Circular Arc Approximation by Quartic H-Bézier Curve

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Abstract

The quartic H-Bézier curve is used for the approximation of circular arcs. It has five control points and one positive real free parameter. The four control points are carried out by G^1-approximation constraints and the remaining control point is dividing the line segment joining the second and fourth control points in the ratio 1:2. Optimized value of free parameter α is obtained by minimizing the maximum value of absolute radius error of the recommended approximation scheme. The developed approximation scheme is found considerably better than the existing approximation schemes for these computed values of control points and optimized value of the free parameter.

Keywords: Quartic H-Bézier curve, Control points, Free parameter, G^1-approximation constraints, Absolute radius error.

AMS Subject Classification 2010: 65D17, 68U07

1. Introduction

Circles and circular arcs are widely applicable in the field of CAD for the designing of various objects. The most common applications can be seen in the designing of highway and railway routs and in the construction of suspension bridges (Lu, 2012). Since the designers can not directly use the parametric equations of circles in CAD rather they use the approximations of circles. So it is of keen interest of many authors to find the optimal approximations of circular arcs. Ahn and Kim (1997) used Bernstein-Bézier curves of degree four and five for $GC^k, k = 2,3,$ approximation of circular arcs. Order of approximation of these schemes was eight and ten respectively. Fang (1998) discussed five circular arcs approximation methods by using polynomial curves of degree 5. The convergence rate of these methods was either 8 or 10. Floater (1995) approximated the conic sections by quadratic splines with continuous curvature. Hur and Kim (2011) used cubic and quartic Bézier curves for the $G^k, k = 1,2$, approximation of circular arcs. The Hausdorff distance between the approximating Bézier curves and circle was least for the proposed approximation schemes. Lee *et al.* (2006) presented the G^0-approximation of circular arc by Bézier curve of degree 2. In (Piegl and Tiller, 2003), the integral B-spline curve of appropriate degree was used as an interpolant for the approximation of circular arcs. The approximation was carried out by interpolating the derivatives at terminal points and a few interior points. Riškus (2006) approximated circular arc by using cubic Bézier curve. The proposed scheme was valuable in CAD system as it supported Bézier curve to interchange data through any data formats.

H-Bézier curves (Lee and Ahn, 2015) have hyperbolic basis functions. These basis functions are known as H-basis functions. H-Bézier curves preserve the favourable characteristics of ordinary Bézier curves like end points and end tangents interpolation property, partition of unity, invariance under affine transformation. H-Bézier curves give better smooth shape-preserving curves as compared to ordinary Bézier curves. In the proposed study, the circular arc approximation scheme by the quartic H-Bézier curve is presented. The existing approximation schemes of circular arcs use G^2-constraints for computing the control points of quartic and quintic approximating polynomials (Fang, 1998). Here the control points $b_i, i = 0,1,3,4,$ are evaluated by G^1-approximation constraints. The control point b_2 is chosen as a point dividing the line segment $\overline{b_1 b_3}$ in the ratio 1:2. The value of free parameter α is evaluated by minimizing the maximum value of absolute radius error of approximation. The absolute radius error of the cultivated circular arc approximation scheme is considerably smaller than the existing schemes (Fang 1998; Lu 2012).

2. Quartic H-Bézier Curve

The quartic H-Bézier curve (Lee and Ahn, 2015)

$$q(t) = \sum_{i=0}^{4} Z_i^4\,(t) b_i,\ t \in [0,1]. \qquad (1)$$

It has five control points $b_i, i = 0,1,2,3,4,$ a free parameter α ($\alpha > 0$), t is H-Bézier parameter, and $Z_i^4(t), i = 0,1,2,3,4,$ are the quartic H-Basis functions given as:

$$Z_0^4(t) = \frac{L}{A},\ Z_1^4(t) = \frac{R}{A} - \frac{M_1 - M_2}{BC},\ Z_2^4(t) = \frac{M_1 - M_2}{BC} - \frac{N_1 - N_2}{BC},$$

$$Z_3^4(t) = \frac{N_1 - N_2}{BC} - \frac{Q}{A},\ Z_4^4(t) = \frac{Q}{A},$$

where

$$L = \alpha^2(1 + t^2) + 2(1 - \cosh\alpha(1-t) - 2\alpha^2 t),\ A = \alpha^2 + 2 - 2\cosh\alpha,$$

