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LINEARIZATION OF POLYNOMIAL MATRIX INEQUALITIES

LINEARIZACE POLYNOMIÁLNÍCH MATICOVÝCH NEROVNOSTÍ

Abstract

This paper proposes a linearization approach for the study of solutions of Polynomial Matrix Inequalities (PMIs). The Polynomial Matrix Inequality is transformed into a set of linear inequalities with compatibility conditions by using a Carleman-like linearization. An insight into these compatibility conditions is proposed by representing them in a logarithmic space, in which they are also linear. The existence of solutions is then reduced to the identification of the range space of a linear operator. Geometrically, the problem can be interpreted as the intersection of a curvilinear cone and some linear hyperplanes.

Abstrakt

Příspěvek popisuje linearizační přístup pro studium řešení polynomiálních maticových nerovností. Polynomiální maticová nerovnost je převedena na soustavu lineárních nerovností s podmínkami kompatibility použitím linearizace podobné Carlemanově linearizaci. Díky vyjádření podmínek kompatibility v logaritmickém prostoru, je možno na ně pohlížet jako na lineární podmínky. Existence řešení je potom omezena na identifikaci rozsahu prostoru lineárních operátorů. Problém může být interpretován geometricky jako průnik mezi zakřiveným kuželem a lineární nadrovinou.

1 INTRODUCTION

The study of Linear and Polynomial Matrix Inequalities is an important problem of actual interest within the control community. This interest was generated from the realization that many control problems can be rewritten in terms of them. Over the last decade, there have been many advances in the study of Linear Matrix Inequalities (LMIs), being the book by Boyd, et.al. [1] one of the most cited monographs. This has allowed their application in different control problems, which are commonly rewritten as convex optimization problems with LMI constraints. Some of the control problems include the stability study of Lotka-Volterra systems with delays [4] and the design of H∞ controllers [3] among many. Despite the wide range of applications of LMIs, many control problems are actually written in terms of Polynomial Matrix Inequalities, e.g. in the case of Bilinear Matrix Inequalities (BMIs), in [5] it was shown that core problems in robust control could be formulated not as linear but rather as BMIs. However, the study of general PMIs is more complicated than the study

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of LMIs, as their feasible set usually lacks convexity and the problems involve more decision variables and dimensions, which means that the computation of solutions is not straightforward. This opened a lot of actual interest and research possibilities under this subject as there is still a long way to go to reach the research level achieved on the study of LMIs and there seems to be no existing texts covering BMIs and PMIs. The research possibilities are many. One can study their geometry, feasibility, the possibility of transforming them to LMI problems, the development of algorithms which solve non-convex optimization problems, the representation of control problems into optimization problems with BMI or PMI constraints, etc.

In this paper, a linearization of the PMI problem is proposed to obtain a set of linear inequalities with linear compatibility conditions. The compatibility conditions are represented in the logarithmic space, in which they are linear. The existence of solutions reduces to the identification of the range space of a linear operator which can be interpreted geometrically as the intersection of some linear hyperplanes with a curvilinear cone. The purpose of the paper is not to give a new algorithm to solve PMI problems but to study the problem using a linearization approach. The paper is organized as follows. Section 2 presents the reduction of the Polynomial Matrix Inequality problem to a set of linear inequalities with compatibility conditions, Section 3 proposes a logarithmic representation of the compatibility conditions to linearize completely the PMI. Section 4 presents a simple example and finally Section 5 presents some conclusions.

