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Collective motion of living organisms: the Vicsek model

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DIPARTIMENTO DI MATEMATICA
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The artist, like the God of creation, remains within or behind or beyond or above his handiwork, invisible, refined out of existence, indifferent, paring his fingernails.

– James Joyce, *A Portrait of the Artist as a Young Man*

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Introduction

The purpose of this work is to study the Vicsek model for self-driven particles, in particular its time-continuous version, and provide a mean-field result for the dynamics.

We start by introducing the original discrete-time version, as proposed by T. Vicsek et al. in [16]. It is an example of what are known as *individual-based models* (or *IBMs*), in the sense that it simulates the movements of a population by considering the individual organisms it is comprised of. Each individual has certain attributes and behaviours (e.g. spatial location, behavioural traits), that may vary among the population, as well as change through time.

The main difference between IBMs and traditional differential equation population models is that these are so-called *bottom-up* models, in which behaviours emerge from the interactions among autonomous individuals.

We consider N particles inside a square of side L, with periodic boundary conditions. The idea of this model is the following: at time $t = 0$ the particles are positioned randomly, with same speed but random orientation. At each time step, a given particle assumes the average direction of the particles in its neighbourhood, with some random noise added.

This model has a number of possible applications in the field of biological systems involving clustering and migration. In fact, many individuals have the tendency of moving as other subjects in their neighbourhood, without the need for a leader. This is certainly the case for schools of fish, herds of mammals or flocks of migrating birds, but also for colonies of bacteria.

The dynamics of the i-th particle is then given by the following discrete-time stochastic system

$$x_i(t + \Delta t) = x_i(t) + v_i(t)\Delta t,$$

where $v_i(t + \Delta t)$ has constant absolute value v and direction given by the angle

$$\theta(t + \Delta t) = \langle \theta(t) \rangle_r + \Delta\theta,$$

with $\langle \theta \rangle_r = \arctan \frac{\langle \sin\theta \rangle_r}{\langle \cos\theta \rangle_r}$ being the average direction of the neighbouring particles, within a circle of radius r, and $\Delta\theta$ the random noise, uniformly distributed on the interval $\left[-\frac{\eta}{2}, \frac{\eta}{2} \right]$. We have three parameters: the noise η, the density $\rho := N/L^2$, and the distance v a particle makes between two time-steps.

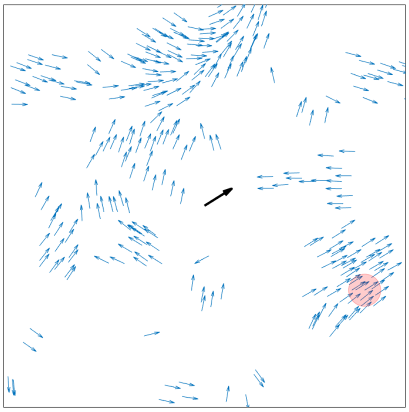

Figure 1.1: Simulation of 300 individuals after 200 time steps of the Vicsek model, with parameters $L = 25$, noise $\eta = 0.1$. In red the interaction circle (of radius $r = 1$) of one individual. The black arrow denotes the average direction.

As we can see in the above figure, the particles tend to form groups moving coherently in random directions. After some time, for large densities and little noise, the motion becomes ordered; in this case – where $\langle \cdot \rangle_r$ is the arithmetic average taken over all particles within a circle of radius r – if there were no noise, the collective final direction would coincide with the initial mean direction.

This work is structured as follows: in the second chapter, using as main reference [2], by Bolley et al., we introduce the time-continuous version of the discrete Vicsek model. We define the empirical distribution

$$f^N(x, v, t) := \frac{1}{N} \sum_{j=1}^{N} \delta_{(X_t^j, V_t^j)}(x, v),$$

where X_t^i and V_t^i represent the position and speed of the i-th particle at time t. We will show that, as $N \to \infty$, its limit is a probability density function f which satisfies a PDE of the form

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \big((\mathbb{I} - v \otimes v) \bar{J}_f f \big) = d \Delta_v f. \tag{1.1}$$

In fact, the main result of the second chapter is the following *propagation of chaos* result: the existence and uniqueness of N independent processes $(\bar{X}_t^i, \bar{V}_t^i)$, $i = 1, \ldots, N$, whose law is given by (1.1), such that, for all $T \geqslant 0$, there exists a constant $C > 0$ such that

$$\mathbb{E}\left[|X_t^i - \bar{X}_t^i|^2 + |V_t^i - \bar{V}_t^i|^2\right] \leqslant \frac{C}{N},$$

for all $0 \leqslant t \leqslant T$, $N \geqslant 1$, $1 \leqslant i \leqslant N$.

In the third chapter, inspired by [5], of Degond, Frouvelle and Liu, we use a *hydrodynamic scaling* to study the large scale behaviour of the model. In particular, we are going to perform the change of variable $\hat{x} = \epsilon x$ and $\hat{t} = \epsilon t$. This allows us to study the rescaled function $f^\epsilon(\hat{x}, \omega, \hat{t}) = f(x, \omega, t)$ and the PDE it satisfies:

$$\epsilon(\partial_t f^\epsilon + \omega \cdot \nabla_x f^\epsilon) = Q(f^\epsilon) + O(\epsilon^2), \tag{1.2}$$

where $Q(f)$ is a *collision operator*. As $\epsilon \to 0$, since Q is the only term of order 0 in ϵ, particular interest lies in the equilibria of this equation, i.e. the functions which belong to the null-space of Q. We find a *phase transition*: the form of the equilibria, as well as the rate of convergence to them, depends on whether the local density $\rho^\epsilon = \int_{\mathbb{S}} f^\epsilon \, d\omega$ is above or below the threshold value n. In particular, if $\rho^\epsilon \leqslant n$, then the only equilibria are the isotropic ones $f = \rho > 0$; if $\rho^\epsilon > n$, then we also have equilibria of the form $f = \rho M_{\kappa\Omega}$, where $M_{\kappa\Omega}$ is the *Von-Mises-Fischer distribution*.
In the first case, the isotropic equilibria are stable; in the second one, the isotropic equilibria become unstable, and there is exponentially fast convergence to the anisotropic ones.

In fact, the fourth chapter consists of the proof of the main result of the previous chapter, which provides the rates of convergence to equilibrium of the solutions to (1.2). Again, these rates will depend on whether the local density $\rho = \int_{\mathbb{S}} f \, d\omega$ is greater or smaller than the dimension n.
The proof will use various notions on Sobolev spaces, which we are going to recall, like the Poincaré inequality on the sphere and the Sobolev embedding theorem.

Finally, in the fifth chapter, using techniques found in [4] and [9], we will present a generalization of the classic Vicsek model, where two populations with different diffusion coefficients are interacting. Also in this case, a phase transition regarding the form of the equilibria will be observed.

I would like to thank my supervisor, Professor Paolo Dai Pra, for the invaluable guidance and precious advice he has offered throughout our work together.

2

Mean-field limit

We now proceed to study the continuous-time model, where the dynamics is not limited to a square, but happens in the infinite space $\mathbb{R}^n$.

First, we are going to provide the stochastic differential system that describes the dynamics. Following the contributions by F. Bolley et al. in [2], we consider N interacting particles in $\mathbb{R}^n$, described by their positions and orientation vectors. The equation for the space variable will be the usual $x_t = v_t dt$, while the SDE for the velocity will include the stochastic integral term; in particular, we will use the operator $\mathbb{I} - v \otimes v$ for the projection orthogonal to v.

Then, we will give various results for the mean-field limit of the model; in particular, existence and uniqueness for the solution of the mean-field system. A number of very helpful tools for this section come from [15].

2.1 *Introduction of the model*

Let $\{(X^i, V^i)\}_{i=1,\dots,n}$ denote the position and orientation vectors of N interacting particles in $\mathbb{R}^n$.
The *mean momentum* J^i of the neighbourhood of the i-th particle is defined by setting

$$J^i = \frac{1}{N} \sum_{j=1}^{N} K\big(|X^j - X^i|\big) V^j.$$

The function K is called an *interaction kernel*, which we suppose to be isotropic, i.e. depending only on the distance $|X^j - X^i|$ between particle i and its neighbours.
We assume that the processes satisfy the following system of coupled Stratonovich stochastic differential equations (the projection operator $P(v) := \mathbb{I} - v \otimes v$ constrains the norm of

the velocity to be constantly equal to 1):

$$\begin{cases} dX_t^i = V_t^i dt, \\ dV_t^i = \sqrt{2d}(\mathbb{I} - V_t^i \otimes V_t^i) \circ dB_t^i + (\mathbb{I} - V_t^i \otimes V_t^i) J_t^i dt. \end{cases} \tag{2.1}$$

Let $\mathbb{S}^{n-1}$ be the sphere of dimension $n-1$ in $\mathbb{R}^n$. We suppose that the initial data $(X_0^i, V_0^i) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$, $1 \leqslant i \leqslant N$, are independent and identically distributed random processes. Moreover, $(B_t^i)_{t\geqslant 0}$ are N independent n-dimensional Brownian motions.[1]

First, we write the previous SDE system as the equivalent Itô-type system. From the Itô-Stratonovich calculus found in [13], we have the following

Proposition 2.1. If $(X_t)_{t\geqslant 0}$ is an $\mathbb{R}^n$-valued random process, and $(B_t)_{t\geqslant 0}$ is a p-dimensional Brownian motion, the Stratonovich SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t) \circ dB_t,$$

is equivalent to the Itô SDE

$$dX_t = \tilde{b}(t, X_t)dt + \sigma(t, X_t)dBt,$$

where the drift term $\tilde{b}(t, X_t)$ is given, component-wise, by

$$\tilde{b}_i(t, X_t) = b_i(t, X_t) + \frac{1}{2} \sum_{j=1}^{p} \sum_{k=1}^{n} \frac{\partial \sigma_{ij}}{\partial x_k} \sigma_{kj}, \quad 1 \leqslant i \leqslant n.$$

Applying this formula to our problem, we have that (2.1) can be written as

$$\begin{cases} dX_t^i = V_t^i dt \\ dV_t^i = \sqrt{2d}(\mathbb{I} - V_t^i \otimes V_t^i) dB_t^i + (\mathbb{I} - V_t^i \otimes V_t^i) J_t^i dt - (n-1) V_t^i dt. \end{cases} \tag{2.2}$$

This system will be the main object of our study.

The mean-field limit. As we have said before, the main purpose of this chapter is to show that, as $N \to \infty$, the N $\mathbb{R}^{2n}$-valued interacting processes $(X_t^i, V_t^i)_{t\geqslant 0}$, solutions of (2.2), behave like the solutions $(\bar{X}_t^i, \bar{V}_t^i)_{t\geqslant 0}$ of the following non-linear[2] SDE system

$$\begin{cases} d\bar{X}_t^i = \bar{V}_t^i dt \\ d\bar{V}_t^i = \sqrt{2d}(\mathbb{I} - \bar{V}_t^i \otimes \bar{V}_t^i) dB_t^i + (\mathbb{I} - \bar{V}_t^i \otimes \bar{V}_t^i) \bar{J}_{f_t}(\bar{X}_t^i) dt - (n-1) \bar{V}_t^i dt \\ (\bar{X}_0^i, \bar{V}_0^i) = (X_0^i, V_0^i),\ f_t = \mathrm{Law}(\bar{X}_t^i, \bar{V}_t^i), \end{cases} \tag{2.3}$$

where $\bar{J}_f(x) := \int_{\mathbb{R}^n \times \mathbb{S}} K(|x-y|) f(y, \omega) \omega \, dy d\omega$, for all $x \in \mathbb{R}^n$.

[1] From now on we will simply write $\mathbb{S}$ for the sphere in $\mathbb{R}^n$.

[2] The non-linearity is due to the fact that we require f_t to be the law of the $2n$-dimensional process.

We recall that a mean-field equation is a model that describes the evolution of a typical particle, subject to the collective interaction created by a large number of other individuals. The state of the typical particle is given by its phase space density; the force field exerted by the other particles is approximated by the average with respect to the phase space density of the force field exerted on that particle from each point in the phase space.

We wish to derive a mean-field limit for (2.2) as the number of particles N tends to infinity. In order to do so, we define the empirical distribution $f^N(x,v,t) := \frac{1}{N}\sum_{j=1}^{N}\delta_{(X_t^j,V_t^j)}(x,v)$.

→ When there is no noise (i.e. $d=0$) it can be easily shown that f^N is a weak solution for the following partial differential equation

$$\partial_t f^N + v\cdot\nabla_x f^N + \nabla_v\cdot((\mathbb{I}-v\otimes v)\bar{J}_{f^N}f^N) = 0.$$

→ When noise is present (i.e. $d\neq 0$) the empirical distribution f^N tends to a probability density function f satisfying

$$\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot\left((\mathbb{I}-v\otimes v)\bar{J}_f f\right) = d\Delta_v f. \tag{2.4}$$

Without loss of generality, we can suppose that the intensity parameter d and the total weight $K_0 := \int_{\mathbb{R}^n} K(x)dx$ are both equal to 1.[3]

In the following section we show in detail this result, under the assumption that the kernel K is Lipschitz and bounded.

2.2 *First results*

Theorem 2.2. Let f_0 be a probability measure on $\mathbb{R}^n\times\mathbb{S}$ with finite second moment in $x\in\mathbb{R}^n$ and let (X_0^i,V_0^i) for $1\leqslant i\leqslant N$ be N independent random variables with law f_0. Let also K be a Lipschitz and bounded map on $\mathbb{R}^n$. Then

i. There exists a pathwise unique global solution to the SDE system (2.2), with initial data (X_0^i,V_0^i) for $i=1,\dots,N$.

ii. There exists a pathwise unique global solution to the nonlinear SDE system (2.3), with initial data (X_0^i,V_0^i) for $i=1,\dots,N$.

iii. There exists a unique global weak solution to the nonlinear PDE (2.4), with initial datum f_0. Moreover, it is the law of the solution to (2.3).

[3]It is enough to consider the rescaled functions $\tilde{f}(x,v,t) = f(\frac{x}{d},v,\frac{t}{d})$ and $\tilde{K}(x) = \frac{1}{K_0 d^n}K(\frac{x}{d})$ and notice that, if we suppose that K is integrable and that K_0 is positive, $\tilde{f}$ satisfies (2.4) with $d=1$ and K replaced by $\tilde{K}$ Moreover, $\int_{\mathbb{R}^n}\tilde{K}(x)dx = 1$.

Proof. We start by setting $\sigma_1(v) \equiv P(v) := \mathbb{I} - \frac{v\otimes v}{|v|^2}$, $\sigma_2(v) := \frac{v}{|v|^2}$, for all v such that $|v| \geqslant 1/2$, and $\sigma_3(v) = v$ if $|v| \leqslant 2$.

i. Consider the SDE system

$$\begin{cases} dX_t^i = V_t^i dt \\ dV_t^i = \sqrt{2}\sigma_1(V_t^i)dB_t^i + \sigma_1(V_t^i)J_t^i dt - (n-1)\sigma_2(V_t^i)dt, \end{cases} \tag{2.5}$$

with initial data $(X_0^i, V_0^i) \in \mathbb{R}^n \times \mathbb{S}$. We notice that the coefficients of this system are locally Lipschitz.

Firstly, we show that $|V_t^i|$ is constantly equal to 1. As long as $|V_t^i| \geqslant \frac{1}{2}$, we have (using the fact that $V^i \cdot P(V^i)y = 0$ for all $y \in \mathbb{R}^n$)

$$\begin{aligned}
d|V^i|^2 &= 2V^i dV_t^i + \frac{1}{2}\frac{\partial^2 |V^i|^2}{\partial (V^i)^2} d\langle V^i \rangle = \\
&= 2\sqrt{2}V^i \cdot P(V^i)dB^i + 2V^i J^i dt - 2(n-1)\frac{V^i \cdot V^i}{|V^i|^2}dt + 2d\langle V^i\rangle = \\
&= -2(n-1)dt + 2d\langle V^i \rangle = \\
&= -2(n-1)dt + 2\sum_{k,l=1}^n \partial_{kl}\, d\left\langle B_k^i - \sum_{p=1}^n \tfrac{V_k^i V_p^i}{|V^i|^2}B_p^i, B_l^i - \sum_{q=1}^n \tfrac{V_l^i V_q^i}{|V^i|^2}B_q^i \right\rangle = \\
&= -2(n-1)dt + 2\sum_{k=1}^n d\langle B_k^i, B_k^i\rangle - 2\sum_{k=1}^n d\left\langle \sum_{p=1}^n \tfrac{V_k^i V_p^i}{|V^i|^2}B_p^i, B_k^i \right\rangle + \\
&\quad - 2\sum_{k=1}^n d\langle B_k^i, \sum_{q=1}^n \tfrac{V_k^i V_q^i}{|V^i|^2}B_q^i\rangle + 2\sum_{k=1}^n d\left\langle \sum_{p=1}^n \tfrac{V_k^i V_p^i}{|V^i|^2}B_p^i, \sum_{q=1}^n \tfrac{V_k^i V_q^i}{|V^i|^2}B_q^i \right\rangle = \\
&= -2(n-1)dt + 2\sum_{k=1}^n \left(1 - 2\tfrac{(V_k^i)^2}{|V^i|^2} + \sum_{p=1}^n \frac{(V_k^i)^2\left(V_p^i\right)^2}{|V^i|^2}\right) dt = \\
&= -2(n-1)dt + (2n-4+2)dt = 0.
\end{aligned}$$

This means that $|V_t^i| = 1$ up to explosion time and, since $dX_t^i = V_t^i dt$, the explosion time is infinite. We then have global existence and pathwise uniqueness for (2.5). Since the solution to this system has velocity of norm 1, we also get global existence for (2.2). Thanks to the same reasoning, pathwise uniqueness holds as well.

ii. Consider the following SDE system

$$\begin{cases} d\bar{X}_t^i = \bar{V}_t^i dt \\ d\bar{V}_t^i = \sqrt{2}(\mathbb{I} - \bar{V}_t^i \otimes \bar{V}_t^i)dB_t^i - (\mathbb{I} + \bar{V}_t^i \otimes \bar{V}_t^i)\bar{J}_{f_t}(\bar{X}_t^i)dt - (n-1)\bar{V}_t^i dt \\ (\bar{X}_0^i, \bar{V}_0^i) = (X_0^i, V_0^i),\ f_t = \mathrm{Law}(\bar{X}_t^i, \bar{V}_t^i). \end{cases}$$

If we redefine

$$\bar{J}_f(x) := \int_{\mathbb{R}^{2n}} K(|x-y|) f(y,\omega)\sigma_3(\omega) dy d\omega, \tag{2.6}$$

and if f_0 is a distribution on $\mathbb{R}^n \times \mathbb{S}$ with finite second moment in $x \in \mathbb{R}^n$, the nonlinear system

$$\begin{cases} d\bar{X}^i_t = \bar{V}^i_t dt \\ d\bar{V}^i_t = \sqrt{2}\sigma_1(\bar{V}^i_t) dB^i_t + \sigma_1(\bar{V}^i_t)\bar{J}_{f_t}(\bar{X}^i_t)dt - (n-1)\sigma_2(\bar{V}^i_t)dt \\ (\bar{X}^i_0, \bar{V}^i_0) = (X^i_0, V^i_0),\ f_t = \text{Law}(\bar{X}^i_t, \bar{V}^i_t), \end{cases} \tag{2.7}$$

has bounded and Lipschitz coefficients on $\mathbb{R}^{2n}$. We wish to prove that the system admits a pathwise unique strong solution.

We consider the processes $Y^i_t = (X^i_t, V^i_t)$ and $\bar{Y}^i_t = (\bar{X}^i_t, \bar{V}^i_t)$, respectively solutions of

$$dY^i_t = \frac{1}{N}\sum_{j=1}^N b(Y^i_t, Y^j_t)dt + \sigma(Y^i_t)dB^i_t, \tag{2.8}$$

and

$$d\bar{Y}^i_t = \int_{y\in\mathbb{R}^{2n}} b(\bar{Y}^i_t, y) f_t(dy)\ dt + \sigma(\bar{Y}^i_t)dB^i_t, \tag{2.9}$$

where $b(\cdot,\cdot)$ and $\sigma(\cdot)$ are Lipschitz functions, defined as

$$b(Y^i_t, y) = \Big(V^i_t, \sigma_1(\bar{V}^i_t)K(|X^i_t - x|)v - (n-1)\sigma_2(V^i_t)\Big) \qquad \text{for } Y^i_t = (X^i_t, V^i_t),$$
$$\sigma(Y^i_t) = \Big(0, \sqrt{2}\sigma_1(\bar{V}^i_t)\Big) \qquad \text{and } y = (x, v).$$

We introduce the *Wasserstein metric* on the set $\{p \in \mathcal{P}(\mathbb{R}^n) : \int_{\mathbb{R}^n}|x|p(dx) < +\infty\}$ of probability measures admitting finite mean value, defined by

$$\text{dist}(p_1, p_2) := \inf\left\{\int |x-y|p(dx,dy) : p \text{ has marginals } p_1 \text{ and } p_2\right\}. \tag{2.10}$$

We now omit the apex i, and use a Picard iteration: let $Y^{(0)}_t = \bar{Y}_0$, and define

$$Y^{(k+1)}_t = \int_0^t \int_{y\in\mathbb{R}^{2n}} b(Y^{(k)}_s, y) f^{(k)}_s(dy)ds + \int_0^t \sigma(Y^{(k)}_s)dB_s,$$

where $f^{(k)}_s$ is the law of $Y^{(k)}_s$. We wish to estimate

$$E^{(k)}_T := \mathbb{E}\left[\sup_{t\in[0,T]} |Y^{(k+1)}_t - Y^{(k)}_t|\right]. \tag{2.11}$$

Notice that, from the Lipschitzianity of $b(\cdot,\cdot)$, we have

$$\left|\int b(x_1,y)p_1(dy) - \int b(x_2,y)p_2(dy)\right| \leqslant L\left(|x_1-x_2| + \int |y_1-y_2| p(dy_1,dy_2)\right),$$

where p is any probability measure with marginals p_1 and p_2. In particular, the above inequality holds for the infimum among such measures. From this we get

$$\begin{aligned}
&\sup_{t\in[0,T]}\left|\int_0^t \int_{\mathbb{R}^{2n}} b(Y_s^{(k)},y)f_s^{(k)}(dy)ds - \int_{\mathbb{R}^{2n}} b(Y_s^{(k-1)},y)f_s^{(k-1)}(dy)ds\right| \leqslant \\
&\leqslant L\int_0^T \left(|Y_s^{(k)} - Y_s^{(k-1)}| + \operatorname{dist}\left(f_s^{(k)}, f_s^{(k-1)}\right)\right) ds \leqslant \\
&\leqslant L\int_0^T \left(|Y_s^{(k)} - Y_s^{(k-1)}| + \mathbb{E}\left[|Y_s^{(k)} - Y_s^{(k-1)}|\right]\right) ds.
\end{aligned}$$

Taking the average yields

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \int_{\mathbb{R}^{2n}} b(Y_s^{(k)},y)f_s(dy)ds - \int_{\mathbb{R}^{2n}} b(Y_s^{(k-1)},y)f_s(dy)ds\right|\right] \leqslant \\
&\leqslant 2L\int_0^T \int_{\mathbb{R}^{2n}} \mathbb{E}\left[|Y_t^{(k)} - Y_t^{(k-1)}|\right] \leqslant 2LTE_T^{(k-1)}.
\end{aligned} \tag{2.12}$$

We recall the *Burkholder-Davis-Gundy inequality*: for any $1 \leqslant p < +\infty$, there exist positive constants c_p, C_p such that, for all local martingales X, with $X_0 = 0$ and stopping times τ, the following inequalities hold

$$c_p\mathbb{E}\left[[X]_\tau^{p/2}\right] \leqslant \mathbb{E}\left[(X_\tau^*)^p\right] \leqslant C_p\mathbb{E}\left[[X]_\tau^{p/2}\right],$$

where $[X]$ denotes the quadratic variation of a process X, and $X_t^* := \sup_{s\leqslant t}|X_s\||$. Furthermore, for continuous local martingales, this statement holds for all $p \in (0,+\infty)$.

By using it on the diffusion term, we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \left(\sigma(Y_s^{(k)}) - \sigma(Y_s^{(k-1)})\right) dB_s\right|\right] \leqslant \\
&\leqslant C\mathbb{E}\left[\left(\int_0^T \left|\sigma(Y_s^{(k)}) - \sigma(Y_s^{(k-1)})\right|^2 ds\right)^{1/2}\right] \leqslant \\
&\leqslant LC\mathbb{E}\left[\left(\int_0^T \left|Y_s^{(k)} - Y_s^{(k-1)}\right|^2 ds\right)^{1/2}\right] \leqslant \\
&\leqslant LC\sqrt{T}\mathbb{E}\left[\sup_{t\in[0,T]}\left|Y_s^{(k)} - Y_s^{(k-1)}\right|\right].
\end{aligned}$$

Summing up the contributions of the drift and diffusion terms, we have that there exists a constant $C > 0$ such that

$$E_T^{(k)} \leqslant C(T + \sqrt{T}) E_T^{(k-1)}. \tag{2.13}$$

By induction on k, $\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{(k)}|\right] < +\infty$. So, if we denote by $\mathcal{M}$ the space of progressively measurable, cadlag, R^n-valued processes such that

$$\|Y\|_{\mathcal{M}} := \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|\right] < +\infty,$$

we have shown that $\sum_{k\geqslant 0} \|Y^{(k)}\|_{\mathcal{M}} < +\infty$, for a sufficiently small $T > 0$.