$$R = 2\alpha^2 t - \alpha^2 t^2 + 2\cosh\alpha(1-t) - 2\cosh\alpha,$$

$$M_1 = (\alpha^2 t^2 - 2\cosh\alpha(1-t) - 2\alpha t\sinh\alpha + 2\cosh\alpha)(\alpha\cosh\alpha + \alpha - 2\sinh\alpha),$$

$$M_2 = (\alpha - \sinh\alpha)(-2\cosh\alpha(1-t) + \alpha^2 t^2\cosh\alpha + \alpha^2 t^2 - 2\cosh\alpha t + 2\cosh\alpha - 2\alpha t\sinh\alpha + 2),$$

$$N_1 = -2\cosh\alpha(1-t) + \alpha^2 t^2\cosh\alpha + \alpha^2 t^2 - 2\cosh\alpha t - 2\alpha t\sinh\alpha + 2,$$

$$N_2 = (\alpha^2 t^2 - 2\cosh\alpha t + 2)(\alpha\cosh\alpha + \alpha - 2\sinh\alpha),\ Q = \alpha^2 t^2 - 2\cosh\alpha t + 2,$$

$$B = \alpha \cosh \alpha + \alpha - 2 \sinh \alpha, C = 2 \cosh \alpha - 2 - \alpha \sinh \alpha.$$

Numerical experiments suggest that the basis functions are either positive or negative depending upon the value of α.

3. Approximation of Circular Arc by Quartic H-Bézier Curve

In this section, the numerical approximation scheme of circular arc by quartic H-Bézier curve is computed. The circular arc is considered in standard position i.e. center at origin $O(0,0)$, radius r, the initial point of the arc is along the positive horizontal axis and the final point is making a counter clockwise angle θ with the same axis. Any circular arc with arbitrary center can be transformed to this position by affine transformations.

The following G^1-approximation constraints are used for the approximation:

$$q(t)|_{t=0} = c_0,\ q(t)|_{t=1} = c_1, \tag{2}$$

$$T_0 = t_0,\ T_1 = t_1. \tag{3}$$

Here, $c_i{}'s$ are the end points and $t_i{}'s$ are the end unit tangent vectors of the circular arc. By using (1), we have $q(0) = b_0$ and $q(1) = b_4$. The end unit tangents of quartic H-Bézier curves are denoted by T_0 and T_1 and are computed by the relation $T_i = \frac{q'(i)}{\|q'(i)\|}$, for $i = 0, 1$, which implies $T_0 = \frac{b_1 - b_0}{\gamma_1}$ and $T_1 = \frac{b_3 - b_2}{\gamma_2}$. Now by using these values of end points and end unit tangents of the quartic H-Bézier curve into (2) and (3) respectively, the following equations are obtained

$$b_0 = c_0, b_4 = c_1, \tag{4}$$

$$\frac{b_1 - b_0}{\gamma_1} = t_0,\ \ \frac{b_4 - b_3}{\gamma_2} = t_1. \tag{5}$$

The values of $\gamma_1 = \|b_1 - b_0\|$ and $\gamma_2 = \|b_4 - b_3\|$ are positive real numbers. The end points and the end unit tangents of the concerned circular arc $c_0 c_1$ are $c_0(r, 0)$, $c_1\ (rcos\theta\ , rsin\theta)$, $t_0(0,1)$, $t_1(-sin\ \theta, cos\ \theta)$. Substituting these values in (4) and (5), the control points of quartic H-Bézier curve are given by

$$\left.\begin{array}{c} b_0 = (r, 0), b_1 = (r, \gamma_1), b_2 = (b_{20}, b_{21}), \\ b_3 = (b_{30} = rcos\theta + \gamma_2 sin\theta, rsin\theta - \gamma_2 cos\theta), b_4 = (rcos\theta, rsin\theta) \end{array}\right\} \tag{6}$$

The values of parameters γ_1 and γ_2 are assumed as:

$$\gamma_1 = \|b_1 - b_0\| = \frac{\|b_4 - b_0\|}{4}, \gamma_2 = \|b_4 - b_3\| = \frac{\|b_4 - b_0\|}{4}.$$

It gives,
$$\gamma_1 = \gamma_2 = \frac{\sqrt{(rcos\theta - r)^2 + (rsin\theta)^2}}{4}. \tag{7}$$

In (6), the control point b_2 can be evaluated by various methods. In this study, b_2 is chosen as a point which divides the line segment $\overline{b_1 b_3}$ in the ration 1:2. The coordinates of b_2 are $b_{20} = \frac{rcos\theta + \gamma_2 sin\theta + 2r}{3}, b_{21} = \frac{rsin\theta - \gamma_2 cos\theta + 2\gamma_1}{3}$.