2 LINEARIZATION OF POLYNOMIAL MATRIX INEQUALITIES

In this section, consider PMIs in the form

$$\sum_{i=1}^{m} \beta_i(x_1,...,x_n)F_i > 0 \tag{1}$$

where the symmetric matrices $F_i,\ldots,F_m \in \mathrm{R}^{\ell\times\ell}$ and $\beta_i,\ldots,\beta_m$ are m polynomials in n variables of order $\leq N\cdot n$. Eq. (1) can be written in the equivalent form.

$$A(x_1,...,x_n) > 0 \tag{2}$$

where A is an $\ell\times\ell$ matrix with polynomial elements of order at most $N\cdot n$. The $(i,j)^{th}$ element of A can be written in the form

$$a_{ij}(x_1,...,x_n) = \sum_{k_1=0}^{N} \cdots \sum_{k_n=0}^{N} \alpha_{ij}{}^{k_1,...,k_n} x_1{}^{k_1} ... x_n{}^{k_n} \tag{3}$$

or in the vector notation,

$$a_{ij}(x_1,...,x_n) = \sum_{k_1=0}^{N} \alpha_{ij}^{k} x^{k} \tag{4}$$

where $k=(k_1,\ldots,k_n)$, $N=(N,\ldots,N)$, $x=(x_1,\ldots,x_n)$. By Sylvester's criterion, the condition for positivity in (2) is simply that

$$\begin{vmatrix} a_{11} & \cdots & a_{1\mu} \\ \cdots & \cdots & \cdots \\ a_{\mu 1} & \cdots & a_{\mu\mu} \end{vmatrix} > 0, \quad \forall \mu \in \{1,...,\ell\} \tag{5}$$

i.e. using the standard determinant equation

$$\sum_{\sigma_\mu \in S_\mu} (\operatorname{sgn} \sigma_\mu) a_{1\sigma_\mu(1)} \ldots a_{\mu\sigma_\mu(\mu)} > 0, \quad 1 \le \mu \le \ell \tag{6}$$

where $S\mu$ is the symmetry group on μ symbols and $sgn\sigma_\mu$ is the sign of the permutation σ_μ. Substituting (4) into (6) gives a set of ℓ inequalities of the form

$$\sum_{v_1=0}^{L} \cdots \sum_{v_n=0}^{L} \beta^{\mu}_{v_1 \ldots v_n} x_1^{v_1} \ldots x_n^{v_n} > 0, \quad 1 \le \mu \le \ell \tag{7}$$

where $L = \mu N$ and

$$\beta^{\mu}_{v} = \sum_{\sigma_\mu \in s_\mu} (\operatorname{sgn} \sigma_\mu) \sum_{j(1)+\ldots+j(\mu)=v} a^{j(1)}_{1\sigma_\mu(1)} \ldots a^{j(\mu)}_{\mu\sigma_\mu(\mu)} \tag{8}$$

By using a Carleman-like linearization technique, the next new variables are defined,

$$y_{v_1,\ldots,v_n} = x_1^{v_1} \ldots x_n^{v_n}, \quad 0 \le v_k \le L, \quad 1 \le k \le n \tag{9}$$

then, the polynomial inequalities (7) become ℓ linear inequalities in y-space:

$$\sum_{v_1=0}^{L} \cdots \sum_{v_n=0}^{L} \beta^{\mu}_{v_1 \ldots v_n} y_{v_1 \ldots v_n} > 0, \quad 1 \le \mu \le \ell \tag{10}$$

In this case, the y-space is $((L+1)^n -1)$-dimensional. Each inequality defines a half-space in y-space; these ℓ linear inequalities have a solution when these half-spaces have non-empty intersection. Of course, any such solution is a solution of (7) if and only if its components are of the form (9).

3 LOGARITHMIC REPRESENTATION OF THE COMPATIBILITY CONDITIONS

Consider the reduced problem of the PMI (1) consisting of the ℓ linear inequalities in y-space (10) with the variables (9).

Remark 3.1. The y-space is of dimension $((L+1)^n -1)$, obtained when $/v/\neq 0$. Note that the cases when $/v/=0$, e.g. the affine case, the variable $y_{0\ldots 0}$, is not included in the compatibility conditions.