But, under this norm, $\mathcal{M}$ is *not* a complete space (because the sup-norm is not complete in the space of cadlag functions), so this condition is not enough to guarantee existence of a solution. We complete it by replacing the distance in sup-norm in the definition of $E_T^{(k)}$ with the *Skorohod distance*[4]

$$d_S(x,y) = \inf_{\lambda\in\Lambda} \left\{ \sup_t |\lambda(t) - t| \vee \sup_t \left| x(t) - y\big(\lambda(t)\big) \right| \right\},$$

where Λ denotes the class of strictly increasing, continuous mappings of $[0,1]$ onto itself:

$$D_S(X,Y) := \mathbb{E}\big[d_S(X,Y)\big].$$

Since d_S is dominated by the distance in sup-norm, $\big(Y^{(k)}\big)_{k\geqslant 0}$ is a Cauchy sequence under the metric D_S as well; we call Y^∞ its limit in $\mathcal{M}$. It is easy to show that Y^∞ solves (2.7), so we have existence of the solution for small T. Since the condition on T does not involve the initial condition, the argument can be iterated on adjacent time intervals, obtaining a solution on any time interval.

We are left with proving uniqueness. Let Φ be the map which associates to a probability density function f the law of the solution of

$$Y_t = \int_0^t \int_{y\in\mathbb{R}^{2n}} b(Y_s, y) f_s(dy) ds + \int_0^t \sigma(Y_s) dB_s. \tag{2.14}$$

We notice that, if $(Y_t)_{t\geqslant 0}$ is a solution of (2.7), then its law is a fixed point of Φ. Conversely, if f is such a fixed point, then (2.14) defines a solution of (2.7) up to time T. So it is enough to show that, at least for a small enough T, Φ is a contraction.

[4] See [1] for a proof of completeness under this metric.

Let f^1 and f^2 be two probability density functions, and Y^1, Y^2 the two corresponding processes. Using Gronwall's Lemma, we have

$$\begin{aligned} \operatorname{dist}\left(\Phi(f_t^1), \Phi(f_t^2)\right) &\leqslant \mathbb{E}\left[|Y_t^1 - Y_t^2|\right] \leqslant \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^1 - Y_t^2|\right] \leqslant \\ &\leqslant L\int_0^T \mathbb{E}\left[|Y_s^1 - Y_s^2| + \operatorname{dist}(f_s^1, f_s^2)\right] ds \leqslant K(T)\int_0^T \operatorname{dist}(f_s^1, f_s^2) ds. \end{aligned}$$

This concludes the proof of existence and uniqueness of the solution of (2.7). Finally, as long as $|\bar{V}_t^i| \geqslant \frac{1}{2}$, we can prove that $|\bar{V}_t^i| = 1$ for all $t \geqslant 0$. In particular, the obtained solution is a global solution for (2.3). Pathwise uniqueness follows as before.

iii. Let f_0 be a distribution on $\mathbb{R}^n \times \mathbb{S}$ with finite second moment in $x \in \mathbb{R}^n$; $(\bar{X}_0, \bar{V}_0)$ with law f_0; and $(\bar{X}_t, \bar{V}_t)_{t\geqslant 0}$ the solution to (2.3) with initial datum $(\bar{X}_0, \bar{V}_0)$. Then its law f_t, as a measure on $\mathbb{R}^{2n}$, satisfies

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^{2n}} \phi\, df_t = \int_{\mathbb{R}^{2n}} \big(&v\cdot\nabla_x\phi + \text{Hess}_v\phi : (\mathbb{I} - v\otimes v) + \\ &- \nabla_v\phi\cdot(\mathbb{I} - v\otimes v)(\bar{J}_{f_t}) - (n-1)v\cdot\nabla_v\phi\big)df_t. \end{aligned} \tag{2.15}$$

In fact, let $\phi \in C^\infty(\mathbb{R}^{2n})$, and consider

$$\begin{aligned} \phi(\bar{X}_t^i, \bar{V}_t^i) = \phi(\bar{X}_0, \bar{V}_0) &+ \int_0^t \tfrac{\partial}{\partial s}\phi\, ds + \int_0^t v\cdot\nabla_x\phi\, ds + \\ &+ \int_0^t \Big(\big(\sigma_1(V_s^i)(\bar{J}_{f_t}) - (n-1)\sigma_2(V_s^i)\big)\cdot\nabla_v\phi\Big)\, dB_s + \\ &+ \tfrac{1}{2}\int_0^t Hess_v\phi : (\mathbb{I} - v\otimes v)ds, \end{aligned}$$

where $\bar{J}_{f_t}$ is redefined as in (2.6).

Deriving the above expression with respect to t and integrating on $\mathbb{R}^{2n}$ w.r.t. the measure f_t, we get precisely (2.15). We recall that $|\bar{V}_t| = 1$ a.s., so f_t is concentrated on $\mathbb{R}^n \times \mathbb{S}$; with this in mind, we define the restriction of f_t on $\mathbb{R}^n \times \mathbb{S}$ as the function F_t such that

$$\int_{\mathbb{R}^n\times\mathbb{S}} \Phi\, dF_t = \int_{\mathbb{R}^{2n}} \phi\, df_t,$$

for all continuous maps Φ on $\mathbb{R}^n \times \mathbb{S}$, where ϕ is any continuous and bounded map on $\mathbb{R}^{2n}$ equal to Φ on $\mathbb{R}^n \times \mathbb{S}$.

Let Φ and ϕ be C_C^∞ on their respective domains, such that $\phi(x, v) = \Phi(x, \frac{v}{|v|})$, for all $\frac{1}{2} \leqslant |v| \leqslant 2$. We have that $v\cdot\nabla_v\phi = 0$ for all (x, v) in the support of f_t (because ϕ is

0-homogeneous in v for $\frac{1}{2} \leqslant v \leqslant 2$), and

$$\frac{d}{dt}\int_{\mathbb{R}^n\times\mathbb{S}} \Phi\, df_t = \frac{d}{dt}\int_{\mathbb{R}^{2n}} \phi\, df_t = \int_{\mathbb{R}^{2n}} \big(v\cdot\nabla_x\phi - \nabla_v\phi\cdot(\mathbb{I}-v\otimes v)(\bar{J}_{f_t}) + \\ + \Delta_v\phi\big)df_t.$$

Since Φ and ϕ have the same x-dependence, we have that $v\cdot\nabla_x\Phi \equiv v\cdot\nabla_x\phi$ on $\mathbb{R}^n\times\mathbb{S}$. Moreover, for $(x,\omega)\in\mathbb{R}^n\times\mathbb{S}$, $\nabla_\omega\Phi = \nabla_v\phi$ and $\Delta_\omega\Phi = \Delta_v\phi$. Finally,

$$\frac{d}{dt}\int_{\mathbb{R}^n\times\mathbb{S}} \Phi\, df_t = \int_{\mathbb{R}^n\times\mathbb{S}} \big(\omega\cdot\nabla_x\phi - \nabla_\omega\phi\cdot(\mathbb{I}-\omega\otimes\omega)(\bar{J}_{f_t}) + \Delta_v\phi\big)df_t, \tag{2.16}$$

which ensures that F_t is a weak solution of (2.4).

We now prove uniqueness. Let f^1, f^2 be two solutions of (2.4) with same initial datum f_0; for each time t, we view them as measures on $\mathbb{R}^n$ which are concentrated on $\mathbb{R}^n\times\mathbb{S}$.

Let $(\bar{X}_t^1,\bar{V}_t^1)$ and $(\bar{X}_t^2,\bar{V}_t^2)$ be the solutions of (2.3), with common initial datum $(\bar{X}_0,\bar{V}_0)$ of law f_0, and with drift given by $\bar{J}_{f_t^1}$ and $\bar{J}_{f_t^2}$ respectively. Then their respective laws g_t^1 and g_t^2, as measures on $\mathbb{R}^{2n}$ are solutions of the following

$$\partial_t g_t^i + v\cdot\nabla_x g_t^i = \sum_{k,l=1}^{n} \frac{\partial^2}{\partial v_k\partial v_l}\big((\sigma_1\sigma_1^T)_{k,l}g_t^i\big) + \nabla_v\cdot\Big(g_t^i\big(\sigma_1(\bar{J}_{f_t^i}) + (n-1)\sigma_2\big)\Big).$$

But also f_t^i $(i=1,2)$ solve this PDE on $\mathbb{R}^{2n}$ with bounded and regular coefficients. Since this is a linear parabolic PDE, uniqueness holds. We then have that $f_t^i = g_t^i$ $(i = 1,2)$, and that $(\bar{X}_t^i,\bar{V}_t^i)_{t\geqslant 0}$ are solutions to the nonlinear SDE (2.7), for which we have already proved uniqueness. We can conclude that $f_t^1 = g_t^1 = g_t^2 = f_t^2$.

□

Under the same assumptions as the above Theorem we can almost immediately prove the following mean-field result:

Theorem 2.3. For all $T\geqslant 0$, there exists a constant $C>0$ such that:

$$\mathbb{E}\left[|X_t^i - \bar{X}_t^i|^2 + |V_t^i - \bar{V}_t^i|^2\right] \leqslant \frac{C}{N}, \tag{2.17}$$

for all $0\leqslant t\leqslant T$, $N\geqslant 1$, $1\leqslant i\leqslant N$.

Proof. Consider the $\mathbb{R}^{2n}$-valued processes $Y_t^i := (X_t^i, V_t^i)$ and $\bar{Y}_t^i := (\bar{X}_t^i,\bar{V}_t^i)$. We also define two Lipschitz and bounded functions $b(\cdot,\cdot)$ and $\sigma(\cdot)$ that represent the drift and diffusion coefficients for the new processes. More precisely, for $y_i = (x_i, v_i), y_j = (x_j, v_j) \in \mathbb{R}^n \times \mathbb{S}$, we have

$$b(y_i, y_j) = \left(v_i, (\mathbb{I} - v_i \otimes v_i)K(x_j - x_i)v_j - (n-1)v_i\right),$$
$$\sigma(y_i) = \left(0, \sqrt{2}(\mathbb{I} - v_i \otimes v_i)\right).$$

We compute

$$\begin{aligned} Y_t^i - \bar{Y}_t^i &= \int_0^t \frac{1}{N}\sum_{j=1}^N \left[\left(b(Y_s^i, Y_s^j) - \int_{\mathbb{R}^{2n}} b(\bar{Y}_s^i, y) f_s(dy)\right)\right] ds + \int_0^t \left(\sigma(Y_s^i) - \sigma(\bar{Y}_s^i)\right) dB_s^i = \\ &= \int_0^t \frac{1}{N}\sum_{j=1}^N \left[\left(b(Y_s^i, Y_s^j) - b(\bar{Y}_s^i, Y_s^j)\right) + \left(b(\bar{Y}_s^i, Y_s^j) - b(\bar{Y}_s^i, \bar{Y}_s^j)\right) + \right. \\ &\qquad \left. + \left(b(\bar{Y}_s^i, \bar{Y}_s^j) - \int_{\mathbb{R}^{2n}} b(\bar{Y}_s^i, y) f_s(dy)\right)\right] ds + \int_0^t \left(\sigma(Y_s^i) - \sigma(\bar{Y}_s^i)\right) dB_s^i. \end{aligned}$$

Our aim is to provide an estimate for the second moment; we consider the process

$$|Y_t^i - \bar{Y}_t^i|^{*2} = \sup_{u \leqslant t} |Y_u^i - \bar{Y}_u^i|^2,$$

and apply the Burkholder-Davis-Gundy inequality on the stochastic integral term. We use the fact that the expected valued of the stochastic integral of an M^2 process is equal to zero:

$$\begin{aligned} &\mathbb{E}\left[|Y_t^i - \bar{Y}_t^i|^{*2}\right] \leqslant \\ &\leqslant \mathbb{E}\left[\left|\int_0^t \frac{1}{N}\sum_{j=1}^N \left[\left(b(Y_s^i, Y_s^j) - b(\bar{Y}_s^i, Y_s^j)\right) + \left(b(\bar{Y}_s^i, Y_s^j) - b(\bar{Y}_s^i, \bar{Y}_s^j)\right) + \right.\right.\right. \\ &+ \left.\left.\left. \left(b(\bar{Y}_s^i, \bar{Y}_s^j) - \int_{\mathbb{R}^{2n}} b(\bar{Y}_s^i, y) f_s(dy)\right)\right] ds\right|^{*2}\right] + \mathbb{E}\left[\left|\int_0^t \left(\sigma(Y_s^i) - \sigma(\bar{Y}_s^i)\right) dB_s^i\right|^{*2}\right] \leqslant \\ &\leqslant L \cdot \int_0^t \mathbb{E}\left[|Y_s^i - \bar{Y}_s^i|^{*2}\right] + \mathbb{E}\left[\frac{1}{N}\sum_{j=1}^N |Y_s^j - \bar{Y}_s^j|^{*2}\right] + \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N b_s(\bar{Y}_s^i, \bar{Y}_s^j)\right|^2\right], \end{aligned}$$

where $b_s(y, y') := b(y, y') - \int b(y, z) f_s(dz)$. Summing over i we find

$$\sum_{i=1}^N \mathbb{E}\left[|Y^i - \bar{Y}^i|_t^{*2}\right] \leqslant L' \int_0^t \sum_{i=1}^N \left(\mathbb{E}\left[|Y^i - \bar{Y}^i|_s^{*2}\right] + \sum_{i=1}^N \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N b_s(\bar{Y}_s^i, \bar{Y}_s^j)\right|^2\right]\right) ds.$$

Applying Gronwall's Inequality with $u(t) := \sum_{i=1}^N \mathbb{E}\left[|Y^i - \bar{Y}^i|_t^{*2}\right]$, we get

$$\sum_{i=1}^N \mathbb{E}\left[|Y^i - \bar{Y}^i|_t^{*2}\right] \leqslant L(t) \int_0^t \sum_{i=1}^N \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N b_s(\bar{Y}_s^i, \bar{Y}_s^j)\right|^2\right] ds.$$

We have

$$\mathbb{E}\Big[\Big|\frac{1}{N}\sum_{j=1}^{N} b_s(\bar{Y}_s^i, \bar{Y}_s^j)\Big|^2\Big] = \frac{1}{N^2}\mathbb{E}\Big[\sum_{j,k=1}^{N} b_s(\bar{Y}_s^i, \bar{Y}_s^j) b_s(\bar{Y}_s^i, \bar{Y}_s^k)\Big] \leqslant \frac{C}{N},$$

where we have used the fact that, since $b_s(\cdot, \cdot)$ is centred with respect to the second variable, then $\mathbb{E}\big[b_s(\bar{Y}_s^i, \bar{Y}_s^j) b_s(\bar{Y}_s^i, \bar{Y}_s^k)\big] = 0$ for $j \neq k$. The claim follows. □

3

Macroscopic limit

The objective of this chapter is to study the large-scale behaviour of the system. In order to do so, we will perform what is known as a *hydrodynamic scaling* on the space and time variables.

Most of the results of this chapter are given under the hypothesis of space-homogeneity of the model. In other words, we will often assume that the probability density function f is just a function of time t and velocity ω, and does not depend on the space variable x. In the homogeneous case, all results are proven rigorously.

We will then perform a couple of ansatz, that will allow us to assume the behaviour of the model in the general non-homogeneous case. The aim of this second part of the chapter is not to provide detailed proofs of all results, but instead to give a reasonable and logical explanation as to why we work in this way.

3.1 *Hydrodynamic scaling*

We perform the *hydrodynamic scaling* by introducing a small parameter $\epsilon > 0$ and implement the change of variables $\hat{x} = \epsilon x$, $\hat{t} = \epsilon t$; we define $f^\epsilon(\hat{x}, \omega, \hat{t}) := f(x, \omega, t)$ and $K^\epsilon(\hat{x}) := \dfrac{K(x)}{\epsilon^n}$. As a result of this scaling, by studying the formal limit as $\epsilon \to 0$, we are observing the system for large times and distances.

From the results of the previous chapter we have that f^ϵ satisfies the following partial differential equation

$$\epsilon\big(\partial_t f^\epsilon + \omega \cdot \nabla_x f^\epsilon\big) = -\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega)\bar{J}_{f^\epsilon} f^\epsilon\big) + \Delta_\omega f^\epsilon, \tag{3.1}$$

where $\bar{J}^\epsilon_f(x,t) = \int_{\mathbb{S}} (K^\epsilon * f)(x, \omega, t)\omega \, d\omega$.

Lemma 3.1. If we suppose that f^ϵ does not present any pathological behaviour as $\epsilon \to 0$, we get the following expansion:

$$\bar{J}^\epsilon_{f^\epsilon}(x,t) = J_{f^\epsilon}(x,t) + O(\epsilon^2),$$

where $J_f(x,t) := \int_{\mathbb{S}} f(x,\omega,t)\omega \, d\omega$ is the local flux.

Proof. We have

$$\begin{aligned}
\bar{J}^\epsilon_{f^\epsilon}(x,t) &= \int_{\mathbb{R}^n\times\mathbb{S}} K^\epsilon(|x-y|)f^\epsilon(y,\omega,t)\omega \, dy \, d\omega = \\
&= \int_{\mathbb{R}^n\times\mathbb{S}} K^\epsilon(|x-y|)\left(f^\epsilon(x,\omega,t) + \epsilon\zeta\cdot\nabla_x f^\epsilon(x,\omega,t) + O(\epsilon^2)\right)\omega \, d\zeta \, d\omega = \\
&= \int_{\mathbb{R}^n\times\mathbb{S}} K(|\zeta|)\left(f^\epsilon(x,\omega,t) + \epsilon\zeta\cdot\nabla_x f^\epsilon(x,\omega,t) + O(\epsilon^2)\right)\omega \, d\zeta \, d\omega = \\
&= J_{f^\epsilon}(x,t) + O(\epsilon^2),
\end{aligned}$$

where we have performed the change of variable $y = x + \epsilon\zeta$, and expanded f to the first order in ϵ. □

Remark. The remainder is of order ϵ^2 because the kernel is isotropic; with an anisotropic kernel (e.g. one favouring the forward direction) the remainder would be of order ϵ. This causes a substantial change in the dynamics.

Definition 3.2. We define the *collision operator*

$$Q(f) := -\nabla_\omega \cdot \left((\mathbb{I} - \omega\otimes\omega)J_f f\right) + \Delta_\omega f.$$

Notice that local conservation of mass holds: $\int_{\mathbb{S}} Q(f)dw = 0$.

Ignoring the $O(\epsilon^2)$ term, we can use this operator, and the previous Lemma, to rewrite (3.1) as

$$\epsilon(\partial_t f^\epsilon + \omega\cdot\nabla_x f^\epsilon) = Q(f^\epsilon). \tag{3.2}$$

We wish to study the limit of this partial differential equation as $\epsilon \to 0$, so particular interest lies on the *equilibria* for the collision operator, i.e. the functions f such that $Q(f) = 0$.

Remark. We notice that the operator Q acts only on the direction variable ω, and leaves the other variables x and t as parameters. It is therefore legitimate to study the properties of Q as an operator acting on functions of ω only.

3.2 *Study of equilibria*

In this section we see how the dynamics of our model, in particular the form of the equilibria, changes according to whether the density is larger or smaller than a given threshold value.

Definition 3.3. We introduce the following notions, which we will need later on:

1. The *local density* of a function $h : \mathbb{S} \to [0,1]$ is given by
$$\rho_h := \int_{\mathbb{S}} h(\omega) d\omega.$$

2. Let $\Omega \in \mathbb{S}$ and $\kappa \geqslant 0$. The *Von-Mises-Fischer distribution* with concentration parameter κ and orientation Ω is the probability density on the sphere defined by
$$M_{\kappa\Omega}(\omega) := \frac{e^{\kappa\omega\cdot\Omega}}{\int_{\mathbb{S}} e^{\kappa w\cdot\Omega} dw}, \quad \omega \in \mathbb{S}.$$
If we denote by $\langle\cdot\rangle_{M\kappa\Omega}$ the average over this probability measure, for functions γ depending only on $\omega \cdot \Omega =: \cos\theta$, their average does not depend on Ω and will be denoted, using spherical coordinates, by
$$\langle\gamma(\omega\cdot\Omega)\rangle_{M_{\kappa\Omega}} \equiv \langle\gamma(\cos\theta)\rangle_{M_\kappa} := \frac{\int_0^\pi \gamma(\cos\theta) e^{\kappa\cos\theta} \sin^{n-2}\theta d\theta}{\int_0^\pi e^{\kappa\cos\theta}\sin^{n-2}\theta d\theta}.$$

3. The *flux* of the Von-Mises-Fischer distribution is defined as the function
$$J_{M\kappa\Omega} = \langle\omega\rangle_{M_{\kappa\Omega}}.$$
Decomposing $\omega = \omega_\| \Omega + \omega_\perp \Omega^\perp$, where $\omega_\|$ and $\omega_\perp$ are respectively its components parallel and orthogonal to Ω, we find that
$$J_{M\kappa\Omega} = \langle\omega\rangle_{M_{\kappa\Omega}} = \langle\omega_\|\rangle_{M_{\kappa\Omega}}\Omega + \langle\omega_\perp\rangle_{M_{\kappa\Omega}}\Omega^\perp = \langle\cos\theta\rangle_{M\kappa}\Omega =: c(\kappa)\Omega.$$
Notice that when $c(\kappa) = 0$ (that is, when $\kappa = 0$), $M_{\kappa\Omega}$ is the uniform distribution; when $c(\kappa) \to 1$, then we have $M_{\kappa\Omega}(\omega) \to \delta_\Omega(\omega)$.

Remark. Notice that, since $M_{\kappa\Omega}$ depends on κ and Ω only through their product, we can consider M_J for any $J \in \mathbb{R}^n$. Furthermore, $J_h = \int_{\mathbb{S}} h(\omega)\omega\, d\omega$ does not depend on ω, so there exist some $\kappa > 0$ and $\Omega \in \mathbb{R}^n$ such that $\kappa\Omega = J_h$.

From $\nabla_\omega(M_J) = (\mathbb{I} - \omega \otimes \omega) J M_J$, we get
$$\int_{\mathbb{S}} Q(h)\frac{g}{M_{J_h}} d\omega = -\int_{\mathbb{S}} \nabla_\omega\Big(\frac{h}{M_{J_h}}\Big) \cdot \nabla_\omega\Big(\frac{g}{M_{J_h}}\Big) M_{J_h} d\omega \Rightarrow$$
$$\Rightarrow \int_{\mathbb{S}} Q(h)\frac{h}{M_{J_h}} d\omega = -\int_{\mathbb{S}} \Big|\nabla_\omega\Big(\frac{h}{M_{J_h}}\Big)\Big|^2 M_{J_h} d\omega \leqslant 0.$$

From the above computations we have that, if h is an equilibrium for Q, then $\frac{h}{M_{J_h}} = \rho$ does not depend on ω. In order words, if h is an equilibrium, then it is of the form $\rho M_{\kappa\Omega}$, for $\rho > 0$, $\kappa \geqslant 0$, $\Omega \in \mathbb{S}$.

By definition, $J_{M_{\kappa\Omega}} = \langle\omega\rangle_{M_{\kappa\Omega}} = c(\kappa)\Omega$, which implies that

$$\kappa\Omega = J_h = \int_{\mathbb{S}} \rho M_{\kappa\Omega}(\omega)\omega \, d\omega = \rho J_{M_{\kappa\Omega}} = \rho c(\kappa)\Omega.$$

From this we find the following *compatibility condition* for κ:

$$\rho c(\kappa) = \kappa. \tag{3.3}$$

Proposition 3.4 (Compatibility condition)**.**

1. If $\rho \leqslant n$, then $\kappa = 0$ is the unique solution of (3.3). The only equilibria are the isotropic ones, $h = \rho$ for an arbitrary $\rho \geqslant 0$.

2. If $\rho > n$, then (3.3) has 2 roots: $\kappa = 0$ and $\kappa(\rho) > 0$. The set of equilibria for $\kappa = 0$ is $h = \rho > n$; the ones associated to $\kappa(\rho)$ consist of the Von-Mises-Fischer distributions $\rho M_{\kappa(\rho)\Omega}$, for $\rho > n$ and $\Omega \in \mathbb{S}$ arbitrary, and it forms a manifold of dimension n.

