV-algebra and X be a non-empty set. Denote by M^X the set of all functions (mappings) from X into M. Then M^X forms, with respect to the pointwise operations, also a GMV-algebra (a direct power of the GMV-algebra M). It is obvious that M^X contains the GMV-subalgebra of constant functions which is isomorphic to M.

For any $p \in M^X$ denote by $R(p) := \{p(x) : x \in X\}$ the range of p. We want to obtain existential and universal quantifiers by means of suprema and infima, respectively, of ranges of functions. But the underlying lattice $(M; \vee, \wedge)$ of a GMV-algebra M need not be complete. In fact, if $M = \Gamma(G, u)$, where (G, u) is a unital ℓ-group, then by [6], $(M; \vee, \wedge)$ is a complete lattice if and only if G is a complete ℓ-group. In

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particular, every complete ℓ-group is still commutative (see for instance [9]), therefore every non-commutative GMV-algebra is not complete.

Hence we will consider any GMV-subalgebra A of the GMV-algebra M^X satisfying following conditions:

(i) for every $p \in A$ there exist $\sup_M R(p) = \bigvee R(p)$, $\inf_M R(p) = \bigwedge R(p)$;
(ii) for every $p \in A$, the constant functions $\exists p$ and $\forall p$ defined such that

$$\exists p(x) := \bigvee R(p), \quad \forall p(x) := \bigwedge R(p),$$

for any $x \in X$, belong to A.

Every such a subalgebra A of M^X is called a *functional monadic GMV-algebra*.

We will use Proposition 3 in the following proofs without further notice.

Lemma 2 *In any functional monadic GMV-algebra A ($p, q \in A$), the following is valid:*

(1) $\exists 0 = 0$.
(2) $p \leq \exists p$.
(3) $\exists(p \wedge \exists q) = \exists p \wedge \exists q$.
(4) $\exists 1 = 1$.
(5) $\exists\exists p = \exists p$.
(6) $p \in \exists A = \{\exists p : p \in A\}$ *if and only if* $\exists p = p$.
(7) *If* $p \leq \exists q$ *then* $\exists p \leq \exists q$.
(8) *If* $p \leq q$ *then* $\exists p \leq \exists q$.
(9) $\exists((\exists p)^-) = (\exists p)^-$, $\exists((\exists p)^\sim) = (\exists p)^\sim$.
(10) $\exists(\exists p \oplus \exists q) = \exists p \oplus \exists q$.
(11) $\exists A$ *is a subalgebra of the GMV-algebra A.*
(12) $\exists(p \vee q) = \exists p \vee \exists q$.
(13) $\exists(p \oplus p) = \exists p \oplus \exists p$.
(14) $\exists(p \odot p) = \exists p \odot \exists p$.

Proof (1) $R(0) = \{0\}$, therefore $\exists 0 = \bigvee R(0) = 0$.
(2) For every $x \in X$, $p(x) \leq \bigvee R(p)$, hence $p \leq \exists p$.
(3) It holds that $p \wedge \exists q = p \wedge \bigvee R(q) = \bigvee\limits_{a \in R(q)} (p \wedge a)$, so

$$\exists(p \wedge \exists q) = \exists(\bigvee_{a \in R(q)} (p \wedge a)) = \bigvee_{b \in R(p)} \bigvee_{a \in R(q)} (b \wedge a).$$

Further,

$$\exists p \wedge \exists q = \bigvee R(p) \wedge \bigvee R(q) = \bigvee_{b \in R(p)} \bigvee_{a \in R(q)} (b \wedge a).$$

Therefore $\exists(p \wedge \exists q) = \exists p \wedge \exists q$.
(4) By (2).
(5) By (3) and (4), we have $\exists(1 \wedge \exists p) = \exists 1 \wedge \exists p = 1 \wedge \exists p$, hence $\exists\exists p = \exists p$.
(6) If $p = \exists q \in \exists A$ then $\exists p = \exists\exists q = \exists q$, from this $\exists p = p$.
The converse implication is obvious.
(7) Let $p \leq \exists q$. Then $p = p \wedge \exists q$ and by (3), $\exists p = \exists(p \wedge \exists q) = \exists p \wedge \exists q$, consequently, $\exists p \leq \exists q$.
(8) If $p \leq q$ then $p \leq \exists q$ and so by (7), $\exists p \leq \exists q$.

Archive for mathematical logic. 2008, vol. 47, no. 3, p. 277-297. http://dx.doi.org/10.1007/s00153-008-0086-2

(9) $(\exists p)^- = (\bigvee R(p))^- = (\bigvee_{x\in X} p(x))^- = \bigwedge_{x\in X} (p(x))^-$, therefore $R((\exists p)^-) = \{\bigwedge_{x\in X} (p(x))^-\}$.
Hence $\exists((\exists p)^-) = \bigvee R((\exists p)^-) = \bigvee\{\bigwedge_{x\in X} (p(x))^-\} = \bigwedge_{x\in X} (p(x))^-$.
From this we obtain $\exists((\exists p)^-) = (\exists p)^-$.
The proof of the other equality is analogous.

(10) $\exists p \oplus \exists q = \bigvee R(p) \oplus \bigvee R(q) = \bigvee_{a\in R(p)} \bigvee_{b\in R(q)} (a \oplus b)$,
$\exists(\exists p \oplus \exists q) = \exists(\bigvee_{a\in R(p)} \bigvee_{b\in R(q)} (a \oplus b)) = \bigvee_{a\in R(p)} \bigvee_{b\in R(q)} (a \oplus b)$.