Suppose that y_v satisfies (9). To obtain linear compatibility conditions, the logarithmic of the variables (9) is obtained, therefore,

$$\ln(y_{v_1,\ldots,v_n}) = v_1(\ln x_1) + v_2(\ln x_2) \ldots + v_n(\ln x_n), \quad 0 \le v_k \le L, \quad 1 \le k \le n \tag{11}$$

and by renumbering the y_v,

$$\begin{aligned} \ln(y_1) &= v_1^1(\ln x_1) + v_2^1(\ln x_2) + \ldots + v_n^1(\ln x_n) \\ \ln(y_2) &= v_1^2(\ln x_1) + v_2^2(\ln x_2) + \ldots + v_n^2(\ln x_n) \\ &\vdots \\ \ln(y_m) &= v_1^m(\ln x_1) + v_2^m(\ln x_2) + \ldots + v_n^m(\ln x_n) \end{aligned} \tag{12}$$

with $m=((L+1)^n-1)$. Define the new set of variables

$$\begin{aligned} z_1 &= \ln x_1, \\ z_2 &= \ln x_2, \\ &\vdots \\ z_n &= \ln x_n \end{aligned} \tag{13}$$

where $z_n \in \mathrm{R}^n$. Therefore, the linear equation

$$\mathrm{A}z = \eta \tag{14}$$

with

$$\mathrm{A} = \begin{pmatrix} v_1^1 & v_2^1 & \cdots & v_n^1 \\ v_1^2 & v_2^2 & \cdots & v_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ v_1^m & v_2^m & \cdots & v_n^m \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \quad \eta = \begin{pmatrix} \ln y_1 \\ \ln y_2 \\ \vdots \\ \ln y_m \end{pmatrix} \tag{15}$$

therefore, $\mathrm{A} \in \mathrm{R}^{m \times n}$, $z \in \mathrm{R}^n$ and $\eta \in \mathrm{R}^m$ where $m=((L+1)^n-1)$.

Remark 3.2. Note that for each particular PMI problem, the dimensions of the linear equation (14) can be reduced by only including the variables (9) that are occurring in the problem. However, in this paper, all variables are included for simplicity of notation

To obtain a solution of the system (14), the generalized inverse of A is used. Note that in general and by Remark 3.2, the matrix A is not square. From linear algebra [2], the generalized inverse of a matrix A is defined as:

$$A^+ = (A^T A)^{-1} A^T \tag{16}$$

and the solution of system (14) is

$$z = A^+ \eta \quad \text{iff} \quad AA^+ \eta = \eta \,. \tag{17}$$

The original PMI problem (1) has now been reduced to the linear problem of finding solutions to the ℓ linear inequalities in y-space

$$\sum_{v_1=0}^{L} \cdots \sum_{v_n=0}^{L} \beta^{\mu}_{v_1 \ldots v_n} y_{v_1 \ldots v_n} > 0, \quad 1 \le \mu \le \ell \tag{18}$$

such that their logarithm belongs to the range space of the linear operator AA^+.

Remark 3.3. Note that negative solutions of the problem (1) can be found by shifting the variables y_v by an appropriate constant before taking the logarithms.

Geometrically, the problem can be interpreted as the intersection of a curvilinear cone obtained from the compatibility conditions and some linear hyperplanes obtained from the linear set of inequalities.

4 EXAMPLE

Consider the following BMI,

$$\begin{pmatrix} -13x_2 - 5x_1x_2 + x_2^2 & x_2 & 0 \\ x_2 & x_1 & 0 \\ 0 & 0 & -13x_1 - 5x_1^2 + x_1x_2 - x_2 \end{pmatrix} > 0 \tag{19}$$

which can be transformed to the following set of linear inequalities

$$\begin{aligned} -13y_1 - 5y_2 + y_3 > 0 \\ -13y_2 - 5y_4 + y_5 - y_3 > 0 \\ -13y_7 - 5y_6 + y_2 - y_1 > 0 \end{aligned} \tag{20}$$