In (1) by substituting the values of control points from (6), the following parametric equations of quartic H-Bézier curve are obtained:

$$x(t) = Z_0^4(t)x_0 + Z_1^4(t)x_1 + Z_2^4(t)x_2 + Z_3^4(t)x_3 + Z_4^4(t)x_4, \quad (8)$$

$$y(t) = Z_0^4(t)y_0 + Z_1^4(t)y_1 + Z_2^4(t)y_2 + Z_3^4(t)y_3 + Z_4^4(t)y_4, \quad (9)$$

where

$$x_0 = r, x_1 = r, x_2 = \frac{rcos\theta + \gamma_2 sin\theta + 2r}{3}, x_3 = rcos\theta + \gamma_2 sin\theta, x_4 = rcos\theta,$$

$$y_0 = 0, y_1 = \gamma_1, y_2 = \frac{rsin\theta - \gamma_2 cos\theta + 2\gamma_1}{3}, y_3 = rsin\theta - \gamma_2 cos\theta, y_4 = rsin\theta.$$

The H-basis functions $Z_i^4(t), i = 0,1,2,3,4,$ have been already defined in Section 2. The free parameter α of quartic H-Bézier curve can assume different values and produces different H-Bézier curves for the approximation of circular arc. Therefore to find optimal approximation the value of α must be optimized. Here, the optimized value of α is obtained by the following optimization problem-I.

Optimization problem-I:

$$\min_{\alpha>0}\left(\max_{0\le t\le 1} \tilde{r}\,(\alpha, t)\right), \quad (10)$$

subject to

$$\alpha \ge u,$$

where, $u = 2.2204 \times 10^{-16}$, $\tilde{r}(\alpha, t) = |x^2(t) + y^2(t) - r^2|$. $\tilde{r}(\alpha, t)$ is the absolute radius error of the developed approximation scheme for circular arc c_0c_1 by quartic H-Bézier curve (1). r is the radius of the concerned circular arc, $x(t)$ and $y(t)$ are defined in (8) and (9).

The optimization problem-I is solved by the MATLAB 7 built in function fminimax of the MATLAB optimization toolbox. The fminimax is based on the sequential quadratic programming technique (Brayton *et al.*, 1979). Sequential quadratic programming technique is not suitable for discontinuous functions. But the objective function of optimization problem-I is continuous, so the problem is solvable.

4. Numerical Example

In this section, the numerical approximation scheme introduced in Section 3 is implemented on the unit circular arc given in Table 1.

Table 1: Unit circular arc

θ	r	c_0	c_1	t_0	t_1
$\pi/4$	1	(1,0)	(0.7071 , 0.7071)	(0,1)	(−0.7071 , 0.7071)

Control points of the quartic H-Bézier curve corresponding to the unit circular arc of Table 1 are calculated by Theorem 1. These computed values of control points and free parameter α are given in Table 2.

Table 2: Control points of the quartic H-Bézier curve

$\boldsymbol{b_0}$	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{b_4}$	$\boldsymbol{\alpha}$
(1,0)	(1, 0.1913)	(0.9475,0.3182)	(0.8424,0.5718)	(0.7071,0.7071)	9.2149

Graph of quartic H-Bézier curve approximating the circular arc of Table 1 is plotted in Figure 1. The reflection of this circular arc about the line $y = x$ is given in Figure 2. By combining the graphs of Figures 1 and 2, a quarter circle of unit radius is obtained (Figure 3). A semi-circle is obtained in Figure 4 by the reflection of the quarter circle of Figure 3 about y-axis. Complete circle is obtained in Figure 5 by the reflection of semi-circle of Figure 4 about x-axis. The maximum value of the absolute radius error of the developed approximation scheme is 8.7×10^{-3}. The plot of absolute radius error of the developed quartic H-Bézier approximation scheme is given in Figure 6.