(11) $0 \in \exists A$, therefore $\exists A \neq \emptyset$.
If $p, q \in \exists A$ then by (6), (10), $\exists(p\oplus q) = \exists(\exists p\oplus\exists q) = \exists p\oplus\exists q = p\oplus q$, so $p\oplus q \in \exists A$.
Let $p \in \exists A$. Then by (9), $\exists p^- = \exists((\exists p)^-) = (\exists p)^- = p^-$, similarly, $\exists p^\sim = p^\sim$.
Hence $\exists A$ is a subalgebra of A.

(12) By (8), $\exists p \vee \exists q \leq \exists(p \vee q)$. Conversely, $\exists p, \exists q \in \exists A$, hence by (11), $\exists p \vee \exists q \in \exists A$, it entails $\exists(\exists p \vee \exists q) = \exists p \vee \exists q$.
Further, $p \vee q \leq \exists p \vee \exists q$, hence by (8), $\exists(p \vee q) \leq \exists(\exists p \vee \exists q) = \exists p \vee \exists q$.

(13) $\exists p\oplus\exists p = \bigvee R(p)\oplus\bigvee R(p) = \bigvee_{a\in R(p)} \bigvee_{b\in R(p)} (a\oplus b) = \bigvee_{a,b\in R(p)} (a\oplus b) = \bigvee_{c\in R(p)} (c\oplus c) = \bigvee R(p \oplus p) = \exists(p \oplus p)$.

(14) It is analogous to the proof of (13).

$\square$

Theorem 1 *If M is a GMV-algebra, X is a non-empty set and $A \subseteq M^X$ is a functional monadic GMV-algebra, then $(A, \exists)$ is a monadic GMV-algebra.*

Proof It follows from (2), (12), (9), (10), (14) and (13). $\square$

5 Quantifiers, relatively complete subalgebras and left adjoint mappings of GMV–algebras

In this section we study connections among quantifiers, infima and suprema in certain subalgebras and left adjoint mappings to canonical embeddings of GMV-algebras. The results of the section extend those of Section 3 of [4] to the non-commutative case and some from proofs are very similar to their commutative originals in [4].

If $(A, \exists)$ is an $MGMV$-algebra, put

$$\exists A := \{x \in A : x = \exists x\}.$$

Proposition 6 *If $(A, \exists)$ is an $MGMV$-algebra then $\exists A$ is a subalgebra of the GMV-algebra A.*

Proof If $x, y \in \exists A$ then $\exists(x \oplus y) = \exists(\exists x \oplus \exists y) = \exists x \oplus \exists y = x \oplus y$, thus $x \oplus y \in \exists A$.
Let $x \in \exists A$. Then $x^- = (\exists x)^- = \exists((\exists x)^-) = \exists x^-$, hence $x^- \in \exists A$.
Analogously, $x \in \exists A$ implies $x^\sim \in \exists A$.
Moreover, $0, 1 \in \exists A$. $\square$

Archive for mathematical logic. 2008, vol. 47, no. 3, p. 277-297. http://dx.doi.org/10.1007/s00153-008-0086-2

Let A be a GMV-algebra and B be its subalgebra. Then B is called *relatively complete* if for each element $a \in A$, the set $\{b \in B : a \leq b\}$ has a least element, denoted by $\inf\{b \in B : a \leq b\}$, or by $\bigwedge_{a \leq b \in B} b$.

A subalgebra B of a GMV-algebra A is called *m-relatively complete* if it is relatively complete and satisfies the following conditions:

(MRC1) For every $a \in A$ and $x \in B$ such that $x \geq a \odot a$ there is an element $v \in B$ such that $v \geq a$ and $v \odot v \leq x$.

(MRC2) For every $a \in A$ and $x \in B$ such that $x \geq a \oplus a$ there is an element $v \in B$ such that $v \geq a$ and $v \oplus v \leq x$.

Proposition 7 *If $(A, \exists)$ is an $MGMV$-algebra then $\exists A$ is an m-relatively complete subalgebra of the GMV-algebra A.*

Proof Let $a \in A$ and $x \in \exists A$. Then $a \leq x = \exists x$ if and only if $\exists a \leq \exists x = x$. We have $\exists a \in \exists A$, hence $\exists a = \inf\{x \in \exists A : a \leq x\}$, and therefore $\exists A$ is relatively complete. Let $a \in A$, $x \in \exists A$ and let $x \geq a \odot a$. Then $x = \exists x \geq \exists(a \odot a) = \exists a \odot \exists a$, hence for $v = \exists a$, the condition (MRC1) is satisfied.

Similarly, $v = \exists a$ also satisfies the condition (MRC2). □

Lemma 3 *Let A be a GMV-algebra and $x_i \in A$, $i \in I$. If $\bigwedge_{i \in I} x_i$ in A exists then $\bigvee_{i \in I} x_i^-$ and $\bigvee_{i \in I} x_i^\sim$ exist too, and*

a) $$\left(\bigwedge_{i \in i} x_i\right)^- = \bigvee_{i \in I} x_i^- ,$$

b) $$\left(\bigwedge_{i \in i} x_i\right)^\sim = \bigvee_{i \in I} x_i^\sim .$$

Proof Let A be a GMV-algebra and let $A = \Gamma(G, u)$, where (G, u) is a unital ℓ-group.

a) Suppose that $x_i \in A$, $i \in I$, and that $\bigwedge_{i \in I} x_i$ exists.

Let $j \in I$. Then in G we have $u - \left(\bigwedge_{i \in I} x_i\right) \geq u - x_j$, hence $\left(\bigwedge_{i \in I} x_i\right)^- \geq x_j^-$.