Proof. Let us denote $\tilde{\sigma}(\kappa) = \frac{c(\kappa)}{\kappa}$. Thanks to the results in Appendix B, we have that $\tilde{\sigma} \xrightarrow{\kappa\to 0} \frac{1}{n}$. Moreover, since $c(\kappa) \leqslant 1$, the function $\tilde{\sigma}$ tends to 0 as $\kappa \to +\infty$. It is then enough to prove that $\tilde{\sigma}$ is decreasing: we would then have a 1-1 correspondence from $\mathbb{R}_+^*$ to $(0, \frac{1}{n})$, and the compatibility condition for $\kappa > 0$ means exactly to solve $\sigma = \tilde{\sigma}(\kappa)$.

We wish to compute $\tilde{\sigma}'(\kappa) = \frac{1}{\kappa}\left(c'(\kappa) - \frac{c(\kappa)}{\kappa}\right)$. We find that

$$\begin{aligned}
\frac{dc}{d\kappa} &= \frac{d}{d\kappa}\frac{\int_0^\pi \cos\theta e^{\kappa\cos\theta}\sin^{n-2}\theta d\theta}{\int_0^\pi e^{\kappa\cos\theta}\sin^{n-2}\theta d\theta} = \\
&= \frac{\int_0^\pi \cos^2\theta e^{\kappa\cos\theta}\sin^{n-2}\theta d\theta}{\int_0^\pi e^{\kappa\cos\theta}\sin^{n-2}\theta d\theta} - \left(\frac{\int_0^\pi \cos\theta e^{\kappa\cos\theta}\sin^{n-2}\theta d\theta}{\int_0^\pi e^{\kappa\cos\theta}\sin^{n-2}\theta d\theta}\right)^2 = \\
&= 1 - \frac{\int_0^\pi \sin^2\theta e^{\kappa\cos\theta}\sin^{n-2}\theta d\theta}{\int_0^\pi e^{\kappa\cos\theta}\sin^{n-2}\theta d\theta} - c^2 = 1 - (n-1)\frac{c}{\kappa} - c^2,
\end{aligned} \tag{3.4}$$

so $\tilde{\sigma}'(\kappa) = \frac{1 - n\tilde{\sigma}(\kappa) - c(\kappa)^2}{\kappa}$. Thanks to the following Lemma, this expression is negative for $\kappa > 0$, and this ends the proof. □

Lemma 3.5. For any $\kappa > 0$, we have that $\beta := c^2(\kappa) + n\tilde{\sigma}(\kappa) - 1 > 0$.

Proof. Define $[\gamma(\cos\theta)]_\kappa = \int_0^\pi \gamma(\cos\theta)e^{\kappa\cos\theta}\sin^{n-2}\theta d\theta$. By definition,

$$\beta = \frac{\kappa[\cos\theta]_\kappa^2 + n[\cos\theta]_\kappa[1]_\kappa - \kappa[1]_\kappa^2}{\kappa[1]_\kappa^2},$$

so it is enough to prove that the numerator is positive. We expand it in κ. Denoting by $a_p = \frac{1}{(2p)!}\int_0^\pi \cos^{2p}\theta \sin^{n-2}\theta d\theta \geqslant 0$, we get

$$[1]_\kappa = \sum_{p=0}^{+\infty} a_p \kappa^{2p}, \qquad [\cos\theta]_\kappa = \sum_{p=0}^{+\infty}(2p+2)a_{p+1}\kappa^{2p+1}. \tag{3.5}$$

Using the integration by parts formula on a_{p+1}, we have

$$a_{p+1} = \frac{2p+1}{n-1}\Big(\frac{a_p}{(2p+1)(2p+2)} - a_{p+1}\Big), \tag{3.6}$$

from which we find the following induction relation:

$$(2p+2)a_{p+1} = \frac{a_p}{2p+n}. \tag{3.7}$$

We then have, for $\kappa > 0$,

$$\begin{aligned}
\beta\kappa[1]_\kappa^2 &= \sum_{k=0}^{+\infty}\kappa^{2k+1}\left(\sum_{p+q=k-1}(2p+2)a_{p+1}(2q+2)a_{q+1} + \sum_{p+q=k} n(2p+2)a_{p+1}a_q - a_p a_q\right) = \\
&= \sum_{k=0}^{+\infty}\kappa^{2k+1}\left(\sum_{p+q=k,p\geqslant 1} 2pa_p\tfrac{1}{2q+n}a_q + \sum_{p+q=k}(\tfrac{n}{2p+n}-1)a_p a_q\right) = \\
&= \sum_{k=0}^{+\infty}\kappa^{2k+1}\left(\sum_{p+q=k} 2p\Big(\tfrac{1}{2q+n} - \tfrac{1}{2p+n}\Big)\right) = \\
&= \sum_{k=0}^{+\infty}\kappa^{2k+1}\left(\sum_{p+q=k}\Big(p\Big(\tfrac{1}{2q+n}-\tfrac{1}{2p+n}\Big) + q\Big(\tfrac{1}{2p+n}-\tfrac{1}{2q+n}\Big)\Big)a_p a_q\right) = \\
&= \sum_{k=0}^{+\infty}\kappa^{2k+1}\left(\sum_{p+q=k}\tfrac{2(p-q)^2}{(2p+n)(2q+n)}a_p a_q\right) > 0, \text{ if } \kappa > 0.
\end{aligned}$$

□

3.3 *Rates of convergence*

Assume *spatial homogeneity* of the system[1].

In an effort to simplify notation, we write ρ^ϵ for ρ_{f^ϵ}. We consider the velocity probability density function $g^\epsilon := \frac{f^\epsilon}{\rho^\epsilon}$ and, using $\nabla_x f^\epsilon = 0$, we rewrite (3.2) as

$$\epsilon\partial_t(\rho^\epsilon g^\epsilon) = -(\rho^\epsilon)^2\nabla_\omega\cdot\big((\mathbb{I}-\omega\otimes\omega)J_{g^\epsilon}g^\epsilon\big) + \rho^\epsilon\Delta_\omega g^\epsilon = Q(\rho^\epsilon g^\epsilon).$$

[1]We will later suppose that in the space-inhomogeneous case analogous results hold.

Since $\int_{\mathbb{S}} Q(f)d\omega = 0$, by integrating the expression above with respect to ω, we find that

$$\partial_t \rho^\epsilon = 0.$$

We can then cancel out ρ^ϵ to get the following partial differential equation for g^ϵ:

$$\epsilon \partial_t (g^\epsilon) = -\rho^\epsilon \nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega) J_{g^\epsilon} g^\epsilon \big) + \Delta_\omega g^\epsilon. \tag{3.8}$$

We are interested in studying the convergence to equilibrium of the solutions to this equation. Before doing so, we recall some general notions on convergence in a Banach space.

Definition 3.6. Let X be a Banach space with norm $\|\cdot\|$ and let $f : \mathbb{R}_+ \to X$ be a given function. We say that f *converges exponentially fast* to a function f_∞ with *global rate* r if there exists a constant $C = C(\|f_0\|)$, such that $\|f(t) - f_\infty\| \leqslant Ce^{-rt}$ for all $t \geqslant 0$.
We say that the convergence is of *asymptotic rate* r if the above holds for a constant $C =$ depending on f_0 and not only on $\|f_0\|$.
We say that the convergence is of *asymptotic algebraic rate* α if there exists a constant $C = C(f_0)$ such that $\|f(t) - f_\infty\| \leqslant C/t^\alpha$.

We have the following Theorem, which we prove in Chapter 4:

Theorem 3.7. *First part.* Suppose g_0 is a probability measure, belonging to $H^s(\mathbb{S})$. Then there exists a unique weak solution g to (3.8), with initial condition $g(0) = g_0$. Furthermore, this solution is a classical one, is positive for all time $t > 0$, and belongs to $C^\infty\big((0, +\infty) \times \mathbb{S}\big)$.

Second part. The long time behaviour of the solution g depends on the value of J_{g_0}, in fact:

1. If $J_{g_0} = 0$ then (3.8) reduces to the heat equation on the sphere, and g converges exponentially fast to the uniform distribution, with global rate $r = \frac{2n}{\epsilon}$, in any H^p form.

2. If $J_{g_0} \neq 0$ then we have 3 possibilities:

   (a) $\rho^\epsilon < n$: g converges exponentially fast to the uniform distribution, with global rate
   $$r(\rho^\epsilon) = \tfrac{(n-1)(n-\rho^\epsilon)}{n\epsilon}, \tag{3.9}$$
   in any H^p norm.

   (b) $\rho^\epsilon > n$: there exists $\Omega \in \mathbb{S}$ such that g converges exponentially fast to $M_{\kappa(\rho^\epsilon)\Omega}$, with asymptotic rate
   $$r(\rho^\epsilon) = \frac{\rho^\epsilon c\big(\kappa(\rho^\epsilon)\big)^2 + n - \rho^\epsilon}{\epsilon} \Lambda_{\kappa(\rho^\epsilon)} > 0, \text{ [2]}$$
   in any H^p norm. Moreover, for $\rho^\epsilon \to n$,
   $$r(\rho^\epsilon) \sim \tfrac{2}{\epsilon}(n-1)(\rho^\epsilon/n - 1). \tag{3.10}$$

[2] Λ_κ is the best constant for the Poincaré Inequality $\langle |\nabla g|^2 \rangle_{M_{\kappa\Omega}} \geqslant \Lambda_\kappa \langle (g - \langle g \rangle_{M_{\kappa\Omega}})^2 \rangle_{M_{\kappa\Omega}}$.

(c) $\rho^\epsilon = n$: g converges to the uniform distribution in any H^p norm, with algebraic asymptotic rate 1/2.

Remark. We have found a phase transition bifurcation. In Chapter 4 we are going to see that the uniform distribution is stable if $\rho^\epsilon \leqslant n$, and unstable if $\rho^\epsilon > n$. In the latter case, the Von-Mises-Fisher distribution is also an equilibrium, and it is stable.

Remark. If $\lim_{\epsilon\to 0} r(\rho^\epsilon) = +\infty$, then f^ϵ converges rapidly to a given equilibrium. From the expressions of the convergence rates in (3.9) and (3.10) we can reasonably conjecture that, away from a region $|\rho^\epsilon - n| \in O(\epsilon)$, this convergence is exponentially fast.

3.4 Ordered and disordered regions

We now return to the space-inhomogeneous setting; in this main section we wish to study the macroscopic behaviour of the model. Again, note that the results of this section are not as rigorously proven as above: that would go beyond the scope of this work.

We begin by observing that the following mass conservation equation holds:

$$\partial_t \rho^\epsilon + \nabla_x \cdot J_{f^\epsilon} = 0. \tag{3.11}$$

We then define two different regions of the space, a *disordered* and an *ordered* one, which we will study separately.

$$\mathcal{R}_d = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}_+ : n - \rho^\epsilon(x,t) \gg \epsilon \text{ as } \epsilon \downarrow 0\};$$
$$\mathcal{R}_o = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}_+ : \rho^\epsilon(x,t) - n \gg \epsilon \text{ as } \epsilon \downarrow 0\}.$$

Inspired by the results for the space-homogeneous case, we perform the following ansatz: we assume that

$$\lim_{\epsilon\to 0} f^\epsilon(x,\omega,t) = \rho(x,t), \text{ for all } (x,t) \in \mathcal{R}_d;$$
$$\lim_{\epsilon\to 0} f^\epsilon(x,\omega,t) = \rho(x,t) M_{\kappa(\rho)\Omega(x,t)}, \text{ for all } (x,t) \in \mathcal{R}_o,$$

where the convergence is as smooth as needed. In particular, we have that

$$\rho(x,t) := \rho_{\lim_{\epsilon\to 0} f^\epsilon}(x,t) = \lim_{\epsilon\to 0} \rho^\epsilon(x,t).$$

3.4.1 The disordered region

In this region we have $\rho \leqslant n$ which, thanks to Proposition 3.4, means that $J_{f^\epsilon} \to J_h = 0$, so that the mass conservation equation (3.11) reduces to $\partial_t \rho = 0$.

Theorem 3.8. For $\epsilon \to 0$, the formal first order approximation to the solution of the rescaled mean-field system (3.2) in the disordered region $\mathcal{R}_d$ is given by

$$f^\epsilon(x,\omega,t) = \rho^\epsilon(x,t) - \epsilon \frac{n\omega \cdot \nabla_x \rho^\epsilon(x,t)}{(n-1)\big(n-\rho^\epsilon(x,t)\big)},$$

where the density ρ^ϵ satisfies the following diffusion equation:

$$\partial_t \rho^\epsilon = \frac{\epsilon}{n-1}\Big(\nabla_x \cdot \frac{\nabla_x \rho^\epsilon}{n-\rho^\epsilon}\Big).$$

Proof. We assume that f^ϵ is of the form $\rho^\epsilon(x,t) + \epsilon f_1^\epsilon(x,\omega,t)$, with $\int_{\mathbb{S}} f_1^\epsilon d\omega = 0$. We then have

$$J_{f^\epsilon} = \int_{\mathbb{S}} f^\epsilon(x,\omega,t)\omega d\omega = \int_{\mathbb{S}} \rho^\epsilon(x,\omega,t)\omega d\omega + \epsilon \int_{\mathbb{S}} f_1^\epsilon(x,\omega,t)\omega d\omega = \epsilon J_{f_1^\epsilon}.$$

We can then rewrite $\epsilon(\partial_t f^\epsilon + \omega \cdot \nabla_x f^\epsilon) = Q(f^\epsilon)$ as

$$\begin{aligned}&\partial_t \rho^\epsilon + \omega \cdot \nabla_x \rho^\epsilon + \epsilon \partial_t f_1^\epsilon + \epsilon\omega \cdot \nabla_x f_1^\epsilon = \\ &= -\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega) J_{f_1^\epsilon} \rho^\epsilon\big) + \Delta_\omega f_1^\epsilon - \epsilon \nabla_\omega \cdot \big((\mathbb{I} - \omega\otimes\omega) J_{f_1^\epsilon} f_1^\epsilon\big).\end{aligned}$$

Using the mass conservation equation 3.11, we find

$$\partial_t \rho^\epsilon + \epsilon \nabla_x \cdot J_{f_1^\epsilon} = 0, \tag{3.12}$$

which gives $\partial_t \rho^\epsilon \in O(\epsilon)$. We notice that, for a constant vector $A \in \mathbb{R}^n$,

$$\nabla_\omega \cdot \big((\mathbb{I} - \omega\otimes\omega)A\big) = -(n-1)A \cdot \omega.$$

We can rewrite (3.12) as $\Delta_\omega f_1^\epsilon = \big(\nabla_x \rho^\epsilon - (n-1)\rho^\epsilon J_{f_1^\epsilon}\big) \cdot \omega + O(\epsilon)$, which can be solved by noticing that its right hand size is of the form $A \cdot \omega$, of zero mean. In fact, by using the fact that $\nabla_\omega \cdot (A \cdot \omega) = -(n-1)A \cdot \omega$, we have

$$f_1^\epsilon = -\tfrac{1}{n-1}\big(\nabla_x \rho^\epsilon - (n-1)\rho^\epsilon J_{f_1^\epsilon}\big) \cdot \omega + O(\epsilon).$$

Now, since $\int_{\mathbb{S}} \omega \otimes \omega d\omega = \frac{1}{n}\mathbb{I}$, we get

$$J_{f_1^\epsilon} = -\tfrac{1}{n(n-1)}\big(\nabla_x \rho^\epsilon - (n-1)\rho^\epsilon J_{f_1^\epsilon}\big) \cdot \omega + O(\epsilon),$$

which implies that

$$J_{f_1^\epsilon} = -\tfrac{1}{(n-1)(n-\rho^\epsilon)}\big(\nabla_x \rho^\epsilon + O(\epsilon)\big).$$

Inserting this into (3.12), ends the proof, since $f_1^\epsilon = -\dfrac{n\omega \cdot \nabla_x \rho^\epsilon(x,t)}{(n-1)\big(n-\rho^\epsilon(x,t)\big)}$. □

3.4.2 *The ordered region*

Theorem 3.9. For $\epsilon \to 0$, the formal limit of the solution $f^\epsilon(x,\omega,t)$ of the rescaled mean-field system (3.2) in the ordered region $\mathcal{R}_o$ is given by

$$h(x,\omega,t) = \rho(x,t) M_{\kappa(\rho(x,t))\Omega(x,t)}(\omega),$$

where $\kappa = \kappa(\rho)$ is the unique positive solution to the compatibility condition (3.3). Moreover, the local density $\rho = \rho_h > n$ and the mean orientation $\Omega \in \mathbb{S}$ satisfy the following first order PDE system

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho c \Omega) = 0 \\ \rho\big(\partial_t \Omega + \tilde{c}(\Omega \cdot \nabla_x)\Omega\big) + \lambda(\mathbb{I} - \Omega \otimes \Omega)\nabla_x \rho = 0, \end{cases} \tag{3.13}$$

for an appropriate coefficient $\tilde{c}\big(\kappa(\rho)\big)$ and a parameter $\lambda(\rho)$.

Remark. The result on the formal limit for f^ϵ follows directly from our previous assumptions on the behaviour in the ordered region. The evolution equation for ρ is also easy to show, since it follows form the fact that $J_f = \rho c \Omega$, by taking the limit of Equation 3.11 as $\epsilon \to 0$.

We then only have to find the evolution equation for $\Omega(x,t)$. The key argument is the determination of the generalized collisional invariants.

Definition 3.10. A *collisional invariant* is a function ψ such that

$$\int_{\mathbb{S}} Q(f)\psi d\omega = 0 \quad \forall f = f(\omega).$$

A *generalized collisional invariant* (GCI) associated to $\kappa \in \mathbb{R}$ and $\Omega \in \mathbb{S}$ is a function ψ such that

$$\int_{\mathbb{S}} L_{\kappa\Omega}(f)\psi d\omega = 0, \quad \forall f \text{ s.t. } (\mathbb{I} - \Omega \otimes \Omega) J_f = 0,$$

where

$$L_{\kappa\Omega}(f) := \Delta_\omega f - \kappa \nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega)\Omega f\big) = \nabla_\omega \cdot \Big[M_{\kappa\Omega} \nabla_\omega \big(\frac{f}{M_{\kappa\Omega}}\big)\Big].$$

Let $C_{\kappa\Omega}$ be the set of all such GCIs.

Remark. If ψ is a GCI associated to κ and Ω, then $\int_{\mathbb{S}} Q(f)\psi d\omega = 0$, $\forall f : J_f = \kappa\Omega$.

Proposition 3.11. The set $C_{\kappa\Omega}$ of GCI's associated to $\kappa \in \mathbb{R}$ and $\Omega \in \mathbb{S}$ is a vector space of dimension n.

Proof. We define the space

$$V = \left\{ g \mid (n-2)\sin^{\frac{n}{2}-2}\theta\, g \ \in L^2(0,\pi),\ (\sin\theta)^{\frac{n}{2}-1} g \ \in H^1_0(0,\pi) \right\},$$

and denote by g_κ the unique solution in V of the elliptic problem $\tilde{L}^*_\kappa g(\theta) = \sin\theta$, where

$$\tilde{L}^*_\kappa g(\theta) := -\sin^{\frac{n}{2}-2}\theta\, e^{-\kappa\cos\theta} \tfrac{d}{d\theta}\left(\sin^{n-2}\theta\, e^{\kappa\cos\theta}\, \tfrac{dg}{d\theta}(\theta) \right) + \tfrac{n-2}{\sin^2\theta}\, g(\theta).$$

Defining ℓ_κ by setting $g_\kappa(\theta) = \ell_\kappa(\cos\theta)\sin\theta$, we get

$$\mathcal{C}_{\kappa\Omega} = \{ C + \ell_\kappa(\omega\cdot\Omega) A\cdot\omega \mid C \in \mathbb{R},\ A \in \mathbb{R}^n,\ A\cdot\Omega = 0 \}.$$

Since the vector A has $n-1$ independent components, $\mathcal{C}_{\kappa\Omega}$ is a vector space of dimension n. ☐

With this result, along with the definition of ℓ_κ, we can provide the proof of the final part of Theorem 3.9.

Proof of Theorem 3.9. Let $J_{f^\epsilon} = \kappa^\epsilon \Omega^\epsilon$. For any vector $A \in \mathbb{R}^n$, with $A\cdot\Omega^\epsilon = 0$, we then have

$$\int_{\mathbb{S}} Q(f^\epsilon)\ell_{\kappa^\epsilon}(\omega\cdot\Omega^\epsilon) A\cdot\omega\, d\omega = 0.$$

This means that the vector $X^\epsilon := \frac{1}{\epsilon}\int_{\mathbb{S}}(Q(f^\epsilon)\ell_{\kappa^\epsilon}(\omega\cdot\Omega^\epsilon)\omega\, d\omega$ is parallel to Ω^ϵ, which is to say that $(\mathbb{I} - \Omega^\epsilon\otimes\Omega^\epsilon)X^\epsilon = 0$. From this we find

$$X^\epsilon = \int_{\mathbb{S}} (\partial_t f^\epsilon + \omega\cdot\nabla_x f^\epsilon)\ell_{\kappa^\epsilon}(\omega\cdot\Omega^\epsilon)\omega\, d\omega.$$

By taking the limit for $\epsilon \to 0$ we get $(\mathbb{I} - \Omega\otimes\Omega)X = 0$, where

$$X = \int_{\mathbb{S}} \big(\partial_t(\rho M_{\kappa\Omega}) + \omega\cdot\nabla_x(\rho M_{\kappa\Omega}) \big) \ell_\kappa(\omega\cdot\Omega)\omega\, d\omega.$$

Thanks to Lemma 3.12, we have that the equation $(\mathbb{I} - \Omega\otimes\Omega)X = 0$ is equivalent to $\rho\big(\partial_t\Omega + \tilde{c}(\Omega\cdot\nabla_x)\Omega\big) + \lambda(\mathbb{I} - \Omega\otimes\Omega)\nabla_x\rho = 0$, where the coefficients $\tilde{c}$ and λ are given by

$$\tilde{c} = \langle\cos\theta\rangle_{M\kappa} := \frac{\int_0^\pi \cos\theta\,\ell_\kappa(\cos\theta)e^{\kappa\cos\theta}\sin^n\theta\, d\theta}{\int_0^\pi \ell_\kappa(\cos\theta)e^{\kappa\cos\theta}\sin^n\theta\, d\theta},$$
$$\lambda = \tfrac{1}{\kappa} + \tfrac{\rho}{\kappa}\tfrac{d\kappa}{d\rho}(\tilde{c} - c).$$

Differentiating the compatibility condition $\rho c(\kappa) = \kappa$ with respect to κ, we get

$$c\tfrac{d\rho}{d\kappa} + \rho\tfrac{dc}{d\kappa} = 1.$$

Recalling (Equation 3.4) that $\frac{dc}{d\kappa} = 1 - (n-1)\frac{c}{\kappa} - c^2$, we have

$$c\frac{d\rho}{d\kappa} = \frac{\kappa}{\rho}\frac{d\rho}{d\kappa} = 1 - \rho\frac{dc}{d\kappa} = 1 - \rho(1 - (n-1)\frac{c}{\kappa} - c^2) = n - \rho + \kappa c,$$

which we can use to rewrite λ as

$$\lambda = \frac{1}{\kappa} + \frac{\rho}{\kappa}\frac{d\kappa}{d\rho}(\tilde{c} - c) = \frac{n - \rho + \kappa\tilde{c}}{\kappa(n - \rho + \kappa c)}.$$

This concludes the proof. □

Lemma 3.12. For $X = \int_{\mathbb{S}} \big(\partial_t(\rho M_{\kappa\Omega}) + \omega \cdot \nabla_x(\rho M_{\kappa\Omega})\big)\ell_\kappa(\omega \cdot \Omega)\omega \, d\omega$, the expression

$$(\mathbb{I} - \Omega \otimes \Omega)X = 0$$

is equivalent to

$$\rho\big(\partial_t\Omega + \tilde{c}(\Omega \cdot \nabla_x)\Omega\big) + \lambda(\mathbb{I} - \Omega \otimes \Omega)\nabla_x\rho = 0,$$

where the coefficients $\tilde{c}$ and λ are as above.

Proof. See Appendix A. □

Hyperbolicity of the hydrodynamic model in the ordered region. We now wish to show that the hydrodynamic model

$$\begin{cases} \partial_t\rho + \nabla_x \cdot (\rho c\Omega) = 0 \\ \rho\big(\partial_t\Omega + \tilde{c}(\Omega \cdot \nabla_x)\Omega\big) + \lambda(\mathbb{I} - \Omega \otimes \Omega)\nabla_x\rho = 0, \end{cases} \tag{3.13}$$

is not hyperbolic in the ordered region. Before doing so, we briefly recall the definition of hyperbolicity for a first order system.

Definition 3.13. Consider the first order system

$$\partial_t U + \sum_{i=1}^{n} A_i(U)\partial_{x_i} U = 0,$$

where $x \in \mathbb{R}^n$, $t \geqslant 0$, $U = (U_1, \dots, U_m)$ and $\big(A_i(U)\big)_{i=1,\dots,n}$ are $m \times m$-dimensional matrices.