with variables

$$\begin{array}{llll} y_1 = x_2 & y_2 = x_1 x_2 & y_3 = x_2^2 & y_4 = x_1^2 x_2 \\ y_5 = x_1 x_2^2 & y_6 = x_1^2 & y_7 = x_1 & \end{array} \tag{21}$$

to obtain the compatibility conditions, variables z are introduced: z_1=ln x_1 and z_2=ln x_2, and the logarithm of (21) is obtained, therefore the following linear equation is obtained,

$$\mathrm{A}z = \eta \tag{22}$$

with

$$\mathrm{A} = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 2 & 0 & 0 \end{pmatrix}^T, \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\eta = \begin{pmatrix} \ln y_1 & \ln y_2 & \ln y_3 & \ln y_4 & \ln y_5 & \ln y_6 & \ln y_7 \end{pmatrix}^T \tag{23}$$

by using $A^+ = (A^T A)^{-1} A^T$, the generalized inverse of A is calculated, as well as the operator AA^+, in this case,

$$\mathrm{AA}^+ = \begin{pmatrix} \frac{11}{96} & \frac{1}{16} & \frac{11}{48} & \frac{1}{96} & \frac{17}{96} & \frac{-5}{96} & \frac{-5}{96} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{8} & \frac{3}{16} & \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{11}{48} & \frac{1}{8} & \frac{11}{24} & \frac{1}{48} & \frac{17}{48} & \frac{-5}{24} & \frac{-5}{48} \\ \frac{1}{96} & \frac{3}{16} & \frac{1}{48} & \frac{35}{96} & \frac{19}{96} & \frac{17}{48} & \frac{17}{96} \\ \frac{17}{48} & \frac{3}{16} & \frac{17}{48} & \frac{19}{96} & \frac{35}{96} & \frac{1}{48} & \frac{1}{96} \\ \frac{-5}{48} & \frac{1}{8} & \frac{-5}{24} & \frac{17}{48} & \frac{1}{48} & \frac{11}{24} & \frac{11}{48} \\ \frac{-5}{96} & \frac{1}{16} & \frac{-5}{48} & \frac{17}{96} & \frac{1}{96} & \frac{11}{48} & \frac{11}{96} \end{pmatrix} \tag{24}$$

The column space of (24) is spanned by the vectors [v_1 v_2],

$$\begin{array}{l} v_1 = \begin{pmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{8} & \frac{3}{16} & \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \end{pmatrix} \\ v_2 = \begin{pmatrix} 0 & \frac{-1}{96} & 0 & \frac{-1}{98} & \frac{-1}{96} & \frac{-1}{48} & \frac{-1}{96} \end{pmatrix} \end{array} \tag{25}$$

As η must belong to the range space of (24), one possible solution is obtained by taking η=v_1+5v_2. The corresponding y's are obtained from $exp(\eta)$, in this case,

$$\begin{array}{l} y_1 = 518.01, \quad y_2 = 1468.01, \quad y_3 = 2.68x10^5 \\ y_4 = 4160.26, \quad y_5 = 7.6x10^5, \quad y6 = 8.03, \quad y_7 = 2.83 \end{array} \tag{26}$$

is easy to check that the obtained y's satisfy the linear inequalities (20). Therefore, one possible solution of the BMI (19) is

$$x_1 = y_7 = 2.833 \text{ and } x_2 = y_1 = 518.01 \tag{27}$$

5 CONCLUSIONS

In this paper, the problem of solving Polynomial Matrix Inequalities was addressed by using a linearization approach. First, the original problem was transformed to the problem of solving a set of linear inequalities with nonlinear compatibility conditions by using a Carleman-like linearization. Then, linear compatibility conditions were found by introducing new variables and obtaining the logarithm of the original compatibility conditions. The original problem was then reduced to the problem of solving a set of linear inequalities together with linear compatibility conditions. It was shown that the problem is actually reduced to finding solutions of the linear inequalities that belong to the range of a linear operator obtained from the compatibility conditions.

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