Let $z \in A$ be such that $z \geq x_j^-$ for every $j \in I$. Then $z \geq u - x_j$ for each $j \in I$, hence $-z + u \leq \bigwedge_{i \in I} x_i$, thus $u - \bigwedge_{i \in I} x_i \leq z$, that means $\left(\bigwedge_{i \in I} x_i\right)^- \leq z$. Therefore in A we get $\left(\bigwedge_{i \in I} x_i\right)^- = \bigvee_{i \in I} x_i^-$.

b) Analogously the second equality. □

Theorem 2 *There exists a one-to-one correspondence between $MGMV$-algebras and pairs (A, B), where B is an m-relatively complete subalgebra of a GMV-algebra A.*

Archive for mathematical logic. 2008, vol. 47, no. 3, p. 277-297. http://dx.doi.org/10.1007/s00153-008-0086-2

Proof a) Let $(A, \exists)$ be an $MGMV$-algebra. Then the m-relatively complete subalgebra $B = \exists A$ is uniquely determined by $\exists$.

b) Conversely, let B be an m-relatively complete subalgebra of a GMV-algebra A. Denote by $\exists = \exists_B : A \longrightarrow A$ the mapping such that

$$\exists a := \inf\{b \in B : a \le b\} = \bigwedge_{a \le b \in B} b.$$

We will show that this uniquely determined mapping is an existential quantifier on A.

(E1) Obvious.

(E2) Let $a, b \in A$. Then

$$\exists(a \vee b) = \bigwedge_{a \vee b \le x \in B} x = \bigwedge_{a \le y \in B} y \vee \bigwedge_{b \le z \in B} z = \exists a \vee \exists b.$$

(E3) Let $a \in A$. Then

$$\exists((\exists a)^-) = \exists\left(\left(\bigwedge_{a \le x \in B} x\right)^-\right) = \exists\left(\bigvee_{a \le x \in B} x^-\right) = \bigvee_{a \le x \in B} x^- = \left(\bigwedge_{a \le x \in B} x\right)^- = (\exists a)^-.$$

Analogously $\exists((\exists a)^\sim) = (\exists a)^\sim$.

(E4) Let $a, b \in A$. Then

$$\exists(\exists a \oplus \exists b) = \exists\left(\bigwedge_{a \le x \in B} x \oplus \bigwedge_{b \le y \in B} y\right) = \bigwedge_{a \le x \in B} x \oplus \bigwedge_{b \le y \in B} y = \exists a \oplus \exists b.$$

(E5) Let $a \in A$. Then

$$\exists a \odot \exists a = \bigwedge_{a \le x \in B} x \odot \bigwedge_{a \le y \in B} y = \bigwedge_{a \le x \in B} \bigwedge_{a \le y \in B} (x \odot y).$$

Since $a \le x$ and $a \le y$, $a \odot a \le x \odot y$, hence by (MRC1) there is $v \in B$ such that $v \ge a$ and $v \odot v \le y$. Thus

$$\bigwedge_{a \le x \in B} \bigwedge_{a \le y \in B} (x \odot y) = \bigwedge_{a \le v \in B} \bigwedge_{a \le v \in B} (v \odot v) = \bigwedge_{a \le v \in B} (v \odot v) = \bigwedge_{a \odot a \le v \odot v} (v \odot v).$$

Let $t \in B$ be such that $t \ge a \odot a$. Then by (MRC1), there is $w \in B$ such that $w \ge a$ and $w \odot w \le t$. Thus

$$\bigwedge_{a \odot a \le w \odot w \in B} (w \odot w) = \bigwedge_{a \odot a \le t \in B} t = \exists(a \odot a).$$

Therefore $\exists a \odot \exists a = \exists(a \odot a)$.

(E6) Let $a \in A$. Then

$$\exists a \oplus \exists a = \bigwedge_{a \le x \in B} x \oplus \bigwedge_{a \le y \in B} y = \bigwedge_{a \le x \in B} \bigwedge_{a \le y \in B} (x \oplus y).$$

Since $a \le x$ and $a \le y$, $a \oplus a \le x \oplus y$, therefore by (MRC2), there is $v \in B$ such that $v \ge a$ and $x \oplus y \ge v \oplus v$. Hence

$$\bigwedge_{a \le x \in B} \bigwedge_{a \le y \in B} (x \oplus y) = \bigwedge_{a \oplus a \le x \oplus y \in B} (x \oplus y) = \bigwedge_{a \oplus a \le v \oplus v \in B} (v \oplus v).$$

Archive for mathematical logic. 2008, vol. 47, no. 3, p. 277-297. http://dx.doi.org/10.1007/s00153-008-0086-2

Thus

$$\bigwedge_{a\oplus a\leq v\oplus v\,\in B} (v\oplus v) = \bigwedge_{a\oplus a\leq u\,\in B} u = \exists(a\oplus a).$$

Therefore $\exists a \oplus \exists a = \exists(a\oplus a)$. □

Let A be a GMV-algebra, B a subalgebra of A and $h : B \longrightarrow A$ a mapping. Then a mapping $\exists_h : A \longrightarrow B$ is called a *left adjoint mapping* to h if

$$\exists_h(a) \leq x \iff a \leq h(x)$$

for each $a \in A$ and $x \in B$.

If $\exists_h$, moreover, satisfies the identities

$$\exists_h(a\odot a) = \exists_h(a) \odot \exists_h(a),$$

$$\exists_h(a\oplus a) = \exists_h(a) \oplus \exists_h(a),$$

then $\exists_h$ is called a *left m-adjoint mapping* to h.