1. The system is *hyperbolic in a neighbourhood of* $U_0 \in \mathbb{R}^m$ if and only if the matrix

$$A(\xi) := \sum_{i=1}^{n} A_i(U_0)\xi_i$$

is diagonalizable with real eigenvalues for all $\xi \in \mathbb{S}$.

2. The system is *hyperbolic* if and only if it is hyperbolic for any state U_0 in the domain of definition of the matrices $A_i(U)$.

Theorem 3.14. Under the previous assumptions we have that System (3.13) is hyperbolic if and only if $\lambda > 0$.

Proof. We consider a system satisfying (3.13), but that evolves only along one space direction, say $e_z \in \mathbb{S}$. We write $\Omega = \cos\theta e_z + \sin\theta w$, where $w \in \mathbb{S}^{n-2}$. Under these assumptions, the hydrodynamic model is equivalent to the following system

$$\begin{cases} \partial_t \rho + \partial_z\big(\rho c(\rho)\cos\theta\big) = 0 \\ \rho\big(\partial_t(\cos\theta) + \tilde{c}(\rho)\cos\theta\partial_z(\cos\theta)\big) + \lambda\sin^2\theta\partial_z\rho = 0 \\ \partial_t w + \tilde{c}(\rho)\cos\theta\partial_z w = 0, \text{ with } |w| = 1 \text{ and } e_z \cdot w = 0, \end{cases} \tag{3.14}$$

which we can also write as the following first order quasilinear system of PDEs

$$\begin{pmatrix} \partial_t \rho \\ \partial_t \cos\theta \\ \partial_t w \end{pmatrix} + A(\rho, \cos\theta, w) \begin{pmatrix} \partial_z \rho \\ \partial_z \cos\theta \\ \partial_z w \end{pmatrix} = 0.$$

By definition, we have that (3.13) is hyperbolic if and only if (3.14) is hyperbolic for all $e_z \in \mathbb{S}$. We show that (3.14) is hyperbolic if and only if $\lambda > 0$ or

$$\begin{cases} |\tan\theta| < \tan\theta_c \dfrac{|\tilde{c} - \frac{c}{n-\rho+\kappa c}|}{2\sqrt{-\lambda c}}, \text{ if } \lambda < 0, \\ \theta \neq 0 \text{ and } \tilde{c} \neq \frac{c}{n-\rho+\kappa c}, \text{ if } \lambda = 0. \end{cases} \tag{3.15}$$

* If $\lambda < 0$, asking the eigenvalues of A to be real and distinct is equivalent to (3.15). In this case A is diagonalizable.
* If $\lambda = 0$, we have that A is diagonalizable if and only if $\tilde{c}(\rho)$ and $\frac{c}{n-\rho+\kappa c}$ are different.
* If $\lambda > 0$, then (3.14) is hyperbolic for all $e_z \in \mathbb{S}$, which means that (3.13) is as well.

We now assume that (3.13) is hyperbolic at some point (ρ, Ω) for $\lambda \leqslant 0$. Since $n \geqslant 2$, we can find $e_z \in \mathbb{S}$ such that $e_z \cdot \Omega = 0$. But then we would have $\cos\theta = 0$, a contradiction with the hyperbolicity of (3.14). □

Proposition 3.15. We have the following expansions:

1. For $\rho \to n$,
$$\lambda = \frac{-1}{4\sqrt{n+2}} \frac{1}{\sqrt{\rho - n}} + O(1).$$

2. For $\rho \to \infty$,
$$\lambda = -\frac{n+1}{6} + O(\rho^{-3}).$$

Proof. Follows from the results in Appendix B. □

Thanks to this result, we can see that, at least for $\rho \to n$ and $\rho \to \infty$, we have $\lambda < 0$. Moreover, through numerical computations (see figure below and Appendix C), we know that $\lambda < 0$ also for $n = 2, 3, 4$.
It is therefore reasonable to assume that $\lambda < 0$ in our model, which implies that system (3.13) is *not* hyperbolic in the ordered region.

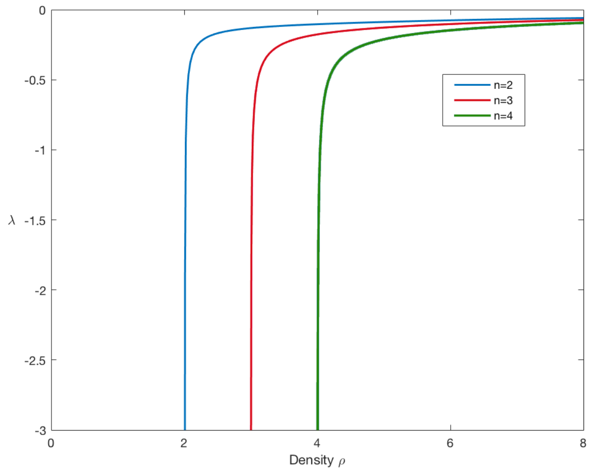

Figure 3.1: Coefficient λ for dimensions $n = 2, 3, 4$, as a function of the density ρ.

4

The convergence theorem

The aim of this chapter is to prove the main result on the rate of convergence of Chapter 3:[1]

Theorem 4.1. *First part.* Suppose f_0 is a probability measure, belonging to $H^s(\mathbb{S})$. Then there exists a unique weak solution f to

$$\epsilon\partial_t(f) = -\rho\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega)J_f f\big) + \Delta_\omega f, \tag{4.1}$$

with initial condition $f(0) = f_0$. Furthermore, this solution is a classical one, is positive for all time $t > 0$, and belongs to $C^\infty\big((0, +\infty) \times \mathbb{S}\big)$.

Second part. The long time behaviour of the solution f depends on the value of J_{f_0}, in fact:

1. If $J_{f_0} = 0$ then (4.1) reduces to the heat equation on the sphere, and f converges exponentially fast to the uniform distribution, with global rate $r = \frac{2n}{\epsilon}$, in any H^s norm.
2. If $J_{f_0} \neq 0$ then we have 3 possibilities:
   (a) *Subcritical case* ($\rho < n$): f converges exponentially fast to the uniform distribution, with global rate
   $$r(\rho) = \tfrac{(n-1)(n-\rho)}{n\epsilon}, \tag{4.2}$$
   in any H^p norm.
   (b) *Supercritical case* ($\rho > n$): there exists $\Omega \in \mathbb{S}$ such that f converges exponentially fast to $M_{\kappa(\rho)\Omega}$, with asymptotic rate
   $$r(\rho) = \frac{\rho c(\kappa(\rho))^2 + n - \rho}{\epsilon}\Lambda_{\kappa(\rho)} > 0, \text{ [2]}$$

[1] In an effort to simplify notations, in this chapter we will stop writing the superscript ϵ; we will denote by f the *velocity* probability distribution, i.e. what was denoted in the previous chapters by f/ρ.

[2] Λ_κ is the best constant for the Poincaré Inequality $\langle|\nabla f|^2\rangle_{M_{\kappa\Omega}} \geqslant \Lambda_\kappa\langle(f - \langle f\rangle_{M_{\kappa\Omega}})^2\rangle_{M_{\kappa\Omega}}$.

In any H^p norm. More precisely, for all $r < r_\infty$, there exists $t_0 > 0$ such that, for all $t \geqslant 0$,

$$\|f(t) - M_{\kappa\Omega}\| \leqslant Ce^{-rt}.$$

Moreover, when ρ is close to n,

$$r(\rho) \sim \tfrac{2}{\epsilon}(n-1)(\rho/n-1). \tag{4.3}$$

(c) *Critical case* ($\rho = n$): f converges to the uniform distribution in any H^p norm, with algebraic asymptotic rate 1/2.

Work plan: We will first provide the notions that are needed in order to allow for an easier study of (4.1). In particular, we are going to see that equilibria can be characterized as the minimizers for a function $\mathcal{F}$ which we call *free energy*, as well as those solutions for which there is no *dissipation* $\mathcal{D}$.
One of the main results is the LaSalle invariance principle, which defines the set of equilibria of (4.1) as $\mathcal{E}_\infty := \{f \in C^\infty(\mathbb{S}) : \mathcal{D}(f) = 0 \text{ and } \mathcal{F}(f) = \mathcal{F}_\infty\}$.

After this, we are going to study separately the three cases (subcritical, supercritical and critical) which we have previously defined.
We will see that in the first case, that is for $\rho < n$, there is global decay to the uniform distribution.
In the second case, we will use the LaSalle invariance principle – as well as Sobolev embeddings and the Poincaré inequality – to define the set of equilibria and determine the asymptotic rate of convergence.
Finally, for the critical case, we will use similar arguments to estimate the algebraic asymptotic rate of convergence to the uniform distribution.

4.1 Preliminary notions

Let $\dot{H}^s(\mathbb{S})$ be the subspace of mean zero functions of the Sobolev space $H^s(\mathbb{S})$. This is a Hilbert space, with inner product given by $\langle \ell, k\rangle_{\dot{H}^s} = \langle (-\Delta)^s \ell, k\rangle$, where Δ is the Laplace-Beltrami operator on the sphere.
We also define the *conformal Laplacian* $\tilde{\Delta}_{n-1}$ on the sphere, as

$$\tilde{\Delta}_{n-1} = \begin{cases} \displaystyle\prod_{0 \leqslant j \leqslant \frac{n-3}{2}} \big(-\Delta + j(n-j-2)\big) & \text{if } n \text{ is odd,} \\ \big(-\Delta + (\tfrac{n}{2}-1)^2\big)^{\frac{1}{2}} \displaystyle\prod_{0 \leqslant j \leqslant \frac{n}{2}-2} \big(-\Delta + j(n-j+2)\big) & \text{if } n \text{ is even.} \end{cases}$$

We recall the following

Theorem 4.2 (Poincaré inequality)**.** Let $1 \leqslant p < +\infty$ and $\mathcal{U}$ a subset, bounded in at least one direction, of a Banach space $\mathcal{B}$. Then there exists a constant $C > 0$, depending only on $\mathcal{U}$ and p, such that

$$\|u\|_{L^q(\mathcal{U})} \leqslant C\|\nabla u\|_{L^p(\mathcal{U})}, \quad \forall q \in [1, \tfrac{pn}{n-p}],$$

for all $u \in W_0^{1,p}(\mathcal{U}) = \overline{C_c^\infty(\mathcal{U})}^{W^{1,p}(\mathcal{U})}$.

On the $(n-1)$-dimensional sphere $\mathbb{S}$, on which we consider the Laplace-Beltrami operator Δ, the Poincaré (Wirtinger) inequality for $p = 2$,

$$\|h\|_2^2 \leqslant C\|\nabla h\|_2^2,$$

holds with optimal constant $C = \frac{1}{n-1}$.

Theorem 4.3 (General Sobolev inequalities)**.** Let U be a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary. Assume $u \in W^{k,p}(U)$.

1. If $k < \frac{n}{p}$, then $u \in L^q(U)$, where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. Moreover, there exists a constant $C > 0$, depending only on k, p, n and U, such that

$$\|u\|_{L^q(U)} \leqslant C\|u\|_{W^{k,p}(U)}.$$

2. If $k > \frac{n}{p}$, then $u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}$, where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, \text{ if } \frac{n}{p} \text{ is not an integer,} \\ \text{any positive number} < 1, \text{ if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

Moreover, there exists a constant $C > 0$, depending only on k, p, n, γ and U, such that

$$\|u\|_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\bar{U})} \leqslant C\|u\|_{W^{k,p}(U)}.$$

4.1.1 Weak and classical solutions

First of all, we rescale time by setting $\tau = \frac{\rho}{\epsilon}t$, so that the following equation

$$\epsilon\partial_t f = -\rho\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega)J_f f\big) + \Delta_\omega f, \tag{4.4}$$

can be rewritten, denoting $\sigma = \rho^{-1}$, as

$$\partial_\tau f = -\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega)J_f f\big) + \sigma\Delta_\omega f =: Q(f). \tag{4.5}$$

This equation is known as the Doi (or Doi-Onsager) equation, and was first introduced by Doi in [6].

We define g by subtracting 1 to the (velocity) probability density function f, thus obtaining a zero-mean function. With this new definition, if f is a weak solution of (4.5), i.e. for all $\phi \in H^{-s+1}(\mathbb{S})$, the following holds

$$\langle \partial_\tau f, \phi \rangle = -\sigma \langle \nabla_\omega f, \nabla_\omega \phi \rangle + \langle f, J_f \cdot \nabla_\omega \phi \rangle, \tag{4.6}$$

then $g \in \dot{H}^{s-1}(\mathbb{S})$ solves the following equation, for all $\phi \in \dot{H}^{-s+1}(\mathbb{S})$:

$$\langle \partial_\tau g, \phi \rangle = -\sigma \langle \nabla_\omega g, \nabla_\omega \phi \rangle + (n-1) J_g \cdot J_\phi + \langle g, J_g \cdot \nabla_\omega \phi \rangle. \tag{4.7}$$

The existence of a solution to this problem is given by the following

Theorem 4.4. Given an initial probability measure $f_0 \in H^s(\mathbb{S})$, there exists a unique weak solution f of (4.5) such that $f(0) = f_0$. This solution is global in time, is in $C^\infty\big((0,+\infty) \times \mathbb{S}\big)$, with $f(t,\omega) > 0$ for all $t > 0$.
Moreover, for all $m \in \mathbb{N}$, we have the following estimate, for all $t > 0$,

$$\|f(t)\|^2_{H^{s+m}} \leqslant C\Big(1 + \tfrac{1}{t^m}\Big) \|f_0\|^2_{H^s}, \tag{4.8}$$

for a constant C depending only on σ, m and s.

Proof. ***Step 1: existence.*** *We want to show that, if $\|g_0\|_{\dot{H}^s} \leqslant K$, then there exists a weak solution on $[0,T]$, for some $T > 0$. Moreover, this solution is uniformly bounded in $L^2\big((0,T), \dot{H}^{s+1}(\mathbb{S})\big) \cap H^1\big((0,T), \dot{H}^{s-1}(\mathbb{S})\big)$.*

Let P_N be the finite dimensional vector space spanned by the first N non-constant eigenvectors of the Laplace-Beltrami operator. We have that $P_N \subseteq \dot{H}^p(\mathbb{S})$ for all p, and it contains all functions of the form $\omega \mapsto V \cdot \omega$.
Let $g^N \in C^1(I, P_N)$ be the unique solution to the following Cauchy problem

$$\begin{cases} \frac{d}{dt} g^N = \Pi_N\big(\sigma \Delta_\omega g^N + (n-1)(1+g^N)\omega \cdot J_{g^N} - J_{g^N} \cdot \nabla_\omega g^N\big) \\ g^N(0) = \Pi_N(g_0), \end{cases}$$

where Π_N is the orthogonal projection on P_N, and $I \subseteq \mathbb{R}^+$ is the maximal interval of existence. Equivalently, g^N solves

$$\frac{d}{dt}\langle g^N, \phi \rangle = -\sigma \langle \nabla_\omega g^N, \nabla_\omega \phi \rangle + (n-1) J_{g^N} \cdot J_\phi + \langle g^N, J_{g^N} \cdot \nabla_\omega \phi \rangle. \tag{4.9}$$

We wish to show that the limit of g^N as $N \to \infty$ is the solution we are looking for.
Taking $\phi = (-\Delta)^s g^N \in P_N$ in the above equation, we get[3]

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|g^N\|^2_{\dot{H}^s} + \sigma \|g^N\|^2_{\dot{H}^{s+1}} &\leqslant C_0 |J_{g^N}| \|g^N\|^2_{\dot{H}^s} + (n-1)^s |J_{g^N}|^2 \leqslant \\ &\leqslant C_1 \|g^N\|^2_{\dot{H}^s}\Big(1 + C_2 \|g^N\|_{\dot{H}^s}\Big). \end{aligned} \tag{4.10}$$

[3]See [9]: if $\phi \in \dot{H}^{-s+1}$ and $g \in \dot{H}^s$, then $\left|\int g \nabla \phi\right| \leqslant C(s,n) \|g\|_{\dot{H}^s} \|\phi\|_{\dot{H}^{-s+1}}$.

Set $T := C_1^{-1} \log\big(1 + (1 + 2C_2K)\big)$. Solving the above inequality, we find that

$$\|g^N\|_{\dot{H}^s} \leqslant \frac{\|\Pi_N(g_0)\|_{\dot{H}^s}}{e^{-C_1 t} - C_2\|\Pi_N(g_0)\|_{\dot{H}^s}(1 - e^{-C_1 t})},$$

for $0 \leqslant t < C_1^{-1} \log\big(1 + (C_2\|\Pi_N(g_0)\|_{\dot{H}^s})^{-1}\big)$. From this we get $\|g^N(t)\|_{\dot{H}^s} \leqslant 2\|g_0\|_{\dot{H}^s}$ for all $t \in [0,T]$, so that (4.9) has a solution on $[0,T]$ for any $N \in \mathbb{N}$.

Suppose $|J_{g^N}| \leqslant M_0$ on $[0,T]$. From (4.10) we get

$$\frac{1}{2}\frac{d}{dt}\|g^N\|^2_{\dot{H}^s} + \sigma\|g^N\|^2_{\dot{H}^{s+1}} \leqslant (1 + M_0)C_3\|g\|^2_{\dot{H}^s},$$

which leads to

$$\|g^N\|_{\dot{H}^s} + \sigma\int_0^T \|g^N\|^2_{\dot{H}^{s+1}} \leqslant \|g_0\|^2_{\dot{H}^s} e^{(1+M-0)C_3 T}.$$

We can then control the derivative of g, since

$$\int_0^T \|\partial_t g^N\|^2_{\dot{H}^{s-1}} \leqslant (C_4 + M_0)\|g_0\|_{\dot{H}^s} e^{(1+M_0)C_3 T}.$$

Taking $M_1^2 = K^2 e^{(1+M_0)C_3 T} \times \max\{\sigma^{-1}; C_4 + M_0\}$, we get that g^N is bounded by M_1 in $L^2\big((0,T), \dot{H}^{s+1}(\mathbb{S})\big) \cap H^1\big((0,T), \dot{H}^{s-1}(\mathbb{S})\big)$.
We now wish to provide estimates on the derivative of J_{g^N}. Taking $\phi = \omega \cdot V$ in (4.9), we find

$$\left|\frac{d}{dt}J_{g^N}\right| = \left|\frac{n-1}{n}(1 - \sigma n)J_{g_N} - \int_{\mathbb{S}}(\mathbb{I} - \omega \otimes \omega)J_{g^N} g^N \, d\omega\right| \leqslant$$
$$\leqslant (C_5 + M_0 C_6)\|g_0\|_{\dot{H}^s} e^{\frac{1}{2}(1+M_0)C_3 T}.$$

Since any component of $(\mathbb{I} - \omega \otimes \omega)$ is in $\dot{H}^{-s}$, we can control the term $\int_{\mathbb{S}}(\mathbb{I} - \omega \otimes \omega)g^N \, d\omega$ by any $\dot{H}^s$ norm of g^N, uniformly in N and in t.

We can now use weak compactness and the Ascoli-Arzela Theorem, to find an increasing sequence N_k, a function $g \in L^2\big((0,T), \dot{H}^{s+1}(\mathbb{S})\big) \cap H^1\big((0,T), \dot{H}^{s-1}(\mathbb{S})\big)$, and a continuous function $J : [0,T] \to \mathbb{R}^n$ such that

* $J_{g^{N_k}} \xrightarrow{k\to\infty} J$ uniformly on $[0,T]$;
* $g^{N_k} \xrightarrow{k\to\infty} g$ weakly in $L^2\big((0,T), \dot{H}^{s+1}(\mathbb{S})\big)$ and in $H^1\big((0,T), \dot{H}^{s-1}(\mathbb{S})\big)$.

Moreover, g is bounded by M_1 in $L^2\big((0,T), \dot{H}^{s+1}(\mathbb{S})\big) \cap H^1\big((0,T), \dot{H}^{s-1}(\mathbb{S})\big)$. It is easy to show that $J = J_g$.
For any M we have that

$$\langle \partial_t g, \phi \rangle = -\sigma\langle \nabla_\omega g, \nabla_\omega \phi \rangle + (n-1)J_g \cdot J_\phi + \langle g, J_g \cdot \nabla_\omega \phi \rangle,$$

for all $\phi \in P_M$, for a.e. $t \in [0, T]$. By density, we conclude that g is a weak solution to our problem.
Let $\phi \in \dot{H}^{-s+1}(\mathbb{S})$. We have that

$$\langle g^N(t) - \Pi_N(g_0), \phi \rangle = \int_0^t \langle \partial_t g^N, \phi \rangle$$

is controlled by $M_1\sqrt{t}\|\phi\|_{\dot{H}^{-s+1}}$, uniformly in N. Passing to the limit we find that $g(t) \xrightarrow{t\to 0} g_0$ in $\dot{H}^{-s+1}(\mathbb{S})$. But $g \in C\big([0,T], H^s(\mathbb{S})\big)$, so we conclude that $g(0) = g_0$.

Step 2: continuity with respect to initial conditions. *Suppose g and $\tilde{g}$ are two solutions, with $\|g(0)\|_{\dot{H}^s} \leqslant K$ and $\|\tilde{g}(0)\|_{\dot{H}^s} \leqslant K$. We show that there exists a constant M_3 such that $g - \tilde{g}$ is bounded in $L^2\big((0,T), \dot{H}^{s+1}(\mathbb{S})\big)$ and $H^1\big((0,T), \dot{H}^{s-1}(\mathbb{S})\big)$ by $M_3\|g(0) - \tilde{g}(0)\|_{\dot{H}^s}$. From this uniqueness follows.*

As before, we have that

$$\|g^N\|_{\dot{H}^s} \leqslant \frac{\|\Pi_N(g_0)\|_{\dot{H}^s}}{e^{-C_1 t} - C_2\|\Pi_N(g_0)\|_{\dot{H}^s}(1 - e^{-C_1 t})},$$

for $0 \leqslant t \leqslant T := C_1^{-1}\log\big(1 + (1 + 2C_2K)\big)$, and these solutions are uniformly bounded by M_1 in $L^2\big((0,T), \dot{H}^{s+1}(\mathbb{S})\big)$ and $H^1\big((0,T), \dot{H}^{s-1}(\mathbb{S})\big)$.
Taking $u := g - \tilde{g}$ and using (4.7), we get

$$\langle \partial_t u, \phi \rangle = -\sigma\langle \nabla_\omega u, \nabla_\omega \phi \rangle + (n-1)J_u \cdot J_\phi + \langle u, J_g \cdot \nabla_\omega \phi \rangle + \langle \tilde{g}, J_u \cdot \nabla_\omega \phi \rangle.$$

Taking $\phi = (-\Delta)^s u$, we find that[4]

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{\dot{H}^s}^2 + \sigma\|u\|_{\dot{H}^{s+1}}^2 &\leqslant (1 + M_1)C_3\|u\|_{\dot{H}^s}^2 + C_7\|u\|_{\dot{H}^s}\|\tilde{g}\|_{\dot{H}^{s+1}}\|(-\Delta)^s u\|_{\dot{H}^{-s}} \leqslant \\ &\leqslant M_2(1 + \|\tilde{g}\|_{\dot{H}^{s+1}})\|u\|_{\dot{H}^s}^2.\end{aligned}$$

Using Gronwall's Lemma, we get

$$\|u\|_{\dot{H}^s}^2 + \sigma\int_0^T \|u\|_{\dot{H}^{s+1}}^2 \leqslant \|u_0\|_{\dot{H}^s}^2 e^{M_2\int_0^T(1+\|\tilde{g}\|_{\dot{H}^{s+1}})} \leqslant \|u_o\|_{\dot{H}^s}^2 e^{M_2(T+M_1^2)},$$

so u is bounded in $L^2\big((0,T), \dot{H}^{s+1}(\mathbb{S})\big) \cap H^1\big((0,T), \dot{H}^{s-1}(\mathbb{S})\big)$ by $M_3\|u(0)\|_{\dot{H}^s}$.

Step 3: positivity. *Let $g_0 \in \dot{H}^s(\mathbb{S})$, for an s sufficiently large so that the solution g is in C^0. Recalling that $f = 1 + g$, we have that, if f_0 is non-negative, then f is positive for any positive time.*

[4] For $g \in \dot{H}^{s+1}$, $\left|\int_{\mathbb{S}} g\nabla(-\Delta)^s g\right| \leqslant C(s,n)\|g\|_{\dot{H}^s}^2$

From (4.7), we find that, as an element of $L^2\big((0,T),\dot{H}^{s-1}(\mathbb{S})\big)$, the function $\partial_t f$ is equal almost everywhere to

$$\partial_t f = \sigma\Delta_\omega f - \nabla_\omega \cdot \big((\mathbb{I} - \omega\otimes\omega)J_f f\big),$$

which is an element of $C^0\big([0,T]\times\mathbb{S}\big)$. So, up to redefining f on a set of measure zero, we have that f satisfies the PDE, and it is in $C^1\big([0,T],C(\mathbb{S})\big)\cap C^0\big([0,T],C^2(\mathbb{S})\big)$.
By maximum principle arguments, we have that $f>0$ on $(0,T]$.