5. Conclusion

In the proposed study, a circular arc quartic H-Bézier curve approximation scheme is introduced. The values of control points are evaluated by G^1-approximation constraints and the value of free parameter is obtained by minimizing the maximum value of absolute radius error of proposed approximation scheme. Absolute radius error of approximation for the proposed scheme is compared to the prevailing schemes (Table 3). It is noted that the absolute radius error of approximation in the proposed scheme is less than (Fang, 1998; Lu 2012).

Table 3: Absolute radius errors

References	Methods	Absolute radius error
Fang, 1998	Fang method IV	1.1788×10^{-2}
Lu, 2012	Lu	1.5213×10^{-2}
Section 4	Quartic H-Bézier approximation scheme for circular arc	8.7×10^{-3}

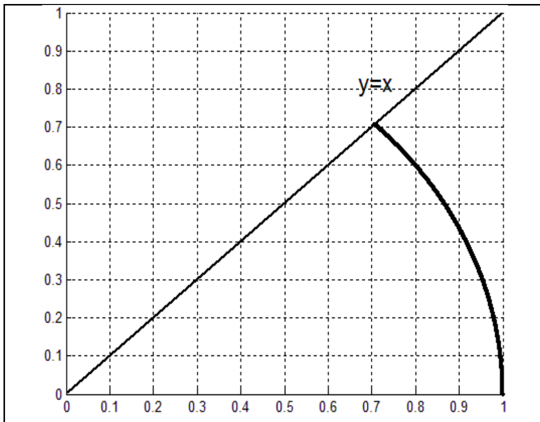

Figure 1: Unit circular arc making angle $\frac{\pi}{4}$ with x-axis

Figure 2: Reflection of Fig. 1 about the line $y = x$

Figure 3: Combination of Figures 1 and 2

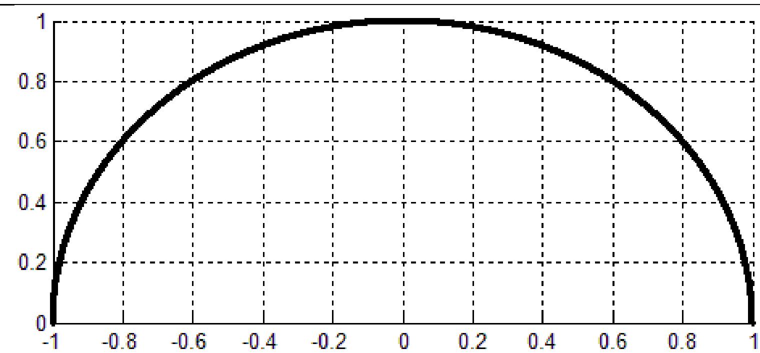

Figure 4: Reflection of Figure 3 about y-axis

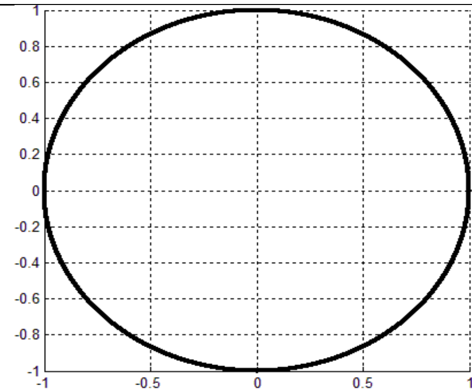

Figure 5: Reflection of Figure 4 about x-axis (circle approximation)

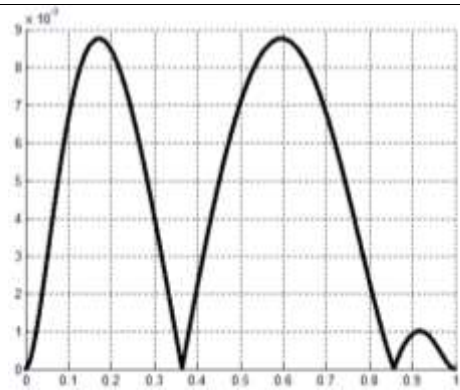

Figure 6: Absolute radius error

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