Theorem 3 *There is a one-to-one correspondence between pairs (A, B), where B is an m-relatively complete subalgebra of a GMV-algebra A, and pairs (A, B), where B is a subalgebra of a GMV-algebra A such that the canonical embedding $h : B \hookrightarrow A$ has a left m-adjoint mapping.*

Proof a) Let B be an m-relatively complete subalgebra of a GMV-algebra A and $h : B \hookrightarrow A$ be the canonical embedding. Put

$$\exists_h(a) := \bigwedge_{a\leq x\,\in B} x$$

for every $a \in A$. Then $\exists_h(a) \leq x$ if and only if $a \leq x = h(x)$, hence $\exists_h$ is a left adjoint mapping to the mapping h. Moreover, $\exists_h(a\odot a) = \exists_h(a) \odot \exists_h(a)$ and $\exists_h(a\oplus a) = \exists_h(a) \oplus \exists_h(a)$, therefore $\exists_h$ is a left m-adjoint mapping to h.

b) Let A be a GMV-algebra and B be a subalgebra of A such that the canonical embedding $h : B \hookrightarrow A$ has a left m-adjoint mapping $\exists_h$. Put $\exists a := h\circ\exists_h(a)$ for each $a \in A$. For any $x \in B$ we have $\exists_h(a) \leq x$ if and only if $a \leq x$, thus $\exists a$ is a least element $x \in B$ such that $a \leq x$, that means $\exists a = \bigwedge_{a\leq x\,\in B} x$. Hence B is a relatively complete subalgebra of A. We will prove that it is also m-relatively complete.

Firstly we will show that the mapping $\exists = \exists_h$ is isotone. Let $a, b \in A$, $a \leq b$. Then

$$\exists_h a = \bigwedge_{a\leq x\,\in B} x \leq \bigwedge_{b\leq y\,\in B} y = \exists_h b.$$

Now, let $a \in A$, $x \in B$ and $x \geq a\odot a$. Put $v = \exists_h a$. Then $v \geq a$ and

$$x = \exists_h x \geq \exists_h(a\odot a) = \exists_h a \odot \exists_h a = v\odot v.$$

Further, let $a \in A$, $x \in B$ and $x \geq a\oplus a$. Then for $v = \exists_h a$ we have $v \geq a$ and

$$x = \exists_h x \geq \exists_h(a\oplus a) = \exists_h a \oplus \exists_h a = v\oplus v.$$

Hence B is m-relatively complete. □

Archive for mathematical logic. 2008, vol. 47, no. 3, p. 277-297. http://dx.doi.org/10.1007/s00153-008-0086-2

The following theorem is an immediate consequence of Theorems 2 and 3.

Theorem 4 *There are one-to-one correspondences among*

1. *$MGMV$-algebras;*
2. *pairs (A, B), where B is an m-relatively complete subalgebra of a GMV-algebra A;*
3. *pairs (A, B), where B is a subalgebra of a GMV-algebra A such that the canonical embedding $h : B \hookrightarrow A$ has a left m-adjoint mapping.*

Now, let us denote by $\mathcal{MGMV}$ the category of $MGMV$-algebras in which morphisms are homomorphisms f of GMV-algebras satisfying the condition $f(\exists a) = \exists f(a)$.

Further, denote by $\mathcal{GMV}^2$ the category in which objects are pairs (A, B_A) of GMV-algebras such that an injective GMV-homomorphism $h : B_A \hookrightarrow A$ has a left m-adjoint mapping $\exists_h$, and morphisms are pairs of mappings $(f, f_B) : (A, B_A) \longrightarrow (A', B_{A'})$ such that

(1) $f : A \longrightarrow A'$ is a GMV-homomorphism;
(2) $f \circ h = h' \circ f_B$;
(3) $f_B \circ \exists_h = \exists_{h'} \circ f$,

where $h' : B_{A'} \hookrightarrow A'$ is an injective GMV-homomorphism having a left m-adjoint mapping $\exists_{h'}$.

From injectivity of h' and from (1) and (2) it follows that f_B is a GMV-homomorphism.

Theorem 5 *The categories $\mathcal{GMV}^2$ and $\mathcal{MGMV}$ are equivalent.*

Proof If $(A, \exists)$ is an $MGMV$-algebra, put $\Phi(A) = (A, B_A)$, where $B_A = \exists A$, and if f is an $MGMV$-homomorphism of an $MGMV$-algebra $A = (A, \exists)$ into an $MGMV$-algebra $A' = (A', \exists)$, put $\Phi(f) = (f, f|B_A)$.

Conversely, if (A, B_A) is an object in $\mathcal{GMV}^2$, put $\Psi(A, B_A) = (A, \exists)$, where $\exists = h \circ \exists_h$, and if (f, f_{B_A}) is a morphism in $\mathcal{GMV}^2$ of (A, B_A) into $(A', B_{A'})$, put $\Psi(f, f_{B_A}) = f$.

Then $\Phi : \mathcal{MGMV} \longrightarrow \mathcal{GMV}^2$ and $\Psi : \mathcal{GMV}^2 \longrightarrow \mathcal{MGMV}$ are functors which give the equivalence between $\mathcal{MGMV}$ and $\mathcal{GMV}^2$. □

The following theorem is the non-commutative generalization of [1, Theorem 3.1] and it gives the possibility of introducing of quantifiers on certain GMV-algebras.

Theorem 6 *Let L be a linearly ordered GMV-algebra, $n \in \mathbb{N}$ and $D = \{\langle a, \ldots, a\rangle : a \in L\}$ be the diagonal subalgebra of a direct power L^n. Let A be a subalgebra of the GMV-algebra L^n containing D. Then there exists an existential quantifier $\exists$ on A such that $\exists A = D \cong L$ holds in the $MGMV$-algebra $(A, \exists)$.*

Proof Let A be a subalgebra of a GMV-algebra L^n such that $D \subseteq A$. For any $a = \langle a_1, \ldots, a_n\rangle \in L^n$, we put $\exists a := \langle c, \ldots, c\rangle$, where $c = \max\{a_1, \ldots, a_n\}$. Then $\exists a \in D$ and $\exists a = a$ if and only if $a \in D$.