Step 4: global existence and uniqueness *These are easily shown by constructing a solution on a sequence of intervals.*

Step 5: regularity and boundedness estimates. *By reasoning over intervals, it is easy to show that the solution f is positive for all $t>0$, and belongs to $C\big([0,+\infty),H^s(\mathbb{S})\big)$. Moreover, it is also in C^k and, through Sobolev embeddings, it is in $C^\infty\big((0,+\infty)\times\mathbb{S}\big)$. We want to show that*

$$\|f(t)\|^2_{H^{s+m}} \leqslant C(\sigma,m,s)\Big(1+\tfrac{1}{t^m}\|f_0\|^2_{H^s}\Big), \tag{4.11}$$

for all $s\in\mathbb{R}$ and $m\geqslant 0$.

Let f^N be the orthogonal projection of f on P_N, and define $g^N = f - f^N$. Noting that the eigenvalues of $-\Delta$ are given by $\ell(\ell+n-2)$ for $\ell\in\mathbb{N}$, we can use

$$\frac{1}{2}\frac{d}{dt}\|g^N\|^2_{\dot{H}^s} + \sigma\|g^N\|^2_{\dot{H}^{s+1}} \leqslant C_0|J_{g^N}|\|g^N\|^2_{\dot{H}^s} + (n-1)^s|J_{g^N}|^2, \tag{4.10}$$

together with the Poincaré inequality $\|g^N\|^2_{\dot{H}^s} \leqslant \frac{1}{(N+1)(N+n-1)}\|g^N\|^2_{\dot{H}^{s+1}}$, to find that

$$\frac{1}{2}\frac{d}{dt}\|g\|^2_{\dot{H}^s} + \sigma\|g\|^2_{\dot{H}^{s+1}} \leqslant \frac{C_0}{(N+1)(N+n-1)}|J_g|\|g\|^2_{\dot{H}^s} + (n-1)^s|J_g|^2 + C_0\|f^N-1\|^2_{\dot{H}^s}. \tag{4.12}$$

Moreover, since f is a probability measure,

$$\|f^N-1\|^2_{\dot{H}^s} = \int_{\mathbb{S}}(-\Delta)^s f^N f\,d\omega \leqslant \|(-\Delta)^s f^N\|_{L^\infty} \leqslant K_N\|f^N-1\|_{\dot{H}^s},$$

which gives a uniform bound on f^N. By taking N large enough, we get

$$\frac{1}{2}\frac{d}{dt}\|g\|^2_{\dot{H}^s} + \frac{\sigma}{2}\|g\|_{\dot{H}^{s+1}} \leqslant C_8.$$

Multiplying by t the above expression at order $s+1$ reads

$$\frac{1}{2}\frac{d}{dt}t\|g\|^2_{\dot{H}^{s+1}} + \frac{\sigma}{2}t\|g\|_{\dot{H}^{s+2}} \leqslant C_9 t + \frac{1}{2}\|g\|^2_{\dot{H}^{s+1}}.$$

Summing the two expressions, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|g\|_{\dot{H}^s}^2+\frac{\sigma}{2}t\|g\|_{\dot{H}^{s+1}}\right)+\frac{\sigma}{4}\left(\|g\|_{\dot{H}^{s+1}}^2+\frac{\sigma}{2}t\|g\|_{\dot{H}^{s+2}}\right)\leqslant C_8+C_9\frac{\sigma}{2}t.$$

Solving this inequality, we find

$$\|g\|_{\dot{H}^s}^2+\frac{\sigma}{2}t\|g\|_{\dot{H}^{s+1}}^2\leqslant\|g_0\|_{\dot{H}^s}^2e^{-(n-1)\frac{\sigma}{4}t}+C_{10}(1+t).$$

So, for $\|f\|_{\dot{H}^s}^2=1+\|g\|_{\dot{H}^s}^2$ and $m=1$, we get

$$\|f\|_{\dot{H}^{s+1}}^2\leqslant C\left(1+\frac{1}{t}\right)\|f_0\|_{\dot{H}^s}^2.$$

It is then easy to show that the claim follows.

□

Stability of the constant state. As anticipated in Chapter 3, we now study the stability of the uniform distribution. As before, $g=f-1$ is a zero-mean function, and we consider the linearized equations for g and J_g:

$$\begin{cases}\partial_t g=\sigma\nabla_\omega g+(n-1)\omega\cdot J_g+O(g^2),\\ \frac{d}{dt}J_g=(n-1)\left(\frac{1}{n}-\sigma\right)J_g+O(g^2).\end{cases}$$

Considering only the linear part of the system, we find that the equation for g reduces to the heat equation with known source term of the form $\exp\left(t(n-1)\left(\frac{1}{n}-\sigma\right)\right)$, so that, around the uniform distribution, the linearized equation is stable if $\sigma\geqslant\frac{1}{n}$, and unstable if $\sigma<\frac{1}{n}$.

4.1.2 Free energy and steady states

Thanks to Theorem 4.4, we know that any solution f of

$$\partial_\tau f=-\nabla_\omega\cdot\left((\mathbb{I}-\omega\otimes\omega)J_f f\right)+\sigma\Delta_\omega f=:Q(f)\tag{4.5}$$

is in $C^\infty\left((0,\infty)\times\mathbb{S}\right)$ and is positive for any $t>0$. This means that we can take the logarithm of such a function, and rewrite (4.5) as

$$\partial_\tau f=-\nabla_\omega\cdot\left(\sigma\nabla_\omega f-\nabla_\omega(\omega\cdot J_f)f\right)=\nabla_\omega\cdot\left(f\nabla_\omega(\sigma\log f-\omega\cdot J_f)\right).$$

For the same reasons, we can apply the integration by parts formula:

$$\int_{\mathbb{S}}\partial_\tau f\left(\sigma\log f-\omega\cdot J_f\right)d\omega=-\int_{\mathbb{S}}f|\nabla_\omega(\sigma\log f-\omega\cdot J_f)|^2d\omega.$$

Defining the *free energy* $\mathcal{F}(f)$ and the *dissipation term* $\mathcal{D}(f)$ as

$$\mathcal{F}(f) := \sigma \int_{\mathbb{S}} f \log f - \tfrac{1}{2}|J_f|^2, \tag{4.13}$$

$$\mathcal{D}(f) := \int_{\mathbb{S}} f|\nabla_\omega(\sigma \log f - \omega \cdot J_f)|^2, \tag{4.14}$$

we find the following conservation relation:

$$\frac{d}{d\tau}\mathcal{F} + \mathcal{D} = 0. \tag{4.15}$$

We now state and prove two results which will help us better understand the steady states of the Doi equation 4.5. The first provides equivalent characterizations of the equilibria; the second, known as the LaSalle invariance principle, tells us that the solution converges to the set of such equilibria.

Proposition 4.5 (Steady states)**.** The steady states of (4.5), i.e. the time-independent solutions, are the probability measures f on $\mathbb{S}$ which satisfy one of the following equivalent conditions:

1. *Equilibrium*: $f \in C^2(\mathbb{S})$ and $Q(f) = 0$;
2. *No dissipation*: $f \in C^1(\mathbb{S})$ and $\mathcal{D}(f) = 0$;
3. The probability density $f \in C^0(\mathbb{S})$ is positive and a critical point of $\mathcal{F}$ under the mean 1 constraint;
4. There exists $C \in \mathbb{R}$ such that $\sigma \log f - J_f \cdot \omega = C$.

Proof. Firstly we show that if f is a steady state then 1-4 are all true; secondly we show that, if any of 1-4 hold, f is a steady state.

First part: Let f be a steady state. We show that 1-4 hold.

By definition, f is a probability density function solving (4.5), which is independent of time. This means that it is positive, C^∞, and such that $Q(f) = 0$, so 1 holds.

From the conservation relation (4.15), we have

$$\mathcal{D}(f) = -\frac{d}{d\tau}\mathcal{F}(f) = 0,$$

since f does not depend on time, so 2 holds. To show that also 4 holds, notice that, from the definition of $\mathcal{D}(f)$, the previous equation reads as

$$\nabla_\omega(\sigma \log f - \omega \cdot J_f) = 0,$$

so there exists a constant $C \in \mathbb{R}$ such that $\sigma \log f - \omega \cdot J_f = C$.

Finally, we show that 3 holds: taking h such that $\int_{\mathbb{S}} h = 0$, we have that $f + h$ is still a probability density function, and

$$\begin{aligned}\mathcal{F}(f+h) = \sigma \int_{\mathbb{S}} (f \log f + h \log f + h) d\omega - \tfrac{1}{2}|J_f|^2 + \\ - J_f \cdot \int_{\mathbb{S}} \omega h \, d\omega + O(\|h\|_\infty^2) = \\ = \mathcal{F}(f) + \int_{\mathbb{S}} h(\sigma \log g - J_f \cdot \omega) d\omega + O(\|h\|_\infty^2) = \\ = \mathcal{F}(f) + O(\|h\|_\infty^2),\end{aligned} \tag{4.16}$$

which shows that f is a critical point of $\mathcal{F}$.

Second part: We show that, if any of 1-4 holds, then f is a steady state.

Suppose that 1 holds: let $f \in C^2(\mathbb{S})$ such that $Q(f) = 0$. Then f is obviously a steady state.

Suppose $\sigma \log f - J_f \cdot \omega = C$. Then $f \in C^2(\mathbb{S})$ and $Q(f) = 0$, which means that if 4 holds, then f is a steady state.

We now show that 2 and 3 can be reduced to this fourth condition.

Suppose that 3 holds: performing the computations in (4.16) for a positive function $f \in C^0(\mathbb{S})$ gives that, if f is a critical point of $\mathcal{F}$, then $\int_{\mathbb{S}} h(\sigma \log f - J_f \cdot \omega) d\omega = 0$ for any zero-mean function h. This means that $\sigma \log f - J_f \cdot \omega$ is constant.

Suppose that 2 holds: let $f \in C^1(\mathbb{S})$ such that $\mathcal{D}(f) = 0$. We have that, in a neighbourhood of any point ω_0 such that $f(\omega_0) > 0$, $\nabla_\omega(\sigma \log f - J_f \cdot \omega)$ is equal to zero.

We define the function $\phi(\omega) = \sigma \log f - J_f \cdot \omega$, which is locally constant at any point where it is finite. For any $C \in \mathbb{R}$, $\phi^{-1}(\{C\})$ is open in $\mathbb{S}$. Let $(w_k)_k$ be a sequence converging to ω_∞, such that $\phi(\omega_k) = C$. We have

$$f(\omega_k) = e^{\frac{1}{\sigma}(C + J_f \cdot \omega_k)} \quad \xrightarrow{k \to \infty} \quad f(\omega_\infty) = e^{\frac{1}{\sigma}(C + J_f \cdot \omega_\infty)}.$$

So $\phi(\omega_\infty) = C$, which means that $\phi^{-1}(\{C\})$ is closed. Moreover, since f is not identically zero, there exists at least one $C \in \mathbb{R}$ such that $\phi^{-1}(\{C\}) \neq \emptyset$. By connectedness of the sphere, $\phi^{-1}(\{C\}) = \mathbb{S}$, or equivalently $\sigma \log f - J_f \cdot \omega = C$.

□

Remark. From point 4 we get that, if f is an equilibrium, then $\sigma \log f - J_f \cdot \omega$ is constant, and $f = Ce^{\sigma^{-1} J_f \cdot \omega}$, from which we get that f is of the form

$$f = M_{\kappa\Omega}, \quad \kappa\Omega = \sigma^{-1} J_f.$$

Moreover, since $J_f = c(\kappa)\Omega$, we get the compatibility condition

$$c(\kappa) = \sigma\kappa. \tag{4.17}$$

Remark. Compared to the previous chapter, we have used a different – but equivalent – strategy to arrive to the form of the equilibria; in particular, we have used the notion of *free energy* and *dissipation*. We have done so because these quantities are going to be of key importance in the proof of convergence to the equilibria.

Proposition 4.6 (Compatibility condition)**.**

1. If $\rho \leqslant n$, there is only one solution to (4.17): $\kappa = 0$. The only equilibrium is the constant function $f = 1$.
2. If $\rho > n$, then (4.17) has 2 roots: $\kappa = 0$ and $\kappa(\rho) > 0$. The only equilibria for $\kappa = 0$ is $f = 1$; the ones associated to $\kappa(\rho)$ consist of the Von-Mises-Fischer distributions $M_{\kappa(\rho)\Omega}$, for an arbitrary $\Omega \in \mathbb{S}$, and they form a manifold of dimension n.

Proof. We provide a different proof from the one in Chapter 3, which does not require the direct computation of the function $\beta = c(\kappa) + n\sigma(\kappa)1$. We compute the second derivative of $\tilde{\sigma} = \frac{c(\kappa)}{\kappa}$:

$$\tilde{\sigma}''(\kappa) = (n-1)\frac{\beta}{\kappa^2} - 2\tilde{\sigma}(\tilde{\sigma} - \beta).$$

We then notice that:

* For $\kappa > 0$, if $\tilde{\sigma}' = -\frac{\beta}{\kappa} = 0$, then $\tilde{\sigma}'' < 0$.
* For $\kappa = 0$, we expand
$$\tilde{\sigma}(\kappa) = \tfrac{1}{n} - \tfrac{1}{n^2(n-2)}\kappa^2 + O(\kappa^4).$$
  We have that any critical point of $\tilde{\sigma}$ is a maximum. Therefore, since $\kappa = 0$ is a local maximum, the function is decreasing.

□

Remark. We have provided a different proof because we will be interested in the behaviour of the convergence rate as σ approaches $\frac{1}{n}$ (see Proposition 4.10). In fact, in this case we have

$$\sigma - \tfrac{1}{n} \sim -\tfrac{1}{n^2(n+2)}\kappa^2(\sigma),$$

and

$$c(\kappa(\sigma)) \sim \tfrac{1}{n}\kappa(\sigma) \sim \sqrt{(n+2)(\tfrac{1}{n} - \sigma)}. \tag{4.18}$$

Theorem 4.7 (LaSalle's invariance principle)**.** Let f_0 be a probability measure on the sphere $\mathbb{S}$. We denote by $\mathcal{F}_\infty$ the limit of $\mathcal{F}\big(f(\tau)\big)$ as τ tends to $+\infty$, where f is the solution of (4.5) with initial condition f_0. Then the set

$$\mathcal{E}_\infty := \{f \in C^\infty(\mathbb{S}) : \mathcal{D}(f) = 0 \text{ and } \mathcal{F}(f) = \mathcal{F}_\infty\}$$

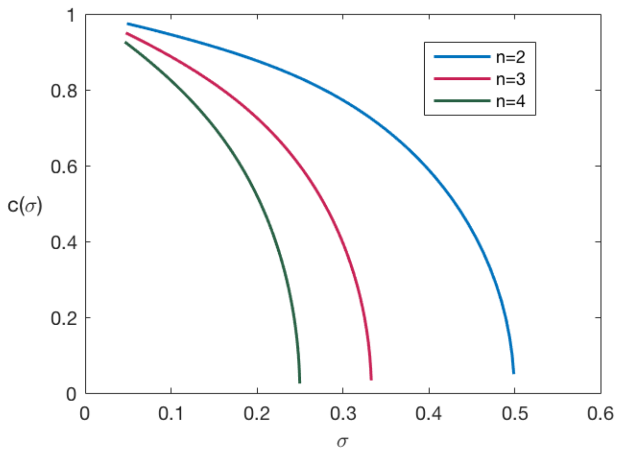

Figure 4.1: Graph of $c(\sigma)$ for dimensions $n = 2, 3, 4$.

is not empty. Furthermore, $f(\tau)$ converges in any H^s norm to this set of equilibria, in the following sense:

$$\lim_{\tau\to\infty} \inf_{g\in\mathcal{E}_\infty} \|f(\tau) - g\|_{H^s} = 0.$$

Proof. We notice that $\mathcal{F}(f(\tau))$ is decreasing in time, and bounded from below by $-1/2$, which means that $\mathcal{F}_\infty$ is well defined.

Let $(\tau_n)_{n\geqslant 0}$ be an unbounded increasing sequence in $\mathbb{R}^+$, and suppose that $f(\tau_n)$ converges to f_∞ in $H^s(\mathbb{S})$, for some $s \in \mathbb{R}$. Using Theorem 4.4, we have that $f(\tau_n)$ is uniformly bounded in $H^{s+2p}(\mathbb{S})$, for any p. Moreover,

$$\|f(\tau_n) - f(\tau_m)\|^2_{H^{s+p}} \leqslant \|f(\tau_n) - f(\tau_m)\|_{H^s} \|f(\tau_n) - f(\tau_m)\|_{H^{s+2p}},$$

so $f(\tau_n)$ converges in $H^{s+p}(\mathbb{S})$, which implies that f_∞ belongs to any $H^s(\mathbb{S})$.

We wish to show that $\mathcal{D}(f_\infty) = 0$. Supposing this is not the case,

$$\begin{aligned}\mathcal{D}(f) &= \sigma^2 \int_{\mathbb{S}} \frac{|\nabla_\omega f|^2}{f}\, d\omega + J_f \cdot \int_{\mathbb{S}} (\mathbb{I} - \omega\otimes\omega) f J_f\, d\omega - 2\sigma J_f \cdot \int_{\mathbb{S}} \nabla_\omega f\, d\omega = \\ &= \sigma^2 \int_{\mathbb{S}} \frac{|\nabla_\omega f|^2}{f}\, d\omega + (1 - 2(n-1)\sigma)|J_f|^2 - \int_{\mathbb{S}} (\omega\cdot J_f)^2 f d\omega.\end{aligned} \tag{4.19}$$

We now take a big enough s, such that $H^s(\mathbb{S}) \subseteq L^\infty(\mathbb{S}) \cap H^1(\mathbb{S})$. If $f_\infty > 0$, then the dissipation term, thought as a function

$$\mathcal{D} : \{\ell \in H^s(\mathbb{S}) : \ell \geqslant 0\} \to [0,\infty),$$

is continuous in f_∞. In particular, since $\mathcal{D}(f_\infty) > 0$, there exist $\delta, M > 0$ such that, if $\|f - f_\infty\|_{H^s} \leqslant \delta$, then $\mathcal{D}(f) \geqslant M$.

We show the same result for $f_\infty \geqslant 0$. We define the function

$$\mathcal{D}_\xi(f) := \sigma^2 \int_{\mathbb{S}} \frac{|\nabla_{n-1}\omega f|^2}{f+\xi} d\omega + \big(1 - 2(n-1)\sigma\big)|J_f|^2 - \int_{\mathbb{S}} (\omega \cdot J_f)^2 f \, d\omega.$$

By monotone convergence, $\mathcal{D}_\xi(f_\infty) \xrightarrow{\xi\to\infty} \mathcal{D}(f_\infty)$, so there exists $\xi > 0$ such that $\mathcal{D}_\xi(f_\infty) > 0$. By continuity of $\mathcal{D}_\xi$ at f_∞ then, there exist $\delta, M > 0$ such that, if $\|f - f_\infty\|_{H^s} \leqslant \delta$, then $\mathcal{D}_\xi(f) \geqslant M$. Since $\mathcal{D} \geqslant \mathcal{D}_\xi$, we conclude.

We now use the fact that $\partial_t f$ is uniformly bounded in H^s: there exists $\bar{\tau} > 0$ such that, if $|\tau - \tau'| \leqslant \bar{\tau}$, then $\|f(\tau) - f(\tau')\|_{H^s} \leqslant \delta/2$. By taking N sufficiently large, we can assume that $\|f(\tau_n) - f(\tau_\infty)\|_{H^s} \leqslant \delta/2$ for all $n \geqslant N$. For such n we have $\mathcal{D}(f) \geqslant M$ on $[\tau_n, \tau_n + \bar{\tau}]$; up to extracting, we can assume that $\tau_{n+1} \geqslant \tau_n + \bar{\tau}$. For all $p > 0$ we get

$$\mathcal{F}(f(\tau_N)) - \mathcal{F}(f(\tau_{N+p})) = \int_{\tau_N}^{\tau_{N+p}} \mathcal{D}(f) \geqslant p\bar{\tau}M.$$

Since the left term is bounded by $\mathcal{F}(f(\tau_N)) - \mathcal{F}_\infty$, taking p sufficiently large leads to a contradiction.

We now suppose that, for a given s, the distance between $f(\tau)$ and $\mathcal{E}_\infty$ does not tend to 0 as $\tau \to \infty$. More precisely, we suppose that there exist $\xi > 0$ and a sequence $(\tau_n)_{n\geqslant 0}$ such that, for all $g \in \mathcal{E}_\infty$,

$$\|f(\tau_n) - g\|_{H^s} \geqslant \xi, \ \forall n \geqslant 0.$$

Since $f(\tau_n)$ is bounded in $H^{s+1}(\mathbb{S})$, using a compact Sobolev embedding, we can assume (up to extracting) that $f(\tau_n)$ converges to f_∞ in $H^s(\mathbb{S})$.

We recall that $f_\infty \in C^\infty(\mathbb{S})$ is such that $\mathcal{D}(f_\infty) = 0$. Moreover, since $\mathcal{F}(f)$ is decreasing in time, we have $\mathcal{F}(f_\infty) = \mathcal{F}_\infty$. So f_∞ belongs to $\mathcal{E}_\infty$ and, by hypothesis, $\|f(\tau_n) - f_\infty\|_{H^s} \geqslant \xi$ for all $n \geqslant 0$, a contradiction.

We have that the distance between $f(\tau)$ and $\mathcal{E}_\infty$ tends to 0, so this set is obviously not empty. □

Proposition 4.8 (Minimum of the free energy)**.**

1. If $\rho \leqslant n$, the minimum of the free energy is 0, only reached by the uniform distribution. Any solution converges to the uniform distribution in any H^s form.

2. If $\rho > n$, the minimum of the free energy is negative, only reached by any non-isotropic equilibrium of the form $M_{\kappa(\rho)\Omega}$.

Proof. By Theorem 4.7, we have that

$$\inf_{f\in C^\infty(\mathbb{S}), f>0} \mathcal{F}(f) = \inf_{f\in C^\infty(\mathbb{S}), f>0, \mathcal{D}(f)=0} \mathcal{F}(f).$$

Since the set of equilibria is compact (consisting of single point or of one point and a manifold homeomorphic to $\mathbb{S}$), this infimum is a minimum.

If f_0 is not an equilibrium, then $\mathcal{D}(f_0) > 0$, which means that $\mathcal{F}(f(\tau))$ is decreasing in the neighbourhood of $\tau = 0$, since $\frac{d}{d\tau}\mathcal{F} + \mathcal{D} = 0$. So the minimum cannot be reached in f_0.

1. If $\rho \leqslant n$, we conclude, since the only equilibrium is the constant function 1 and by LaSalle's principle we know that the solution is converging in any H^s form.
2. If $\rho > n$, we already know that the uniform distribution is unstable, therefore it is not a minimizer for $\mathcal{F}$.
   Moreover, since $\mathcal{F}$ is decreasing and regular, its minimizers are given only by its critical points. But $\mathcal{F}(M_{\kappa(\rho)\Omega})$ is independent of Ω, so all Von-Mises-Fischer equilibria yield the same value of $\mathcal{F}$. Therefore $M_{\kappa(\rho)\Omega}$ is the only global minimizer.

□

4.1.3 *A new entropy*

We define two different norms:

1. The norm $\|\cdot\|_{\tilde{H}^{-\frac{n-1}{2}}}$ by setting, for $g \in \dot{H}^{-\frac{n-1}{2}}(\mathbb{S})$
$$\|g\|^2_{\tilde{H}^{-\frac{n-1}{2}}} = \int_{\mathbb{S}} g\tilde{\Delta}^{-1}_{n-1}g;$$
   This norm is equivalent to $\|\cdot\|_{\dot{H}^{-\frac{n-1}{2}}}$
2. The norm $\|\cdot\|_{\tilde{H}^{-\frac{n-3}{2}}}$ by setting, for $g \in \dot{H}^{-\frac{n-3}{2}}(\mathbb{S})$,
$$\|g\|^2_{\tilde{H}^{-\frac{n-3}{2}}} = \int_{\mathbb{S}} \Delta g\tilde{\Delta}^{-1}_{n-1}g.$$
   This norm is equivalent to $\|\cdot\|_{\dot{H}^{-\frac{n-3}{2}}}$

By taking $\phi = \tilde{\nabla}^{-1}_{n-1}g$ in (4.7), we obtain the following conservation equation[5]:

$$\frac{1}{2}\frac{d}{d\tau}\|g\|^2_{\tilde{H}^{-\frac{n-1}{2}}} = -\sigma\|g\|^2_{\tilde{H}^{-\frac{n-3}{2}}} + \frac{1}{(n-2)!}|J_g|^2, \tag{4.20}$$

which we can rewrite as

$$\frac{d}{dt}\mathcal{H}(f) + \tilde{\mathcal{D}}(f) = 0, \tag{4.21}$$

where f is a probability density function, $\mathcal{H}(f) = \|f-1\|^2_{\tilde{H}^{-\frac{n-1}{2}}}$, and $\tilde{\mathcal{D}}(f) = 2\sigma\|f - 1\|^2_{\tilde{H}^{-\frac{n-3}{2}}} - \frac{2}{(n-2)!}|J_f|^2$.