The axioms (E1)–(E6) can be verified analogously as in [1]. □

Example 1 Let G be the group of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \text{ where } a, b \in \mathbb{R}, \ a > 0,$$

Archive for mathematical logic. 2008, vol. 47, no. 3, p. 277-297. http://dx.doi.org/10.1007/s00153-008-0086-2

and where the group binary operation is the common multiplication of matrices. Set

$$(a,b) := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Then $(a,b)^{-1} = \left(\frac{1}{a}, -\frac{b}{a}\right)$ and $(1,0)$ is the neutral element. For any $(a,b),\ (c,d) \in G$ we put

$$(a,b) \le (c,d) \ :\Longleftrightarrow\ a < c \ \text{ or } \ a = c,\ b \le d.$$

Then by [5], $G = (G, \le)$ is a linearly ordered (non-commutative) group in which for the positive cone G^+ it is satisfied $G^+ = \{(a,b) :\ a > 1 \ \text{ or } \ a = 1,\ b \ge 0\}$ and e.g. $u = (2,0)$ is its strong order unit. Hence by [6], $A = \Gamma(G,u)$ is a linearly ordered non-commutative GMV-algebra in which among others it holds

$(a,b) \oplus (c,d) = (\min(ac,2),\ \min(ad+b,\ 0))$,

$(a,b)^- = \left(\frac{2}{a}, -\frac{2b}{a}\right)$,

$(a,b)^\sim = \left(\frac{2}{a}, -\frac{b}{a}\right)$.

Let us now consider the (non-commutative) GMV-algebra $M = A^2$. For any $((a,b),\ (c,d)) \in M$ we put $\exists((a,b),\ (c,d)) = (\max\{(a,b),(c,d)\}, \max\{(a,b),(c,d)\})$. Then by the previous theorem, we obtain that $\exists : M \longrightarrow M$ is an existential quantifier on the non-commutative GMV-algebra M and, moreover, $\exists M$ is isomorphic with A.

6 Ideals and congruences of monadic GMV–algebras

Let A be a GMV-algebra and $\emptyset \ne I \subseteq A$. Then I is called an *ideal of* A if the following conditions are satisfied:

(I1) if $x,\ y \in I$ then $x \oplus y \in I$;
(I2) if $x \in I,\ y \in A$ and $y \le x$ then $y \in I$.

If $X \subseteq A$, denote by $\mathrm{Id}(X)$ the ideal of A generated by X. For $X = \emptyset$, we have $\mathrm{Id}(\emptyset) = \{0\}$.

Let $1 \cdot x = x$ and $(n+1)x = nx \oplus x$ for any $x \in A$ and $n \in \mathbb{N}$. Then, by [7], for any ideal I of a GMV-algebra A and each $a \in A$ we have

$\mathrm{Id}(I \cup \{a\}) = \{x \in A :\ x \le (b_1 \oplus n_1 a) \oplus (b_2 \oplus n_2 a) \oplus \cdots \oplus (b_m \oplus n_m a)$ for some $m \in \mathbb{N},\ b_1, \ldots, b_m \in I,\ n_1, \ldots, n_m \in \mathbb{N}_0\}$.

The set $\mathcal{I}(A)$ of all ideals in a GMV-algebra A ordered by set inclusion is a complete lattice (a Brouwerian lattice, moreover).

Let $(A, \exists)$ be an $MGMV$-algebra and let I be an ideal of the GMV-algebra A. Then I is called a *monadic ideal* (in short: *m-ideal*) of $(A, \exists)$ if the following condition is valid:

$$x \in I \implies \exists x \in I.$$

Proposition 8 *If* $(A, \exists)$ *is an* $MGMV$*-algebra,* $a \in \exists A$ *and* I *is an* m*-ideal of* $(A, \exists)$ *then* $\mathrm{Id}(\{a\} \cup I)$ *is also an* m*-ideal of* $(A, \exists)$.

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Proof If $x \in \mathrm{Id}(\{a\} \cup I)$ then there are $m \in \mathbb{N}$, $b_1, \ldots, b_m \in I$, $n_1, \ldots, n_m \in \mathbb{N}_0$ such that $x \leq b_1 \oplus n_1 a \oplus b_2 \oplus n_2 a \oplus \cdots \oplus b_m \oplus n_m a$. Hence

$$\begin{aligned}\exists x &\leq \exists(b_1 \oplus n_1 a \oplus b_2 \oplus n_2 a \oplus \cdots \oplus b_m \oplus n_m a)\\ &\leq \exists b_1 \oplus n_1 \exists a \oplus \exists b_2 \oplus n_2 \exists a \oplus \cdots \oplus \exists b_m \oplus n_m \exists a\\ &= \exists b_1 \oplus n_1 a \oplus \exists b_2 \oplus n_2 a \oplus \cdots \oplus \exists b_m \oplus n_m a,\end{aligned}$$

thus $\exists x \in \mathrm{Id}(\{a\} \cup I)$, and therefore $\mathrm{Id}(\{a\} \cup I)$ is an m-ideal of $(A, \exists)$. □

Proposition 9 *If $(A, \exists)$ is an MGMV-algebra and $I \in \mathcal{I}(A)$ then I is an m-ideal of $(A, \exists)$ if and only if $I = \mathrm{Id}(I \cap \exists A)$.*

Proof Let I be an m-ideal. If $a \in I$ then $\exists a \in I$, and thus $\exists a \in I \cap \exists A$. Since $a \leq \exists a$, we have $a \in \mathrm{Id}(I \cap \exists A)$.

Conversely, if $a \in \mathrm{Id}(I \cap \exists A)$ then $a \leq b$ for some $b \in I \cap \exists A$, hence $a \leq \exists a \leq \exists b = b$, and so $a \in I$.

Therefore for every m-ideal I of $(A, \exists)$ we get $I = \mathrm{Id}(I \cap \exists A)$.

Let now $I \in \mathcal{I}(A)$ be such that $I = \mathrm{Id}(I \cap \exists A)$. If $a \in I$ then $a \leq b_1 \oplus \cdots \oplus b_n$, where $n \in \mathbb{N}$ and $b_1, \ldots, b_n \in I \cap \exists A$. From this, $\exists a \leq \exists(b_1 \oplus \cdots \oplus b_n) \leq \exists b_1 \oplus \cdots \oplus \exists b_n = b_1 \oplus \cdots \oplus b_n$, thus $\exists a \in I$. That means, I is an m-ideal of $(A, \exists)$. □

It is obvious that the set of m-ideals of any $MGMV$-algebra $(A, \exists)$ is a complete lattice with respect to the order by set inclusion. We will denote it by $\mathcal{I}(A, \exists)$.

Theorem 7 *If $(A, \exists)$ is a MGMV-algebra then the lattice $\mathcal{I}(A, \exists)$ is isomorphic to the lattice $\mathcal{I}(\exists A)$ of ideals of the GMV-algebra $\exists A$.*

Proof For any $J \in \mathcal{I}(\exists A)$ we put $\varphi(J) := \mathrm{Id}_A(J)$, where $\mathrm{Id}_A(J)$ is the ideal of A generated by J. If $x \in \varphi(J)$ then $x \leq a$ for some $a \in J$, hence $\exists x \leq \exists a = a$, and thus $\exists x \in \varphi(J)$. Therefore φ is a mapping of the lattice $\mathcal{I}(\exists A)$ into the lattice $\mathcal{I}(A, \exists)$. Moreover, $\varphi(J) \cap \exists A = J$, hence φ is injective.

Let now $K \in \mathcal{I}(A, \exists)$. Then $K = \mathrm{Id}_A(K \cap \exists A)$ and since $K \cap \exists A \in \mathcal{I}(\exists A)$, we get $K = \varphi(K \cap \exists A)$. Therefore φ is a surjective mapping of $\mathcal{I}(\exists A)$ onto $\mathcal{I}(A, \exists)$.

Moreover, it is obvious that for each $J_1, J_2 \in \mathcal{I}(\exists A)$, $J_1 \subseteq J_2$ if and only if $\varphi(J_1) \subseteq \varphi(J_2)$, hence φ is an isomorphism of the lattice $\mathcal{I}(\exists A)$ onto the lattice $\mathcal{I}(A, \exists)$ (and $\varphi^{-1}(K) = K \cap \exists A$ for every $K \in \mathcal{I}(A, \exists)$). □

Recall that if A is a GMV-algebra and $I \in \mathcal{I}(A)$ then I is called a *normal ideal* of A if

$$x^{-} \odot y \in I \iff y \odot x^{\sim} \in I,$$

for every $x, y \in A$.

Proposition 10 *If $(A, \exists)$ is an MGMV-algebra and I is an m-ideal of $(A, \exists)$ which is normal in A, then $\varphi^{-1}(I)$ is a normal ideal in $\exists A$.*

Proof Let us suppose that I is a normal m-ideal of $(A, \exists)$. Let $x, y \in \exists A$. If $x^{-} \odot y \in \varphi^{-1}(I)$, then $x^{-} \odot y \in I$ and $x^{-} \odot y \in \exists A$, hence $y \odot x^{\sim} \in I$ and $y \odot x^{\sim} \in \exists A$, i.e., $y \odot x^{\sim} \in \varphi^{-1}(I)$. Analogously, $y \odot x^{\sim} \in \varphi^{-1}(I)$ implies $x^{-} \odot y \in \varphi^{-1}(I)$. □

Question 1 If $(A, \exists)$ is any $MGMV$-algebra and J is an arbitrary normal ideal of $\exists A$, is $\varphi(J)$ normal in A?

Archive for mathematical logic. 2008, vol. 47, no. 3, p. 277-297. http://dx.doi.org/10.1007/s00153-008-0086-2

If $(A, \exists)$ is an $MGMV$-algebra and θ is a congruence on A, then θ is called an *m-congruence* on $(A, \exists)$ provided

$$(x,\, y) \in \theta \implies (\exists x,\, \exists y) \in \theta,$$

for every $x,\, y \in A$.

Theorem 8 *For any MGMV-algebra there is a one-to-one correspondence between its m-congruences and normal m-ideals.*

Proof Let $(A, \exists)$ be an $MGMV$-algebra. Recall that normal ideals of A are in a one-to-one correspondence with congruences on A, and that the corresponding congruence $\theta(I)$ to a normal ideal I of A is such that

$$(x,\, y) \in \theta(I) \iff (x^- \odot y) \oplus (y^- \odot x) \in I \iff (y \odot x^\sim) \oplus (x \odot y^\sim) \in I.$$

Now, let I be a normal m-ideal on $(A, \exists)$ and $x,\, y \in A$. Then

$$\begin{aligned}(x,\, y) \in \theta(I) &\implies (x^- \odot y) \oplus (y^- \odot x) \in I \implies y \oslash x,\ x \oslash y \in I \\ &\implies \exists(y \oslash x),\ \exists(x \oslash y) \in I \implies \exists y \oslash \exists x,\ \exists x \oslash \exists y \in I \\ &\implies (\exists y \oslash \exists x) \oplus (\exists x \oslash \exists y) \in I \implies ((\exists x)^- \odot \exists y) \oplus ((\exists y)^- \odot \exists x) \in I \\ &\implies (\exists x,\, \exists y) \in \theta(I),\end{aligned}$$

hence $\theta(I)$ is an m-congruence on $(A, \exists)$.