Remark. For $\sigma \geqslant \frac{1}{n}$, we have $\tilde{\mathcal{D}}(f) \geqslant 0$, which means that the new entropy $\mathcal{H}(f)$ is decreasing in time.

[5] We use the fact that, if $g \in \dot{H}^{\frac{n-3}{2}}$, then $\int_{\mathbb{S}} g\nabla\tilde{\Delta}^{-1}_{n-1}g = 0$.

4.2 *The subcritical case*

In this subsection we derive a convex entropy, which shows global decay to the uniform distribution, in the subcritical case $\rho < n$.

By using the Poincaré inequality on (4.20), we find

$$\frac{1}{2}\frac{d}{d\tau}\|g\|^2_{\tilde{H}^{-\frac{n-1}{2}}} \leqslant (n-1)(\tfrac{1}{n}-\sigma)\|g\|_{\tilde{H}^{-\frac{n-1}{2}}}, \tag{4.22}$$

which means that, if $\rho < n$, there is an exponential decay of rate $\frac{(n-1)(n-\rho)}{n\epsilon}$, for the norm $\|\cdot\|_{\tilde{H}^{-\frac{n-1}{2}}}$:

$$\|g\|_{\tilde{H}^{-\frac{n-1}{2}}} \leqslant \|g_0\|_{\tilde{H}^{-\frac{n-1}{2}}} e^{\frac{(n-1)(n-\rho)}{n\rho}\tau} = \|g_0\|_{\tilde{H}^{-\frac{n-1}{2}}} e^{\frac{(n-1)(n-\rho)}{n\epsilon}t}.$$

We wish to find a similar result for $\|\cdot\|_{H^s}$. If $f_0 \in H^s(\mathbb{S})$ with $s > -\frac{n-1}{2}$, we can use

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|g\|^2_{\dot{H}^s} + \frac{1}{\epsilon}\|g\|^2_{\dot{H}^{s+1}} &\leqslant C_0|J_g|\|g\|^2_{\dot{H}^s} + \tfrac{\rho}{\epsilon}(n-1)^s|J_g|^2 \leqslant \\ &\leqslant \tfrac{C_0}{(N+1)(N+n-1)}\|g\|^2_{\dot{H}^{s+1}} + \tfrac{\rho}{\epsilon}(n-1)^s|J_g|^2 + C_0\|f^N-1\|^2_{\dot{H}^s},\end{aligned} \tag{4.12}$$

the Poincaré inequality, and (since f is a probability measure)

$$(n-1)^s|J_g|^2 + \|f^N-1\|^2_{\dot{H}^s} \leqslant K_N\|f^N-1\|^2_{\tilde{H}^{-\frac{n-1}{2}}} \leqslant K_N\|g_0\|^2_{\tilde{H}^{-\frac{n-1}{2}}} e^{-2(n-1)\left(\sigma-\frac{1}{n}\right)\tau},$$

to get that, for any $\xi < \frac{1}{n}$, there exists $C = C(s)$ such that, if N is sufficiently large,

$$\frac{1}{2}\frac{d}{d\tau}\|g\|^2_{\dot{H}^s} + (n-1)\left(\sigma-\xi\right)\|g\|^2_{\dot{H}^s} \leqslant C\|g_0\|^2_{\tilde{H}^{-\frac{n-1}{2}}} e^{-2(n-1)\left(\sigma-\frac{1}{n}\right)\tau}.$$

From this inequality we have

$$\|g\|^2_{\dot{H}^s} \leqslant \|g_0\|^2_{\dot{H}^s} e^{-2(n-1)\left(\sigma-\xi\right)\tau} + \frac{C}{(n-1)(\frac{1}{n}-\xi)}\|g_0\|_{\tilde{H}^{-\frac{n-1}{2}}} e^{-2(n-1)\left(\sigma-\frac{1}{n}\right)\tau}.$$

Taking $\xi = \frac{1}{2n}$, since $s > -\frac{n-1}{2}$, we finally get

$$\|g\|^2_{\dot{H}^s} \leqslant C_1\|g_0\|^2_{\dot{H}^s} e^{-2(n-1)\left(\sigma-\frac{1}{n}\right)\tau} = C_1\|g_0\|^2_{\dot{H}^s} e^{-2\frac{(n-1)(n-\rho)}{n\epsilon}t}.$$

In conclusion, there exists a constant $C = C(s)$ such that, if $f_0 \in H^s(\mathbb{S})$, we have

$$\|f(t)-1\|^2_{H^s} \leqslant C\|f_0-1\|_{H^s} e^{-\frac{(n-1)(n-\rho)}{n\epsilon}t}.$$

4.3 *The supercritical case*

We now fix $\rho > n$ and study the behaviour of solutions to

$$\partial_\tau f = -\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega) J_f f\big) + \sigma \Delta_\omega f =: Q(f) \tag{4.5}$$

as $t \to \infty$.
In this section we want to show exponential rate of convergence of the solution to the set of equilibria, in any H^s norm. This is immediate if $J_{f_0} = 0$: as we will see in the next Proposition, there is convergence to the uniform distribution. It is more complicated when $J_{f_0} \neq 0$: in this case, the equilibria are given by a Von-Mises-Fisher distribution; in order to determine the rate of convergence, we will first show that there is exponential convergence in L^2 and then, by using interpolation, we will show convergence in any H^s norm.

From the LaSalle principle, we have the following [6]

Proposition 4.9. The limit set of equilibria

$$\mathcal{E}_\infty = \{f \in C^\infty(\mathbb{S}) : \mathcal{D}(f) = 0 \text{ and } \mathcal{F}(f) = \mathcal{F}_\infty\}$$

depends only on whether J_{f_0} is zero or not:

* If $J_{f_0} = 0$, then $\mathcal{E}_\infty$ is reduced to the uniform distribution; equation (4.5) becomes the heat equation; there is exponential decay to the uniform distribution, with rate $2n/\epsilon$, in any $H^s(\mathbb{S})$.

* If $J_{f_0} \neq 0$, then $J_{f(t)} \neq 0$ for all $t > 0$; the limit set $\mathcal{E}_\infty = \{M_{\kappa\Omega}, \Omega \in \mathbb{S}\}$ consists of all the non-isotropic equilibria. Furthermore, for any $s \in \mathbb{R}$, we have

  $$\lim_{t \to \infty} \|f(t) - M_{\kappa\Omega(t)}\|_{H^s} = 0,$$

  where $\Omega(t) = \frac{J_{f(t)}}{|J_{f(t)}|}$ is the direction of $J_{f(t)}$.

Proof. We start by writing a differential equation for J_f. In order to do so, we multiply (4.5)

[6] When there is no confusion, we write $\kappa = \kappa(\rho)$ and $c = c(\rho)$.

by ω and integrate it on the $(n-1)$-dimensional sphere $\mathbb{S}$. We get

$$
\begin{aligned}
\frac{d}{d\tau}J_f &= \int_{\mathbb{S}} \omega \partial_t f \, d\omega = \\
&= -\int_{\mathbb{S}} \omega \nabla_\omega \cdot \big((\mathbb{I} - \omega\otimes\omega)J_f f\big)\, d\omega + \sigma \int_{\mathbb{S}} \omega \Delta_\omega f \, d\omega = \\
&= -J_f \int_{\mathbb{S}} \omega \nabla_\omega \cdot ((\mathbb{I} - \omega\otimes\omega) f)\, d\omega - \sigma \int_{\mathbb{S}} \nabla_\omega \omega \cdot \nabla_\omega f \, d\omega = \\
&= -J_f \int_{\mathbb{S}} \Big(\nabla_\omega \cdot \big((\mathbb{I} - \omega\otimes\omega)\omega f\big) - \big((\mathbb{I} - \omega\otimes\omega) f\big)\Big) d\omega + \\
&\qquad\qquad\qquad - \sigma \int_{\mathbb{S}} \nabla_\omega f \, d\omega = \\
&= J_f \int_{\mathbb{S}} (\mathbb{I} - \omega\otimes\omega) f \, d\omega - \sigma(n-1) \int_{\mathbb{S}} \omega f \, d\omega = \\
&= -\sigma(n-1)J_f + \Big(\int_{\mathbb{S}} (\mathbb{I} - \omega\otimes\omega) f \, d\omega\Big) J_f = \\
&= \Big((1-(n-1)\sigma)\mathbb{I} - \int_{\mathbb{S}} \omega\otimes\omega f \, d\omega\Big) J_f,
\end{aligned}
\tag{4.23}
$$

where we have used the divergence theorem (since the border of $\mathbb{S}$ is empty), along with the fact that $\int \nabla_\omega f = (n-1)\int \omega f$.

We write the above expression as a first order linear ODE of the form $\frac{d}{d\tau}J_f = M(\tau)J_f$; the matrix M is a smooth function of time, so we have a global unique solution.

* If $J_{f(\tau_0)} = 0$ for $\tau_0 \geqslant 0$, then we have $J_{f(\tau)} = 0$ for all $\tau \geqslant 0$, and (4.5) reduces to the heat equation. The distribution f has no component on the first eigenvalue of the Laplace-Beltrami operator, and second eigenvalue equal to $2n$, which means that we have exponential decay with rate $2n/\rho$ in any H^s form, when time is parametrized by τ. Recalling that $\tau = \frac{\rho}{\epsilon}t$, we find that the rate is given by $2n/\epsilon$.

* If $J_{f(\tau_0)} \neq 0$, then $J_{f(\tau)} \neq 0$ for all $\tau \geqslant 0$. From Proposition 4.8, we know that in this case the limiting set can either be given by the uniform distribution, or by $\{M_{\kappa\Omega} : \Omega \in \mathbb{S}\}$.

  In the first case, by LaSalle's principle, $f(\tau)$ converges to the uniform distribution. We then have that $M(\tau) = \big(1-(n-1)\sigma\big)\mathbb{I} - \int_{\mathbb{S}} \omega\otimes\omega f(\tau)\, d\omega$ converges to $(n-1)(\frac{1}{n} - \sigma)\mathbb{I}$. Using (4.23), we find

$$
\frac{d}{d\tau}|J_f|^2 = J_f \cdot M(\tau) J_f \geqslant \big((n-1)(\tfrac{1}{n} - \sigma) - \xi\big)|J_f|^2, \tag{4.24}
$$

  for τ sufficiently large. For a sufficiently small ξ, we get that $|J_f|$ tends to infinity, a contradiction. So we have $\mathcal{E}_\infty = \{M_{\kappa\Omega} : \Omega \in \mathbb{S}\}$.

Suppose that $\|f(\tau) - M_{\kappa\Omega(\tau)}\|_{H^s}$ does not tend to 0 as $t \to \infty$; we can then take a sequence $\tau_n \to \infty$ such that $\|f(\tau_n) - M_{\kappa\Omega(\tau_n)}\|_{H^s} \geqslant \xi > 0$. By LaSalle's principle, there exists $\Omega_n \in \mathbb{S}$ such that $\|f(\tau_n) - M_{\kappa\Omega_n}\|_{H^s} \to 0$.
Up to extracting, we can assume $\Omega_n \to \Omega_\infty \in \mathbb{S}$, so that $f(\tau_n) \to M_{\kappa\Omega_\infty}$ in $H^s(\mathbb{S})$. In particular, $J_{f(\tau_n)}$ converges to $c(\kappa)\Omega_\infty$, and then $\Omega(\tau_n) \to \Omega_\infty$. Finally, we find that $M_{\kappa\Omega(\tau_n)} \to M_{\kappa\Omega_\infty}$, and $\|f(\tau_n) - M_{\kappa\Omega(\tau_n)}\|_{H^s} \to 0$, a contradiction.

□

Now we focus on the case $J_{f_0} \neq 0$. We define $\Omega(\tau)$ as in the previous Proposition; we wish to expand the solution f around $M_{\kappa\Omega(\tau)}$.

We first show convergence in $L^2(\mathbb{S})$ to a given equilibrium, with exponential rate, under appropriate conditions on the initial data.

Proposition 4.10. Let $\Omega(t)$ be as in the previous Theorem. There exists an asymptotic rate $r_\infty(\sigma) > 0$ such that, if $\|f(\tau) - M_{\kappa\Omega(\tau)}\|_{H^s}$ is uniformly bounded on $[\tau_0, +\infty)$ by a constant K, with $s > \frac{3(n-1)}{2}$, then $\forall r < r_\infty(\sigma)$, there exist $\Omega_\infty \in \mathbb{S}$ and $\delta, C > 0$, such that if $\|f(\tau_0) - M_{\kappa\Omega(\tau_0)}\|_{L^2} \leqslant \delta$, we have

$$\|f(\tau) - M_{\kappa\Omega_\infty}\|_{L^2} \leqslant C\|f(\tau_0) - M_{\kappa\Omega(\tau_0)}\|_{L^2} e^{-r(\tau-\tau_0)}.$$

Moreover, as $\sigma \to \frac{1}{n}$, we have that $r_\infty(\sigma) \geqslant 2(n-1)(\frac{1}{n} - \sigma) + O\big((\frac{1}{n} - \sigma)^{3/2}\big)$.

Proof. Let $\Omega = \Omega(\tau)$, and suppose $\tau \geqslant \tau_0$; we write $\cos\theta = \omega \cdot \Omega$. Recalling that $\beta = c^2 + n/\rho - 1 > 0$, we have the following identities ($\langle \cdot \rangle_{M_{\kappa\Omega}}$ denotes the average over the Von-Mises-Fisher distribution):

1. $\langle \omega \rangle_{M\kappa\Omega} = \langle \cos\theta \rangle_{M\kappa\Omega}\Omega = c\Omega$,
2. $\langle \cos^2\theta \rangle_{M\kappa\Omega} = 1 - (n-1)\sigma$,
3. $\langle (\cos\theta - c)^2 \rangle_{M\kappa\Omega} = 1 - (n-1)\sigma - c^2 = \sigma - \beta > 0$.

If we write $f = (1+h)M_{\kappa\Omega}$, with $\langle h \rangle_{M\kappa\Omega} = 0$, since Ω is the direction of $J_f = \langle (1+h)\omega \rangle_{M\kappa\Omega}$, we get that $\langle h\omega \rangle_{M\kappa\Omega} = \Omega \langle h\cos\theta \rangle_{M\kappa\Omega}$.
We wish to expand the free energy and the dissipation in terms of h.

Since $M_{\kappa\Omega}$ is a minimum of $\mathcal{F}(f)$ (see Proposition 4.8), we know that the expansion $\mathcal{F}\big((1+h)M_{\kappa\Omega}\big) - \mathcal{F}(M_{\kappa\Omega})$ does not contain any terms of order 0 and 1 in h. From the definition of free energy (4.13), we get

$$\mathcal{F}\big((1+h)M_{\kappa\Omega}\big) - \mathcal{F}(M_{\kappa\Omega}) = \tfrac{\sigma}{2}\langle h^2 \rangle_{M\kappa\Omega} - \tfrac{1}{2}|\langle h\omega \rangle_{M\kappa\Omega}|^2 + O(\|h\|_\infty^3).$$

Using Sobolev embedding and interpolation, we have

$$\|f - M_{\kappa\Omega}\|_\infty \leqslant C\|f - M_{\kappa\Omega}\|_{H^{\frac{n-1}{2}}} \leqslant C\|f - M_{\kappa\Omega}\|_{L^2}^{1-\frac{n-1}{2s}} K^{\frac{n-1}{2s}}. \tag{4.25}$$

Since

1. $1 - \frac{n-1}{2s} > \frac{2}{3}$,
2. $f - M_{\kappa\Omega} = hM_{\kappa\Omega}$,
3. $M_{\kappa\Omega}$ is uniformly bounded both from below and above,

we have that $\|h\|_{L^2}^2 = \int_S h^2 M_{\kappa\Omega} d\omega \leqslant C \int_S h^2 M_{\kappa\Omega}^2 d\omega = C\langle h^2 \rangle_{M\kappa\Omega}$. Using this in (4.25), we get that $\|h\|_\infty^3 \in o(\langle h^2 \rangle_{M\kappa\Omega})$.
We then have

$$\mathcal{F}(f) - \mathcal{F}(M_{\kappa\Omega}) = \frac{\sigma}{2}\langle h^2 \rangle_{M\kappa\Omega} - \frac{1}{2}\langle h\cos\theta \rangle_{M\kappa\Omega}^2 + o(\|h^2\|_{M_{\kappa\Omega}}). \tag{4.26}$$

From the definition of the dissipation term (4.14), we get:

$$\begin{aligned}
\mathcal{D}(f) &= \int_S f|\nabla_\omega(\sigma \log f - \omega \cdot J_f)|^2 \, d\omega = \\
&= \int_S (1+h)|\nabla_\omega \sigma \log\big((1+h)M_{\kappa\Omega}\big) - \omega \cdot \langle (1+h)\omega \rangle_{M\kappa\Omega}|^2 M_{\kappa\Omega} \, d\omega = \\
&= \langle (1+h)|\nabla_\omega\big(\sigma \log\big((1+h)M_{\kappa\Omega}\big) - \omega \cdot \langle (1+h)\omega \rangle_{M\kappa\Omega}\big)|^2 \rangle_{M_{\kappa\Omega}} = \\
&= \langle (1+h)|\nabla_\omega\big(\sigma \log(1+h)\big) + \sigma\kappa\nabla_\omega \cos\theta + \\
&\qquad - \nabla_\omega\big(\langle h\cos\theta \rangle_{M\kappa\Omega} \cos\theta\big) + \nabla_\omega\big(\cos\theta \langle \cos\theta \rangle_{M\kappa\Omega}\big)|^2 \rangle_{M_{\kappa\Omega}} = \\
&= \langle (1+h)|\nabla_\omega\big(\sigma \log(1+h)\big) + \langle \cos\theta \rangle_{M\kappa\Omega}\Omega + \\
&\qquad - \nabla_\omega\big(\langle h\cos\theta \rangle_{M\kappa\Omega} \cos\theta\big) - \langle \cos\theta \rangle_{M\kappa\Omega}\Omega|^2 \rangle_{M_{\kappa\Omega}} = \\
&= \langle (1+h)|\nabla_\omega\big(\sigma \log(1+h) - \langle h\cos\theta \rangle_{M\kappa\Omega} \cos\theta\big)|^2 \rangle_{M_{\kappa\Omega}} = \\
&= (1 - \|h\|_\infty)\langle |\nabla_\omega\big(\sigma \log(1+h) - \langle h\cos\theta \rangle_{M_{\kappa\Omega}} \cos\theta\big)|^2 \rangle_{M\kappa\Omega}.
\end{aligned}$$

Using the Poincaré-Wirtinger inequality on the sphere, and the fact that $M_{\kappa\Omega}$ is positive and bounded, for any function ℓ, we find

$$\begin{aligned}
\langle |\nabla \ell|^2 \rangle_{M\kappa\Omega} &\geqslant \min M_{\kappa\Omega} \int_S |\nabla \ell|^2 \geqslant \min M_{\kappa\Omega}(n-1) \int_S (\ell - \textstyle\int_S \ell)^2 \geqslant \\
&\geqslant \frac{\min M_{\kappa\Omega}}{\max M_{\kappa\Omega}}(n-1)\langle (\ell - \textstyle\int_S \ell)^2 \rangle_{M_{\kappa\Omega}} \geqslant e^{-2k}(n-1)\langle (\ell - \textstyle\int_S \ell)^2 \rangle_{M_{\kappa\Omega}},
\end{aligned}$$

which means that the optimal Poincaré constant is bounded from below by

$$\Lambda_k \geqslant (n-1)e^{-2\kappa}. \tag{4.27}$$

Applying this estimate to the above expression for $\mathcal{D}(f)$, we get

$$\begin{aligned}
\mathcal{D}(f) &\geqslant (1 - \|h\|_\infty)\Lambda_\kappa \langle \big(\sigma \log(1+h) - \sigma\langle \log(1+h) \rangle_{M\kappa\Omega} + \\
&\qquad - \langle h\cos\theta \rangle_{M\kappa\Omega}(\cos\theta - c)\big)^2 \rangle_{M\kappa\Omega} \geqslant \\
&\geqslant (1 - \|h\|_\infty)\Lambda_\kappa \langle \big(\sigma h - \langle h\cos\theta \rangle_{M\kappa\Omega}(\cos\theta - c) + O(\|h\|_\infty^2)\big)^2 \rangle_{M\kappa\Omega} \geqslant \\
&\geqslant (1 - \|h\|_\infty)\Lambda_\kappa \big(\sigma^2 \langle h^2 \rangle_{M\kappa\Omega} - (\beta + \sigma)\langle h\cos\theta \rangle_{M_{\kappa\Omega}}^2 + O(\|h\|_\infty^2)\big) + O(\|h\|_\infty^3).
\end{aligned}$$

Using the same argument as we did in the case of $\mathcal{F}(f)$, we find

$$\mathcal{D}(f) \geqslant \Lambda_\kappa \big(\sigma^2 \langle h^2 \rangle_{M\kappa\Omega} - \big(\beta+\sigma\big) \langle h \cos\theta \rangle^2_{M_{\kappa\Omega}} + O(\|h\|^2_\infty)\big) + o(\langle h^2 \rangle_{M\kappa\Omega}). \tag{4.28}$$

We set $\alpha = \frac{1}{\sigma-\beta}\langle h\cos\theta\rangle_{M\kappa\Omega}$ (well defined since $\sigma-\beta>0$), and write $h = \alpha(\cos\theta - c) + g$, for some function g. We notice that $\langle g \rangle_{M\kappa\Omega} = \langle g\omega \rangle_{M\kappa\Omega} = 0$.
We can then write $\langle h^2 \rangle_{M_{\kappa\Omega}} = (\sigma-\beta)\alpha^2 + \langle g^2 \rangle_{M\kappa\Omega}$, and

$$\mathcal{F}(f) - \mathcal{F}(M_{\kappa\Omega}) = \tfrac{1}{2}\big(\beta(\sigma-\beta)\alpha^2 + \sigma\langle g^2\rangle_{M_{\kappa\Omega}}\big) + o(\langle h^2\rangle_{M_{\kappa\Omega}}), \tag{4.29}$$

$$\begin{aligned} \mathcal{D}(f) &\geqslant \Lambda_k\big(\beta^2(\sigma-\beta)\alpha^2 + \sigma^2\langle g^2\rangle_{M\kappa\Omega}\big) + o(\langle h^2\rangle_{M\kappa\Omega}) \geqslant \\ &\geqslant \Lambda_k\beta\big(\beta(\sigma-\beta)\alpha^2 + \sigma\langle g^2\rangle_{M\kappa\Omega}\big) + o(\langle h^2\rangle_{M\kappa\Omega}). \end{aligned} \tag{4.30}$$

So, if $\langle h^2 \rangle_{M\kappa\Omega}$ is sufficiently small, we have

$$\mathcal{D}(f) \geqslant 2r\big(\mathcal{F}(f) - \mathcal{F}(M_{\kappa\Omega})\big),$$

for all $r < \Lambda_\kappa\beta$. So there exists $\delta_0 > 0$ such that, if $\|f(t) - M_{\kappa\Omega(t)}\|_{L^2} \leqslant \delta_0$, then we have

$$\frac{d}{d\tau}\big((\mathcal{F}(f) - \mathcal{F}(M_{\kappa\Omega})\big) = -\mathcal{D}(f) \leqslant 2r\big(\mathcal{F}(f) - \mathcal{F}(M_{\kappa\Omega})\big).$$

For all T such that $\|f(\tau) - M_{\kappa\Omega(\tau)}\|_{L^2} \leqslant \delta_0$ on $[\tau_0, T]$, we have

$$\mathcal{F}(f(T)) - \mathcal{F}(M_{\kappa\Omega(T)}) \leqslant \Big(\mathcal{F}\big(f(\tau_0)\big) - \mathcal{F}\big(M_{\kappa\Omega(\tau_0)}\big)\Big)e^{-2r(T-\tau_0)}.$$

Moreover, since

$$\|f(\tau) - M_{\kappa\Omega}\|_{L^2} \leqslant C\sqrt{\langle h^2\rangle_{M\kappa\Omega}} \leqslant C_0\|f(\tau_0) - M_{\kappa\Omega(\tau_0)}\|_{L^2}e^{-r(\tau-\tau_0)}, \tag{4.31}$$

if τ_0 is such that $\|f(\tau_0) - M_{\kappa\Omega(\tau_0)}\|_{L^2} \leqslant \delta$ for a $\delta < \frac{\delta_0}{C_0} \leqslant \delta_0$, we have that (4.31) holds for all $\tau \geqslant \tau_0$.