Conversely, if θ is an m-congruence on $(A, \exists)$ then $0/\theta$ is a normal m-ideal of $(A, \exists)$. □

If I is a normal ideal of a GMV-algebra A, put $A/I := A/\theta(I)$.

Let $(A, \exists)$ be an $MGMV$-algebra and I be its normal m-ideal. We define the mapping $\exists_I : A/I \longrightarrow A/I$ such that

$$\exists_I(x/I) := (\exists x)/I,$$

for each $x \in A$.

Proposition 11 *If I is a normal m-ideal of an MGMV-algebra $(A, \exists)$ then $(A/I, \exists_I)$ is an MGMV-algebra.*

Let us consider the class $\mathcal{MGMV}$ of all $MGMV$-algebras. By the definition of an $MGMV$-algebra it is clear that $\mathcal{MGMV}$ is a variety of algebras of type $\langle 2, 1, 1, 0, 1\rangle$.

Theorem 9 *The variety $\mathcal{MGMV}$ is arithmetical.*

Proof By [18], the variety $\mathcal{GMV}$ of all GMV-algebras (of type $\langle 2, 1, 1, 0\rangle$) is arithmetical, hence the variety $\mathcal{MGMV}$ is arithmetical, too. □

An ideal P of a GMV-algebra A is called *prime* if I is a finitely meet-irreducible element in the lattice $\mathcal{I}(A)$. A prime ideal P is called *minimal* if P is a minimal element in the set of prime ideals of A ordered by inclusion. By Zorn's lemma, every prime ideal contains a minimal prime ideal.

Let A be a GMV-algebra and $X \subseteq A$. The set

$$X^\perp = \{a \in A :\ a \wedge x = 0,\ \text{for each } x \in X\}$$

is called the *polar of X in A*. For any $a \in A$, we write $a^\perp$ instead of $\{a\}^\perp$.

Archive for mathematical logic. 2008, vol. 47, no. 3, p. 277-297. http://dx.doi.org/10.1007/s00153-008-0086-2

Proposition 12 *(See [7, Theorem 2.20].) For $P \in \mathcal{I}(A)$, the following conditions are equivalent:*

(1) P *is a minimal prime.*
(2) $P = \bigcup\{a^{\perp} : a \notin P\}$.

A GMV-algebra A is called *representable* if A is isomorphic to a subdirect product of linearly ordered GMV-algebras.

Proposition 13 *(See [7, Proposition 3.13].) For a GMV-algebra A the following conditions are equivalent:*

(1) A *is representable.*
(2) *There exists a set $\mathcal{S}$ of normal prime ideals such that $\bigcap \mathcal{S} = \{0\}$.*
(3) *Every minimal prime ideal is normal.*

Theorem 10 *Let $(A, \exists)$ be an $MGMV$-algebra satisfying the identity $\exists(x \wedge y) = \exists x \wedge \exists y$. Then $(A, \exists)$ is a subdirect product of linearly ordered $MGMV$-algebras if and only if A is a representable GMV-algebra.*

Proof Let us consider an $MGMV$-algebra $(A, \exists)$ which satisfies $\exists(x \wedge y) = \exists x \wedge \exists y$, for every $x, y \in A$. Let us suppose that the GMV-algebra A is representable. Then by Proposition 13, there exists a system $\mathcal{S}$ of normal prime ideals of A such that $\bigcap \mathcal{S} = \{0\}$, and, moreover, all minimal prime ideals of A are normal. Since every prime ideal of A contains a minimal prime ideal, we get that in our case the intersection of all minimal prime ideals is equal to $\{0\}$.

We will show that every minimal prime ideal of A is an m-ideal in $(A, \exists)$. Let P be a minimal prime ideal of A. Then by Proposition 12, $P = \bigcup\{a^{\perp} : a \notin P\}$. If $x \in P$, then there is $a \notin P$ such that $x \wedge a = 0$, hence $0 = \exists 0 = \exists(x \wedge a) = \exists x \wedge \exists a$. Since $a \notin P$, we get $\exists a \notin P$, therefore $\exists x \in P$. That means, P is an m-ideal in $(A, \exists)$.

The converse implication is trivial. $\square$

7 Polyadic GMV-algebras

In this section we will deal with polyadic GMV-algebras as special cases of polyadic (Λ, I)-algebras in the sense of [17].

Let I be a nonempty set. Any mapping $\sigma : I \longrightarrow I$ is called a *transformation* of I. The set of transformations of I is denoted by I^I and the identity transformation by ι. If $J \subseteq I$ and $\sigma, \tau \in I^I$ then $\sigma J \tau$ means that $\sigma_i = \tau_i$ for each $i \in J$, and $\sigma J_* \tau$ means that $\sigma_i = \tau_i$ for each $i \in I \setminus J$. We say that J *supports* σ if $\sigma J_* \iota$. Further, σ is *of finite support* if it has a finite support set. The set of all transformations of finite support is denoted by $I^{(I)}$. The denotation $J \subseteq_\omega I$ means that J is a finite subset of I. The set of all finite subsets of I is denoted by $\mathrm{Sb}_\omega I$.

Let $\Lambda = \langle \mathcal{N}, \mathcal{B}, \rho \rangle$ be a first-order language with two disjoint sets $\mathcal{N}$ and $\mathcal{B}$ of operation symbols, where $\mathcal{N} = \langle \oplus, \odot, ^-, ^\sim, 0, 1 \rangle$ is the set of operation symbols of GMV-algebras, and $\rho : \mathcal{N} \longrightarrow \omega$ denotes their usual arities. Let I be a nonempty set.

Now we consider a further first-order language closely related to Λ such that its nonlogical symbols are divided to the following categories.

(a) The operation symbols $\langle \oplus, \odot, ^-, ^\sim, 0, 1 \rangle$, called *nonbinding operations* or *propositional connectives.*

(b) An $\mathrm{Sb}_\omega I$-indexed system of unary operation symbols $Q := \langle Q_J : J \subseteq_\omega I \rangle$ for each $Q \in \mathcal{B}$. These are called *binding operations* or *generalized quantifiers*.
(c) A unary operation symbol S_σ for each $\sigma \in I^{(I)}$. These are called *substitution operators*.

Let I be a nonempty set and Λ be a language as above. Then a *polyadic GMV-algebra* (over (Λ, I)) is any algebra of the form

$$\mathbb{A} := \left\langle A;\ \langle \oplus, \odot, {}^-, {}^\sim, 0, 1 \rangle,\ \langle Q_J : Q \in \mathcal{B}, J \subseteq_\omega I \rangle,\ \langle S_\sigma : \sigma \in I^{(I)} \rangle \right\rangle$$

satisfying the following axioms (universally quantified):

(PGMV$_1$) $S_\iota x = x$;

(PGMV$_2$) $S_\sigma(S_\tau x) = S_{\sigma\tau} x$,
for all σ, $\tau \in I^{(I)}$;

(PGMV$_3$) $S_\sigma(x_1 \oplus x_2) = S_\sigma x_1 \oplus S_\sigma x_2$,
$S_\sigma(x_1 \odot x_2) = S_\sigma x_1 \odot S_\sigma x_2$,
$S_\sigma x^- = (S_\sigma x)^-$, $\quad S_\sigma x^\sim = (S_\sigma x)^\sim$,
$S_\sigma 0 = 0$, $\quad S_\sigma 1 = 1$,
for all $\sigma \in I^{(I)}$;

(PGMV$_4$) $S_\sigma Q_J x = S_\tau Q_J x$,
for all $Q \in \mathcal{B}$, $J \subseteq_\omega I$ and σ, $\tau \in I^{(I)}$ such that $\sigma J_* \tau$;

(PGMV$_5$) $Q_J S_\sigma x = S_\sigma Q_{\sigma^{-1}(J)} x$,
for all $Q \in \mathcal{B}$, $J \subseteq_\omega I$ and $\sigma \in I^{(I)}$ such that σ is one-to-one on $\sigma^{-1}(J)$.

A polyadic *GMV*-algebra $\mathbb{A}$ over (Λ, I) is called *pseudomonotonic* if it satisfies the following axiom:

(PGMV$_6$) $Q_J x = Q_J S_\sigma y \wedge Q_J y = Q_J S_\tau y \to Q_J x = Q_J y$,
for all $Q \in \mathcal{B}$, $J \subseteq_\omega I$ and σ, $\tau \in I^{(I)}$ such that $\sigma J_* \iota$ and $\tau J_* \iota$.

A polyadic *GMV*-algebra $\mathbb{A}$ over (Λ, I) is called *infinite-dimensional* if I is infinite.

Remark 2 Every monadic *GMV*-algebra can be considered as a special case of polyadic *GMV*-algebra, where $\mathrm{card} I = 1$ and $\mathcal{B} = \langle \exists \rangle$.

Remark 3 The notion of a polyadic *GMV*-algebra is based on an essentially more general notion of a polyadic (Λ, I)-algebra, where in a language $\Lambda = \langle \mathcal{N}, \mathcal{B}, \rho \rangle$, the set $\mathcal{N}$ of operation symbols is not specified, $\rho : \mathcal{N} \longrightarrow \omega$ is an arbitrary mapping, and instead of the axiom (PGMV$_3$) it is used more general axiom related to all operation symbols in $\mathcal{N}$. (See [17, Definition 1.1].)

Let now $\mathbb{A}$ be a polyadic *GMV*-algebra over (Λ, I). Then $J \subseteq I$ *supports* an element $a \in A$ if $S_\sigma a = S_\tau a$ for all σ, $\tau \in I^{(I)}$ such that $\sigma J \tau$. An element $a \in A$ is of *finite support* if it has a finite support set. $\mathbb{A}$ is called *locally finite* if every element in A is of finite support.

Functional polyadic (Λ, I)-algebras are introduced and studied in [17]. It is shown ([17, Theorems 1.11, 1.12]) that every functional polyadic (Λ, I)-algebra is a polyadic (Λ, I)-algebra and that every locally finite polyadic (Λ, I)-algebra of infinite dimension is isomorphic to a functional polyadic (Λ, I)-algebra.

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Furthermore, Pigozzi and Salibra in [17] deal with the first-order extensions of the so-called standard systems of implicational extensional propositional calculi (SIC's) considered by Rasiowa in [19]. These include many of non-classical logics (classical and intuitionistic and their various weakenings and fragments, the Post and Łukasiewicz multiple-valued logics, modal logics that admit the rule of necessitation, BCK-logic, ...). Note that every SIC $\mathcal{S}$ is algebraizable in the sense of [2].

For any SIC $\mathcal{S}$, polyadic $\mathcal{S}$-algebras and function-representable polyadic $\mathcal{S}$-algebras are introduced in [17], and it is proved ([17, Theorem 3.7]) that every locally finite polyadic $\mathcal{S}$-algebra of infinite dimension is isomorphic to a function-representable polyadic $\mathcal{S}$-algebra.

Now, it is a question how to introduce an analogue of the notion of SIC for non-commutative logics (including the non-commutative Łukasiewicz infinite valued logic) and whether, in such a case, there is an analogous representation for some class of non-commutative polyadic $\mathcal{S}$-algebras as for locally finite polyadic $\mathcal{S}$-algebras.

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