In order to prove strong convergence to a given steady state, we need to prove that $\Omega(\tau)$ converges to an Ω_∞ as $\tau \to \infty$. From

$$J_f = \langle (1+h)\omega \rangle_{M\kappa\Omega} = c\Omega + \langle h\omega\rangle_{M\kappa\Omega} = \big(c + \alpha(\sigma-\beta)\big)\Omega,$$

we get

$$\frac{d}{d\tau}J_f = \big(c+\alpha(\sigma-\beta)\big)\frac{d}{d\tau}\Omega(\tau) + (\sigma-\beta)\Omega\frac{d}{d\tau}\alpha(\tau).$$

Putting this together with (4.23), we find

$$\begin{aligned} \frac{d}{d\tau}J_f &= \big(c+\alpha(\sigma-\beta)\big)\frac{d}{d\tau}\Omega(\tau) + (\sigma-\beta)\Omega\frac{d}{d\tau}\alpha(\tau) = \\ &= \Big(\big(1-(n-1)\sigma\big)\mathbb{I} - \int_{\mathbb{S}} \omega\otimes\omega f\, d\omega\Big)J_f. \end{aligned}$$

We now apply $\mathbb{I} - \Omega \otimes \Omega$ to the members of (4.23). We get

$$\begin{aligned}(\mathbb{I} - \Omega \otimes \Omega)\frac{d}{d\tau}J_f &= (\mathbb{I} - \Omega \otimes \Omega)\Big(- \int_{\mathbb{S}} \omega \otimes \omega f \, d\omega\Big) J_f = \\ &= -\big(c + \alpha(\sigma - \beta)\big)(\mathbb{I} - \Omega \otimes \Omega) \left(\int_{\mathbb{S}} \omega^2 M_{\kappa\Omega} \, d\omega + \int_{\mathbb{S}} \omega^2 h M_{\kappa\Omega} \, d\omega \right) = \\ &= -\big(c + \alpha(\sigma - \beta)\big)(\mathbb{I} - \Omega \otimes \Omega) \left(\langle \cos\theta\omega \rangle_{M\kappa\Omega} + \langle h \cos\theta\omega \rangle_{M\kappa\Omega} \right).\end{aligned}$$

We also have

$$\big(c + \alpha(\sigma - \beta)\big)\frac{d}{d\tau}\Omega = -\big(c + \alpha(\sigma - \beta)\big)(\mathbb{I} - \Omega \otimes \Omega)\langle g \cos\theta\omega \rangle_{M\kappa\Omega}.$$

Since $\big(c + \alpha(\sigma - \beta)\big)$ is equal to the norm of J_f, it is never zero, so we can simplify and get

$$\left|\frac{d}{d\tau}\Omega\right| \leqslant C\sqrt{\langle g^2 \rangle_{M\kappa\Omega}} \leqslant C\|f - M_{\kappa\Omega}\|_{L^2} \leqslant C\|f(\tau_0) - M_{\kappa\Omega(\tau_0)}\|_{L^2} e^{-r(\tau - \tau_0)},$$

for a constant $C = C(r, s, \sigma, K)$. This shows that $\frac{d}{d\tau}\Omega$ has exponential decay of rate r, in particular there exists an $\Omega_\infty \in \mathbb{S}$, such that $\Omega \xrightarrow{\tau \to \infty} \Omega_\infty$.

Since $\Omega \mapsto e^{\kappa\omega\cdot\Omega}$ is globally Lipschitz, with a constant which is independent of $\omega \in \mathbb{S}$, we have that

$$\begin{aligned}\|M_{\kappa\Omega(\tau)} - M_{\kappa\Omega_\infty}\|_{L^2} &\leqslant C|\Omega(\tau) - \Omega_\infty| \leqslant \\ &\leqslant \int_\tau^{+\infty} \left|\frac{d}{d\tau}\Omega\right| dt \leqslant C\|f(\tau_0) - M_{\kappa\Omega(\tau_0)}\|_{L^2} e^{-r(\tau - \tau_0)}.\end{aligned}$$

This allows us to conclude:

$$\begin{aligned}\|f - M_{\kappa\Omega_\infty}\|_{L^2} &\leqslant \|f - M\kappa\Omega(\tau)\|_{L^2} + \|M_{\kappa\Omega(\tau)} - M_{\kappa\Omega_\infty}\|_{L^2} \leqslant \\ &\leqslant C\|f(\tau_0) - M_{\kappa\Omega(\tau_0)}\|_{L^2} e^{-r(\tau - \tau_0)}.\end{aligned}$$

So we have that the Proposition holds true, with $r_\infty(\sigma) = \Lambda_\kappa \beta > 0$. Moreover, from (4.27), we know that $\Lambda_\kappa \geqslant (n-1)e^{-2\kappa}$, and thanks to (4.18), we get that $r_\infty(\sigma) \geqslant 2(n-1)(\frac{1}{n} - \sigma) + O\big((\frac{1}{n} - \sigma)^{3/2}\big)$. □

Remark. Since, by Proposition 4.9, $f(t) - M_{\kappa\Omega(t)} \to 0$ in any H^s, we have that there exists a $t_0 > 0$ which satisfies the hypotheses of the previous Proposition. Using interpolation, we get

$$\|f - M_{\kappa\Omega_\infty}\|_{H^s} \leqslant C\|f - M_{\kappa\Omega_\infty}\|_{L^2}^{1-\frac{s}{p}} \|f - M_{\kappa\Omega_\infty}\|_{H^p}^{\frac{s}{p}} \leqslant C\|f(\tau_0) - M_{\kappa\Omega(t_0)}\|_{L^2}^{1-\frac{s}{p}} e^{-r(1-\frac{s}{p}(\tau - \tau_0))}.$$

Finally, we have that for all $r < r_\infty$ and for all s, there exists some time τ_0 and $C > 0$ such that $\|f - M_{\kappa\Omega_\infty}\|_{H^s} \leqslant Ce^{-r\tau}$, for all $\tau \geqslant \tau_0$. Since $t = \frac{\epsilon}{\rho}\tau$, this proves the claim of Theorem 4.1(ii).

4.4 *The critical case*

We have seen that for any $\rho \in (0, +\infty) \setminus \{n\}$ there is exponential convergence to some equilibrium. However, if $J_{f_0} \neq 0$, the rate that we have found tends to 0 as $\rho \to n$. Therefore we expect there to be a different behaviour in the critical case.

In this section we want to estimate the rate of convergence to the uniform distribution (which we know is an equilibrium) as time goes to infinity. In order to do so, we first provide a result which estimates the L^2 distance between the solution f and the uniform distribution. We will then use an interpolation argument to show the algebraic asymptotic rate of convergence in any H^s norm.

Proposition 4.11. Suppose that $\|f(\tau) - 1\|_{H^s}$ is uniformly bounded on $[\tau_0, +\infty)$ by a constant K, with $s > \frac{7(n-1)}{2}$. Then, for all $C > 1$, there exists $\delta = \delta(\rho, s, K, C) > 0$ such that if $\|f(t_0) - 1\|_{L^2} \leqslant \delta$, we have

$$\|f(\tau) - 1\|_{L^2} \leqslant \frac{C}{\sqrt{\frac{1}{\sqrt{2(n+2)}\|f(\tau_0)-1\|_{L^2}} + \frac{2(n-1)}{n(n+2)}(\tau - \tau_0)}},$$

for all $\tau \geqslant \tau_0$.

Proof. Again, we work with $\tau \geqslant t_0$, write $f = 1 + h$, and suppose $J_{f_0} \neq 0$. From this we get that $J_{f(\tau)}$ is non-zero for all $\tau > 0$, so we can define $\Omega(\tau) := \frac{J_{f(\tau)}}{|J_{f(\tau)}|}$. We also denote by $\langle \cdot \rangle$ the average on the sphere, and $\cos\theta := \omega \cdot \Omega$.

We have $\langle h \rangle = 0$, $J_f = \langle (1+h)\omega \rangle = \langle h\omega \rangle$, and $\langle h\omega \rangle = \langle h \cos\theta \rangle \Omega$, since Ω is the direction of J_f.

As we did in the supercritical case, we now perform an expansion of the free energy $\mathcal{F}$ and of the dissipation $\mathcal{D}$ in terms of h. For $\sigma = \frac{1}{n}$, we find

$$\begin{aligned}
\mathcal{F}(1+h) &= \frac{1}{n}\int_{\mathbb{S}} (1+h)\log(1+h)d\omega - \frac{1}{2}\langle h\cos\theta\rangle^2 = \\
&= \frac{1}{n}\int_{\mathbb{S}} \left(h + \frac{h^2}{2} - \frac{h^3}{6} + \frac{h^4}{12}\right)d\omega - \frac{1}{2}\langle h\cos\theta\rangle^2 + O(\|h\|_\infty^5) = \\
&= \frac{1}{n}\left(\frac{1}{2}\langle h^2\rangle - \frac{1}{6}\langle h^3\rangle + \frac{1}{12}\langle h^4\rangle\right) - \frac{1}{2}\langle h\cos\theta\rangle^2 + O(\|h\|_\infty^5).
\end{aligned}$$

We write $\alpha = n\langle h\cos\theta\rangle$, and define the function

$$g = h - \alpha\cos\theta - \frac{1}{2}\alpha^2\left(\cos^2\theta - \frac{1}{n}\right) - \frac{1}{6}\alpha^3\left(\cos^3\theta - \frac{3}{n+2}\cos\theta\right). \tag{4.32}$$

We now perform some computations which we will need later on, when we will expand $\mathcal{F}$ and $\mathcal{D}$ in terms of α.

Recalling that $(2p+2)a_{p+1} = \frac{a_p}{2p+n}$, where $a_p = \frac{1}{(2p)!}\int_0^\pi \cos^{2p}\theta \sin^{n-2}\theta\, d\theta$, we find that $\langle\cos^4\theta\rangle = 4!\frac{a_2}{a_0} = \frac{3}{n(n+2)}$. Moreover, since $\langle\cos^3\theta\rangle = \langle\cos\theta\rangle = 0$, and $\langle\cos^2\theta\rangle = \frac{1}{n}$, we get

$$\begin{aligned}\langle g\rangle &= \langle h\rangle - \alpha\langle\cos\theta\rangle - \tfrac{1}{2}\alpha^2\big(\langle\cos^2\theta\rangle - \tfrac{1}{n}\big) - \tfrac{1}{6}\alpha^3\big(\langle\cos^3\theta\rangle - \tfrac{3}{n+2}\langle\cos\theta\rangle\big) = 0,\\ \langle g\cos\theta\rangle &= \tfrac{\alpha}{n} - \tfrac{\alpha}{n} - \tfrac{1}{6}\alpha^3\big(\tfrac{3}{n(n+2)} - \tfrac{3}{n(n+2)}\big) = 0.\end{aligned}$$

The following expansions are needed in order to expand the free energy and the dissipation term; they are obtained by taking the second (third, fourth) power of the expression for h in (4.32) and then considering the mean on the sphere:

$$\begin{aligned}\tfrac{1}{2}\langle h^2\rangle &= \tfrac{1}{2}\langle g^2\rangle + \tfrac{1}{2n}\alpha^2 + \tfrac{n-1}{4n^2(n+2)}\alpha^4 + \tfrac{1}{2}\alpha^2\langle g\cos^2\theta\rangle + O(\alpha^3\|g\|_\infty + \alpha^5), \qquad (4.33)\\ -\tfrac{1}{6}\langle h^3\rangle &= -\tfrac{n-1}{2n^2(n+2)}\alpha^4 - \tfrac{1}{2}\alpha^2\langle g\cos^2\theta\rangle + O(\|g\|_\infty^3 + \alpha\|g\|_\infty^2 + \alpha^2\|g\|_\infty + \alpha^5),\\ \tfrac{1}{12}\langle h^4\rangle &= \tfrac{1}{4n(n+2)}\alpha^4 + O(\|g\|_\infty^4 + \alpha\|g\|_\infty^3 + \alpha^2\|g\|_\infty^2 + \alpha^3\|g\|_\infty + \alpha^5).\end{aligned}$$

Inserting these expansions in the above expression for $\mathcal{F}(1+h)$, we get

$$\mathcal{F}(1+h) = \tfrac{1}{2n}\langle g^2\rangle + \tfrac{1}{4n^3(n+2)}\alpha^4 + O(\|g\|_\infty^3 + \alpha\|g\|_\infty^2 + \alpha^3\|g\|_\infty + \alpha^5). \qquad (4.34)$$

Using the inequality $a^pb^q \leqslant sa^{\frac{p}{s}} + (1-s)b^{\frac{q}{1-s}}$ for $s\in(0,1)$, with $a=\alpha$ and $b=\|g\|_\infty$, we find that

$$\alpha\|g\|_\infty^2 \leqslant \tfrac{1}{5}\alpha^5 + \tfrac{4}{5}\|g\|_\infty^{2+\frac{1}{2}}, \qquad \alpha^3\|g\|_\infty \leqslant \tfrac{3}{5}\alpha^5 + \tfrac{2}{5}\|g\|_\infty^{2+\frac{1}{2}}.$$

As we have done previously, by Sobolev embedding and interpolation, we find that

$$\|g\|_\infty \leqslant C\|g\|_{L^2}^{1-\frac{n-1}{2s}}\|g\|_{H^s}^{\frac{n-1}{2s}}, \qquad (4.35)$$

where $1-\frac{n-1}{2s} > \frac{6}{7}$. Since $\|h\|_{H^s}$ is uniformly bounded by K), we get that $\|g\|_{H^s}$ is bounded, and $\|g\|_\infty^{2+\frac{1}{2}} \leqslant C\langle g^2\rangle^\mu$, for $\mu > \frac{1}{2}(2+\frac{1}{2})\frac{6}{7} > 1$. We can now use the estimate on $\langle h^2\rangle$ in (4.33), to find that, for any $\xi>0$, there exists $\delta>0$ such that, if $\|h\|_{L^2}\leqslant\delta$, then we have

$$\begin{aligned}(1-\xi)\big(\langle g^2\rangle + \tfrac{1}{n}\alpha^2\big) \leqslant \langle h^2\rangle &\leqslant (1+\xi)\big(\langle g^2\rangle + \tfrac{1}{n}\alpha^2\big),\\ (1-\xi)\big(\tfrac{1}{2n}\langle g^2\rangle + \tfrac{1}{4n^3(n+2)}\alpha^4\big) \leqslant \mathcal{F}(1+h) &\leqslant \tfrac{1+\xi}{4n^3(n+2)}\big(2n^2(n+2)\langle g^2\rangle + \alpha^4\big). \qquad (4.36)\end{aligned}$$

Combining the two inequalities, up to taking a smaller δ, we have[7]:

$$\tfrac{1-\xi}{1+\xi}2n\mathcal{F}(1+h) \leqslant \langle h^2\rangle \leqslant \tfrac{1+\xi}{\sqrt{1-\xi}}2\sqrt{n(n+2)\mathcal{F}(1+h)}. \qquad (4.37)$$

[7]We need $\delta \ll 1$ such that

$$\tfrac{(1-\xi)\alpha^4}{4n^3(n+2)} + \tfrac{1-\xi)\alpha^2}{n} \geqslant 0, \quad (1+\xi)\langle g^2\rangle + \tfrac{1+\xi}{\sqrt{1-\xi}}2\sqrt{n(n+2)\mathcal{F}(1+h) - \tfrac{1-\xi}{2n}\langle g^2\rangle} \geqslant \tfrac{1+\xi}{\sqrt{1-\xi}}2\sqrt{n(n+2)\mathcal{F}(1+h)}.$$

We now wish to provide an estimate for the dissipation term. Using the Poincaré inequality, we find

$$\begin{aligned}\mathcal{D}(f) &= \langle (1+h) \left| \nabla_\omega \left(\tfrac{1}{n} \log(1+h) - \langle (1+h)\omega \rangle \cdot \omega \right) \right|^2 \rangle = \\ &= \langle (1+h) \left| \nabla_\omega \left(\tfrac{1}{n} \log(1+h) - \langle h \cos\theta \rangle \cos\theta \right) \right|^2 \rangle \geqslant \\ &\geqslant \tfrac{n}{n-1} (1 - \|h\|_\infty) \langle \left(\log(1+h) - \langle (1+h) \rangle - n \langle h \cos\theta \rangle \cos\theta \right)^2 \rangle.\end{aligned} \tag{4.38}$$

We also notice that

$$\begin{aligned}\mathcal{S}(h) :&= \log(1+h) - \langle (1+h) \rangle - n \langle h \cos\theta \rangle \cos\theta = \\ &= h - \langle h \rangle - \alpha \cos\theta - \tfrac{1}{2} \left(h^2 - \langle h^2 \rangle \right) + \tfrac{1}{3} \left(h^3 - \langle h^3 \rangle \right) + O(\|h\|_\infty^4).\end{aligned}$$

We compute the expansions

$$\begin{aligned} h - \alpha\cos\theta &= g + \tfrac{1}{2}\alpha^2 (\cos^2\theta + \tfrac{1}{n}) + \tfrac{1}{6}\alpha^3 (\cos^3\theta - \tfrac{3}{n+2}\cos\theta), \\ -\tfrac{1}{2}\left(h^2 - \langle h^2 \rangle\right) &= -\tfrac{1}{2}(\alpha^2 + \alpha^3\cos\theta)\left(\cos^2\theta - \tfrac{1}{n}\right) + O(\|g\|_\infty^2 + \alpha\|g\|_{H^{\frac{n-1}{2}}}\infty + \alpha^4), \\ \tfrac{1}{3}\left(h^3 - \langle h^3 \rangle\right) &= \tfrac{1}{3}\alpha^3\cos^3\theta + O(\|g\|_\infty^3 + \alpha\|g\|_\infty^2 + \alpha^4),\end{aligned}$$

and write

$$\begin{aligned}\langle \mathcal{S}^2(h) \rangle &= \langle \left(g + \tfrac{1}{6}\alpha^3 (\tfrac{3}{n} - \tfrac{3}{n+2}\cos\theta) \right)^2 \rangle + O(\|g\|_\infty^3 + \alpha\|g\|_\infty^2 + \alpha^4\|g\|_\infty + \alpha^7) = \\ &= \langle g^2 \rangle + \tfrac{1}{n^3(n+2)^2}\alpha^6 + O(\|g\|_\infty^3 + \alpha\|g\|_\infty^2 + \alpha^4\|g\|_\infty + \alpha^7).\end{aligned} \tag{4.39}$$

As before, we get

$$\alpha\|g\|_\infty^2 \leqslant \tfrac{1}{7}\alpha^7 + \tfrac{6}{7}\|g\|_\infty^{2+\frac{1}{3}}, \qquad \alpha^4\|g\|_\infty \leqslant \tfrac{4}{7}\alpha^7 + \tfrac{3}{7}\|g\|_\infty^{2+\frac{1}{3}},$$

and, using (4.35), we find that $\|g\|_\infty^{2+\frac{1}{3}} \leqslant C\langle g^2 \rangle^\mu$, with $\mu > \frac{1}{2}(2+\frac{1}{3})\frac{6}{7} = 1$. For $\|h\|_{L^2} < \delta$, up to taking a smaller δ, we have

$$\mathcal{D}(f) \geqslant (1-\xi)\tfrac{n-1}{n^2}\left(\langle g^2 \rangle + \tfrac{1}{n^3(n+2)^2}\alpha^6 \right).$$

For any $C, C' > 0$, up to taking a smaller δ (we need α and g small), we have that $C\langle g^2 \rangle + \alpha^6 \geqslant \left(C'\langle g^2 \rangle + \alpha^4 \right)^{\frac{3}{2}}$, and

$$\mathcal{D}(f) \geqslant (1-\xi)\tfrac{n-1}{n^5(n+2)^2}\left(2n^2(n+2)\langle g^2 \rangle + \alpha^4 \right)^{\frac{3}{2}}.$$

If we combine this with (4.36) and (4.15), we get that for any $0 < \xi < 1$, there exists $\delta_0 > 0$ such that, if $\|h\|_{L^2} \leqslant \delta_0$, we have

$$\frac{d}{d\tau}\mathcal{F}(f) = -\mathcal{D}(f) \leqslant -\frac{8(n-1)(1-\xi)}{(1+\xi)^{3/2}\sqrt{n(n+2)}}\left(\mathcal{F}(f)\right)^{3/2},$$

which we can solve to find

$$\mathcal{F}\big(f(T)\big)^{-1/2} \geqslant \mathcal{F}\big(f(\tau_0)\big)^{-1/2} + \frac{4(n-1)(1-\xi)}{(1+\xi)^{3/2}\sqrt{n(n+2)}}(\tau - \tau_0), \tag{4.40}$$

for all T such that $\|h\|_{L^2} \leqslant \delta_0$ on $[\tau_0, T]$. Using this in (4.37), we get

$$\|h\|_{L^2}^{-2} \geqslant \frac{\sqrt{1-\xi}}{(1+\xi)2\sqrt{n(n+2)}} \left(\sqrt{\frac{2n(1-\xi)}{1+\xi}} \|h(t_0)\|_{L^2}^{-1} + \frac{4(n-1)(1-\xi)}{(1+\xi)^{3/2}\sqrt{n(n+2)}}(\tau - \tau_0) \right).$$

Writing $C = \frac{(1+\xi)^{5/4}}{(1-\xi)^{3/4}}$, we find

$$\|h\|_{L^2} \leqslant C \left(\frac{1}{\sqrt{2(n+2)}\|h(t_0)\|_{L^2}} + \frac{2(n-1)}{n(n+2)}(\tau - \tau_0) \right)^{-1/2}. \tag{4.41}$$

If we now take $\delta < \min\{\delta_0, \frac{1}{C^2\sqrt{2(n+2)}}\delta_0\}$, and $\|h(t_0)\|_{L^2} \leqslant \delta$, we get that $\|h\|_{L^2} \leqslant \delta$ on $[\tau_0, T]$, $\forall\, T \geqslant \tau_0$. In fact, if this were not the case, the largest of such T would satisfy

$$\delta_0 = \|h(\tau)\|_{L^2} \leqslant C \left(\frac{1}{\sqrt{2(n+2)}\delta} \right)^{-1/2} \leqslant \delta_0.$$

So the inequality (4.41) holds for all $\tau \geqslant \tau_0$, which ends the proof. □

Remark. Since f tends to the uniform distribution in any $H^s(\mathbb{S})$, we get that for any $r < \frac{2(n-1)}{n(n+2)}$, there exists $\tau_0 \geqslant 0$ such that

$$\|f(\tau) - 1\|_{L^2} \leqslant \frac{1}{\sqrt{r(\tau - \tau_0)}}, \quad \forall\, \tau \geqslant \tau_0.$$

Moreover, since for any $r < \tilde{r} < \frac{2(n-1)}{n(n+2)}$ and for τ large enough, $\frac{1}{\sqrt{\tilde{r}(\tau-\tau_0)}} \leqslant \frac{1}{\sqrt{r\tau}}$, we actually have $\|f(\tau) - 1\|_{L^2} \leqslant \frac{1}{\sqrt{r\tau}}$.
If we were to use interpolation to get an inequality in the Sobolev norms, as we have done in the previous section, we would get something of the form

$$\|f(\tau) - 1\|_{H^s} \leqslant C_\eta \tau^{-1/2+\eta},$$

but we can do slightly better.
Using $h = g + \alpha\cos\theta + \frac{1}{2}\alpha^2(\cos^2\theta - \frac{1}{n}) + \frac{1}{6}\alpha^3(\cos^3\theta - \frac{3}{n+2}\cos\theta)$, we write

$$\|h\|_{H^s} \leqslant \|g\|_{H^s} + |\alpha|\|\cos\theta\|_{H^s} + C_2\alpha^2 + C_3|\alpha|^3.$$

We take $\tau_0 > 0$ satisfying the conditions of the previous Proposition, and such that $\|h\|_{L^2} \leqslant \delta$. Since g is uniformly bounded in any $H^s(\mathbb{S})$, by interpolation we have $\|g\|_{H^s} \leqslant C_\eta \|g\|_{L^2}^{1-\eta}$ for any $\eta > 0$. Using (4.40), we get

$$\left(\frac{1}{2n}\langle g^2\rangle + \frac{1}{4n^3(n+2)}\alpha^4\right)^{-1/2} \geqslant \frac{4(n-1)(1-\xi)^{3/2}}{(1+\xi)^{3/2}\sqrt{n(n+2)}}(\tau - \tau_0),$$

from which we have $\|g\|_{L^2} \in O(\tau^{-1})$ and $\alpha^2 \leqslant \frac{(1+\xi)^{3/2}}{(1-\xi)^{3/2}}\frac{n(n+2)}{2(n-1)(\tau-\tau_0)}$. Finally, for any $\eta > 0$,

$$\|h\|_{H^s} \leqslant (n-1)^{s/2}\sqrt{\frac{(1+\xi)^{3/2}}{(1-\xi)^{3/2}}\frac{n(n+2)}{2(n-1)(\tau-\tau_0)}} + O(\tau^{-1+\eta}).$$

So there exists $\tau_1 \geqslant \tau_0$ such that, for all $\tau \geqslant t_1$, the following holds for any $\eta > 0$:

$$\|h\|_{H^s} \leqslant (1+\xi)(n-1)^{s/2}\sqrt{\frac{(1+\xi)^{3/2}}{(1-\xi)^{3/2}}\frac{n(n+2)}{2(n-1)(\tau-\tau_0)}},$$

In conclusion, we have that for any $r < \frac{2}{n(n-1)^{s-1}(n+2)}$, there exists $t_1 \geqslant 0$ such that, for all $\tau \geqslant \tau_1$, we have $\|f(\tau) - 1\|_{H^s} \leqslant \frac{1}{\sqrt{r\tau}}$, i.e. there is asymptotic algebraic rate of convergence equal to 1/2.

5

A generalization

We consider the same model as before, but now we assume that the flock is comprised of two different populations, let's say A and B. We then have different interactions: between individuals of the same population, and of opposite one.

We assume that the probability an individual belongs to A rather than to B is $1/2$; in other words, the *type* of a particle is given by a random variable $T^i \sim \mathrm{Be}(1/2)$. Let (X^i, V^i) denote an individual of the first population, (Y^i, W^i) and individual of the second one.
Let $N = N_A + N_B$ be the total population size, with N_A being the number of individuals of A, and N_B the number of individuals of B. We assume that the two families have different diffusion coefficients. More precisely, the dynamics of our model is given by the coupled system

$$\begin{cases} dX_t^i = V_t^i dt,\ dY_t^i = W_t^i dt \\ dV_t^i = \sqrt{2d}(\mathbb{I} - V_t^i \otimes V_t^i) \circ dB_t^i + (\mathbb{I} - V_t^i \otimes V_t^i) J_t^i(X_t^i) dt \\ dW_t^i = \sqrt{2b}(\mathbb{I} - W_t^i \otimes W_t^i) \circ dB_t^i + (\mathbb{I} - W_t^i \otimes W_t^i) J_t^i(Y_t^i) dt, \end{cases} \tag{5.1}$$

where, for $Z_t^i = X_t^i$ or Y_t^i, the function J is defined as

$$J_t^i(Z_t^i) = \frac{1}{N_A} \sum_{j=1}^{N_A} K(|Z_t^i - X_t^j|) V_t^j + \frac{1}{N_B} \sum_{j=1}^{N_B} K(|Z_t^i - Y_t^j|) W_t^j.$$

We would like to prove the same mean-field and macroscopic results as in the previous chapters. In particular, we wish to show that the empirical distributions f^{N_A} and g^{N_B} tend, as $N \to \infty$, to certain probability density functions f and g, which satisfy a particular PDE system. Moreover, we would like to see if it is possible to determine the equilibria of such a model.

It is fairly easy to show that, in the hydrodynamic limit, the presence of different interactions between the two populations does not affect the form of the equilibria. We can then assume that the interaction kernel K does not distinguish between the two.

5.1 *Mean-field limit*

Theorem 5.1. There exists a unique solution for the following non-linear system

$$\begin{cases} d\bar{X}_t^i = \bar{V}_t^i dt,\ d\bar{Y}_t^i = \bar{W}_t^i dt \\ d\bar{V}_t^i = \sqrt{2d}(\mathbb{I} - \bar{V}_t^i \otimes \bar{V}_t^i) dB_t^i + (\mathbb{I} - \bar{V}_t^i \otimes \bar{V}_t^i)\bar{J}_{f+g}(\bar{X}_t^i)dt - (n-1)\bar{V}_t^i dt \\ d\bar{W}_t^i = \sqrt{2b}(\mathbb{I} - \bar{W}_t \otimes \bar{W}_t^i) dB_t^i + (\mathbb{I} - \bar{W}_t \otimes \bar{W}_t^i)\bar{J}_{f+g}(\bar{Y}_t^i)dt - (n-1)\bar{W}_t^i dt \\ f_t = \mathrm{Law}(\bar{X}_t^i, \bar{V}_t^i),\ g_t = \mathrm{Law}(\bar{Y}_t^i, \bar{W}_t^i), \end{cases} \tag{5.2}$$

with $(B_t^i)_{t\geqslant 0}$ N independent n-dimensional Brownian motions, and

$$\bar{J}_f(x) = \int_{\mathbb{S}} (K * f)(x, \omega, t)\omega \, d\omega,$$

where $*$ denotes the convolution on the space variable.

Remark. Since all the coefficients are Lipschitz and bounded, we have existence and uniqueness of the solutions $(\bar{X}_t^i, \bar{V}_t^i, \bar{Y}_t^i, \bar{W}_t^i)_{t\geqslant 0}$. It only remains to show that f_t and g_t are weak solutions of a certain PDE system.

Following what we have done in Chapter 2, it is easy to obtain (cf. Equation 2.4)

$$\begin{cases} \partial_t f_t + \omega \cdot \nabla_x f_t = -\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega)\bar{J}_{f+g} f_t\big) + d\,\Delta_\omega f_t \\ \partial_t g_t + \omega \cdot \nabla_x g_t = -\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega)\bar{J}_{f+g} g_t\big) + b\,\Delta_\omega g_t. \end{cases} \tag{5.3}$$

We expect the equilibria for this model (which we are going to define better in the next section) to somehow reflect the difference in the diffusion terms between the two equations.

5.2 *Hydrodynamic scaling and equilibria*

We introduce a small parameter $\epsilon > 0$ and perform the scaling $x' = \epsilon x$ and $t' = \epsilon t$. Writing $f^\epsilon(x', \omega, t') = f(x, \omega, t)$, $g^\epsilon(x', \omega, t') = g(x, \omega, t)$, $K^\epsilon(x') = \frac{1}{\epsilon^n} K(x)$, and using (5.3), we have

$$\begin{cases} \epsilon\big(\partial_t f^\epsilon + \omega \cdot \nabla_x f^\epsilon\big) = -\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega)\bar{J}^\epsilon_{f+g} f^\epsilon\big) + d\,\Delta_\omega f^\epsilon \\ \epsilon\big(\partial_t g^\epsilon + \omega \cdot \nabla_x g^\epsilon\big) = -\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega)\bar{J}^\epsilon_{f+g} g^\epsilon\big) + b\,\Delta_\omega g^\epsilon, \end{cases} \tag{5.4}$$

where $\bar{J}^\epsilon_{f+g}(x,t) = \int_{\mathbb{S}} (K^\epsilon * (f+g)^\epsilon)(x, \omega, t)\omega \, d\omega$.

For the time being, we study the equation for f; the same reasoning will apply to g.

Definition 5.2. For two given probability density functions f and g on $\mathbb{S}$, we define

$$Q(f) = -\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega) J_{f+g} f\big) + d\,\Delta_\omega f,$$

where $J_{f+g}(x,t) = \int_{\mathbb{S}} (f+g)(x,\omega,t)\omega\, d\omega$.

Notice that

$$\bar{J}^{\epsilon}_{f+g} = J^{\epsilon}_{f+g} + O(\epsilon^2),$$

for some constant α, with $J^{\epsilon}_{f+g} = \int_{\mathbb{S}} (f^\epsilon + g^\epsilon)(\omega) d\omega$. So, by dropping the $O(\epsilon^2)$ term, we can rewrite (5.4) as

$$\epsilon\big(\partial_t f^\epsilon + \omega \cdot \nabla_x f^\epsilon\big) = Q(f^\epsilon) + O(\epsilon^2). \tag{5.5}$$

Since $Q(f^\epsilon)$ is the only term of order 0 in ϵ, we are again interested in finding its null-space. In order to do so, we introduce the Von-Mises-Fischer distribution

$$M^d_{\kappa\Omega}(\omega) = \frac{e^{\frac{\kappa\omega\cdot\Omega}{d}}}{\int_{\mathbb{S}} e^{\frac{\kappa v\cdot\Omega}{d}} dv}.$$

We also recall that, since $M_{\kappa\Omega}$ depends on κ and Ω only through their product, we can consider M_J for any $J \in \mathbb{R}^n$. Since $\nabla_\omega(M_J) = (\mathbb{I} - \omega \otimes \omega) J M_J$, the collision operator can be written as

$$Q(f) = d\,\nabla_\omega \cdot \left(M_{J_{f+g}} \nabla_\omega \left(\frac{f}{M_{J_{f+g}}} \right) \right).$$

Using Green's formula, we get

$$\int_{\mathbb{S}} Q(f) \frac{f}{M_{J_{f+g}}} d\omega = -d \int_{\mathbb{S}} \left| \nabla_\omega \Big(\frac{f}{M_{J_{f+g}}} \Big) \right|^2 M_{J_{f+g}} d\omega \leqslant 0.$$

This means that $\dfrac{f}{M_{J_{f+g}}}$ is constant in ω, so we can write $f = \rho_f M_{\kappa\Omega}$, with $\kappa\Omega = J_{f+g}$. In other words, if f and g are such that $Q(f)$ and $Q(g)$ are equal to zero, then they are of the form

$$f = \rho_f\, C_1 \exp\left(\frac{\kappa\omega\cdot\Omega}{d}\right), \quad g = \rho_g\, C_2 \exp\left(\frac{\kappa\omega\cdot\Omega}{b}\right). \tag{5.6}$$

We then have that, for any $\kappa\Omega \in \mathbb{R}^n$, there exist constant C_1 and C_2 such that the functions defined in (5.6) are equilibria.

5.3 *Phase transition*

In this section we would like to find, as we did in the one-population case, a phase transition for our model. In order to do so, we search for a compatibility condition of the form $C(\kappa) = \kappa$. From the above computations, we have:

$$J_{M^d_{\kappa\Omega}} = \int_{\mathbb{S}} \omega M^d_{\kappa\Omega}\, d\omega = \langle \omega \rangle_{M^d_{\kappa\Omega}} = \langle \cos\theta \rangle_{M^d_{\kappa\Omega}} \Omega =: c(\tfrac{\kappa}{d})\Omega$$
$$J_{M^b_{\kappa\Omega}} = c(\tfrac{\kappa}{b})\Omega,$$

where $\cos\theta = \omega \cdot \Omega$. We get

$$\kappa\Omega = J_{f+g} = \int_{\mathbb{S}} \omega (f+g)\, d\omega = \int_{\mathbb{S}} \omega \left(\rho_f M^d_{\kappa\Omega} + \rho_g M^b_{\kappa\Omega}\right)\, d\omega =$$
$$= \rho_f J_{M^d_{\kappa\Omega}} + \rho_g J_{M^b_{\kappa\Omega}} = \rho_f\, c(\tfrac{\kappa}{d})\Omega + \rho_g\, c(\tfrac{\kappa}{b})\Omega.$$

This yields the following

$$1 = \frac{\rho_f\, c(\frac{\kappa}{d}) + \rho_g\, c(\frac{\kappa}{b})}{\kappa} = d\rho_f \frac{c(\kappa/d)}{\kappa/d} + b\rho_g \frac{c(\kappa/b)}{\kappa/b}. \tag{5.7}$$

Since we have already seen that $\frac{c(\kappa)}{\kappa} \xrightarrow{\kappa\to 0} \frac{1}{n}$, we get that, in order for there to be a positive solution to (5.7), the following limit has to be greater than 1:

$$\lim_{\kappa\downarrow 0} d\rho_f \frac{c(\kappa/d)}{\kappa/d} + b\rho_g \frac{c(\kappa/b)}{\kappa/b} = \frac{d\rho_f}{n} + \frac{b\rho_g}{n} \overset{!}{>} 1.$$

Summarizing, we have found the following

Proposition 5.3 (Compatibility condition)**.**

1. If $d\rho_f + b\rho_g \leqslant n$, then $\kappa = 0$ is the unique solution of (5.7). The only equilibria are the isotropic ones, $f = \rho_f$ and $g = \rho_g$.
2. If $d\rho_f + b\rho_g > n$, then (5.7) has 2 roots: $\kappa = 0$ and $\kappa(\rho) > 0$. The equilibria for $\kappa = 0$ are $f = \rho_f$ and $g = \rho_g$; the ones associated to $\kappa(\rho)$ consist of the Von-Mises-Fischer distributions $\rho_f M^d_{\kappa(\rho)\Omega}$ and $\rho_g M^b_{\kappa(\rho)\Omega}$, for $\Omega \in \mathbb{S}$ arbitrary.

A

Proof of Lemma 3.12

Lemma A.1. For $X = \int_{\mathbb{S}} \big(\partial_t(\rho M_{\kappa\Omega}) + \omega \cdot \nabla_x(\rho M_{\kappa\Omega})\big)\ell_\kappa(\omega \cdot \Omega)\omega \, d\omega$, the expression

$$(\mathbb{I} - \Omega \otimes \Omega)X = 0$$

is equivalent to

$$\rho\big(\partial_t\Omega + \tilde{c}(\Omega \cdot \nabla_x)\Omega\big) + \lambda(\mathbb{I} - \Omega \otimes \Omega)\nabla_x\rho = 0,$$

where the coefficients $\tilde{c}$ and λ are given by

$$\tilde{c} = \langle \cos\theta \rangle_{\tilde{M}\kappa} := \frac{\int_0^\pi \cos\theta \ell_\kappa(\cos\theta) e^{\kappa\cos\theta} \sin^n\theta \, d\theta}{\int_0^\pi \ell_\kappa(\cos\theta) e^{\kappa\cos\theta} \sin^n\theta \, d\theta},$$
$$\lambda = \tfrac{1}{\kappa} + \tfrac{\rho}{\kappa}\tfrac{d\kappa}{d\rho}(\tilde{c} - c).$$

Proof. We begin by providing some useful formulas for the following.
For any constant vector $V \in \mathbb{R}^n$, we have

$$\nabla_\omega(\omega \cdot V) = (\mathbb{I} - \omega \otimes \omega)V,$$
$$\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega)V\big) = -(n-1)\omega \cdot V.$$

For any constant matrix A, we then have

$$\nabla_\omega \cdot \big((\mathbb{I} - \omega \otimes \omega)A\omega\big) = A : (\mathbb{I} - n\omega \otimes \omega).$$

As usual, we write $\omega \cdot \Omega = \cos\theta$, and

$$\nabla_\omega M_{\kappa\Omega} = \kappa(\mathbb{I} - \omega \otimes \omega)\Omega M_{\kappa\Omega},$$
$$\nabla_\Omega M_{\kappa\Omega} = \kappa(\mathbb{I} - \Omega \otimes \Omega)\omega M_{\kappa\Omega},$$
$$\partial_\kappa M_{\kappa\Omega} = (\cos\theta - \langle \cos\theta \rangle_{M\kappa})\, M_{\kappa\Omega}.$$

Using these and the chain rule, and denoting with $\dot\kappa = \frac{d}{d\rho}\kappa$, we get

$$\begin{aligned}(\partial_t + \omega\cdot\nabla_x)(\rho M_{\kappa\Omega}) &= \big(1 + (\cos\theta - \langle\cos\theta\rangle_{M\kappa})\rho\dot\kappa\big)M_{\kappa\Omega}(\partial_t + \omega\cdot\nabla_x)\rho + \\ &\quad + \rho\kappa(\mathbb{I} - \Omega\otimes\Omega)\omega M_{\kappa\Omega}\cdot(\partial_t + \omega\cdot\nabla_x)\Omega = \\ &= \big(1 + (\cos\theta - \langle\cos\theta\rangle_{M\kappa})\rho\dot\kappa\big)M_{\kappa\Omega}(\partial_t\rho + \omega\cdot\nabla_x\rho) + \\ &\quad + \rho\kappa M_{\kappa\Omega}(\omega\cdot\partial_t\Omega + \omega\otimes\omega : \nabla_x\Omega).\end{aligned}$$

We write $X = X_1 + X_2 + X_3$, where

$$\begin{aligned}X_1 &= \int_{\mathbb{S}} \ell_\kappa(\cos\theta)\gamma_1(\cos\theta)\omega M_{\kappa\Omega}\, d\omega, \\ X_2 &= \int_{\mathbb{S}} \ell_\kappa(\cos\theta)\omega\otimes\omega\big(\gamma_2(\cos\theta)\nabla_x\rho + \rho\kappa\partial_t\Omega\big)M_{\kappa\Omega}\, d\omega, \\ X_3 &= \int_{\mathbb{S}} \ell_\kappa(\cos\theta)\gamma_3(\cos\theta)\omega(\omega\otimes\omega : \nabla_x\Omega)M_{\kappa\Omega}\, d\omega,\end{aligned}$$

with

$$\begin{aligned}\gamma_1(\cos\theta) &= \big(1 + (\cos\theta - c)\rho\dot\kappa\big)\partial_t\rho, \\ \gamma_2(\cos\theta) &= 1 + (\cos\theta - c)\rho\dot\kappa, \\ \gamma_3(\cos\theta) &= \rho\kappa.\end{aligned}$$

We now write $\omega = \cos\theta\Omega + \sin\theta v$, for a $v \in \mathbb{S}^{n-2}$. Supposing that $\int_{\mathbb{S}^{n-2}} dv = 1$, we have the following formulas:

$$\begin{gathered}\int_{\mathbb{S}^{n-1}} a(\omega)d\omega = C\int_0^\pi \int_{\mathbb{S}^{n-1}} a(\theta, v)\sin^{n-2}\, dvd\theta, \\ \int_{\mathbb{S}^{n-2}} v\, dv = 0, \\ \int_{\mathbb{S}^{n-2}} v\otimes v\, dv = \tfrac{1}{n-1}(\mathbb{I} - \Omega\otimes\Omega).\end{gathered}$$

We use the fact that, for any function $\gamma(\cos\theta)$, the following equalities hold:

$$\begin{gathered}\int_{\omega\in\mathbb{S}} \gamma(\cos\theta)M_{\kappa\Omega}\omega\, d\omega = \langle\cos\theta\gamma(\cos\theta)\rangle_{M_\kappa}\Omega \\ \int_{\omega\in\mathbb{S}} \omega\otimes\omega\gamma(\cos\theta)M_{\kappa\Omega}\omega\, d\omega = \langle\cos^2(\theta)\gamma\rangle_{M\kappa}\Omega\otimes\Omega + \frac{\langle\sin^2\theta\gamma\rangle_{M_\kappa}}{n-1}(\mathbb{I} - \Omega\otimes\Omega).\end{gathered}$$

We then have:

1. Since $\partial_t\Omega$ and Ω are orthogonal, $(\mathbb{I} - \Omega\otimes\Omega)X_1 = 0$.

2. $(\mathbb{I} - \Omega \otimes \Omega) X_2 = \frac{\langle \sin^2 \theta \gamma_2 \ell_\kappa \rangle_{M\kappa}}{n-1} (\mathbb{I} - \Omega \otimes \Omega) \nabla_x \rho + \frac{\rho \kappa \langle \sin^2 \theta \ell_\kappa \rangle_{M_\kappa}}{n-1} \partial_t \Omega.$

3. Since $(\omega \cdot \nabla_x)\Omega$ and Ω are orthogonal, we have

$$(\mathbb{I} - \Omega \otimes \Omega)\omega(\omega \otimes \omega : \nabla_x \Omega) = \sin\theta v \big(\omega \cdot (\omega \cdot \nabla_x)\Omega\big) = \sin^2\theta v \big(v \cdot (\omega \cdot \nabla_x)\Omega\big).$$

Moreover, from $\int_{v \in \mathbb{S}^{n-2}} v(v \otimes v : \nabla_x \Omega) dv = 0$ we get, since $(\Omega \cdot \nabla_x)\Omega$ and Ω are orthogonal,

$$\begin{aligned}(\mathbb{I} - \Omega \otimes \Omega) X_3 &= \langle \sin^2 \theta \cos\theta \gamma_3 \ell_\kappa \rangle_{M\kappa} \int_{v \in \mathbb{S}^{n-2}} v \otimes v \, dv (\Omega \cdot \nabla_x)\Omega = \\ &= \frac{\langle \sin^2 \theta \cos\theta \gamma_3 \ell_\kappa \rangle_{M\kappa}}{n-1} (\Omega \cdot \nabla_x)\Omega.\end{aligned}$$

We have that $(\mathbb{I} - \Omega \otimes \Omega) X = 0$ is equivalent to the following equation:

$$\rho\kappa \langle \sin^2 \theta \ell_\kappa \kappa \rangle_{M\kappa} \partial_t \Omega + \langle \sin^2 \theta \cos\theta \gamma_3 \ell_\kappa \rangle_{M\kappa} (\Omega \cdot \nabla_x)\Omega + \langle \sin^2 \theta \gamma_2 \ell_\kappa \rangle_{M_\kappa} \nabla_x \rho = 0.$$

We remark that for any function $\gamma(\cos\theta)$, the weighted mean

$$\langle \gamma(\cos\theta) \rangle_{\tilde{M}\kappa} := \frac{\int_0^\pi \gamma(\cos\theta) \ell_\kappa(\cos\theta) e^{\kappa \cos\theta} \sin^n \theta \, d\theta}{\int_0^\pi \ell_\kappa(\cos\theta) e^{\kappa \cos\theta} \sin^n \theta \, d\theta},$$

can be written in terms of $\langle \cdot \rangle_{M\kappa}$:

$$\langle \gamma(\cos\theta) \rangle_{\tilde{M}\kappa} = \frac{\sin^2 \theta \ell_\kappa(\cos\theta) \gamma(\cos\theta) \rangle_{M\kappa}}{\langle \sin^2 \theta \ell(\cos\theta) \rangle_{M\kappa}}.$$

Dividing by $\kappa \langle \sin^2 \theta \ell_\kappa(\cos\theta) \rangle_{M\kappa}$ concludes the proof, as we get that $(\mathbb{I} - \Omega \otimes \Omega) X = 0$ is equivalent to

$$\rho \big(\partial_t \Omega + \tilde{c}(\Omega \cdot \nabla_x)\Omega\big) + \lambda (\mathbb{I} - \Omega \otimes \Omega) \nabla_x \rho = 0,$$

for coefficients $\tilde{c}$ and λ as above. □

B

Taylor expansions

We wish to provide Taylor expansions, as $\kappa \to 0$ and $\kappa \to \infty$, for the following averages:

1. $\langle f(\theta) \rangle_{M_\kappa} = \dfrac{\int_0^\pi f(\theta) e^{\kappa \cos\theta} \sin^{n-2}\theta d\theta}{\int_0^\pi e^{\kappa \cos\theta} \sin^{n-2}\theta d\theta}$,

2. $\langle f(\theta) \rangle_{M_\kappa} = \dfrac{\int_0^\pi f(\theta) \ell_\kappa(\cos\theta) e^{\kappa \cos\theta} \sin^{n}\theta d\theta}{\int_0^\pi \ell_\kappa(\cos\theta) e^{\kappa \cos\theta} \sin^{n}\theta d\theta}$.

B.1 Asymptotics of $\langle f(\theta) \rangle_{M_\kappa}$

For a given function f such that $f \sin^{n-2}\theta \in L^1(0,\pi)$, we define

$$a_p = \frac{1}{p!} \int_0^\pi \cos^p\theta \sin^{n-2}\theta d\theta,$$
$$b_p = \frac{1}{p!} \int_0^\pi f(\theta) \cos^p\theta \sin^{n-2}\theta d\theta.$$

We can then write

$$\langle f(\theta) \rangle_{M_\kappa} = \frac{\sum_{i=0}^N b_p \kappa^p + O(\kappa^{N+1})}{\sum_{i=0}^N a_p \kappa^p + O(\kappa^{N+1})}.$$

For $f(\theta) = cos\theta$, $b_p = (p+1)a_{p+1}$ and, using the integration by parts formula, we find the following induction relation:

$$(p+2)(p+n)a_{p+2} = a_p.$$

Since $a_1 = 0$, the odd terms vanish and we can write

$$c(\kappa) = \langle \cos\theta \rangle_{M\kappa} = \frac{a_0 \frac{\kappa}{n} + a_0 \frac{\kappa^3}{2n(n+2)} + O(\kappa^5)}{a_0 + a_0 \frac{\kappa^2}{2n} + O(\kappa^4)} =$$

$$= \frac{\frac{\kappa}{n} + \frac{\kappa^3}{2n(n+2)} + O(\kappa^5)}{1 + \frac{\kappa^2}{2n} + O(\kappa^4)} = \frac{\kappa}{n} - \frac{\kappa^3}{n^2(n+2)} + O(\kappa^5).$$

For the expansion as $\kappa \to \infty$, we need the following

Lemma B.1 (Watson's Lemma)**.** Let p be a function in $L^1(0, T)$, with $T > 0$, and let $I_k(p) = \int_0^T p(t) e^{-\kappa t} dt$. Supposed that, in a neighbourhood of 0, we have $p(t) = t^\beta \big(\sum_{i=0}^{N-1} a_i t^i + O(t^N) \big)$, with $\beta > -1$. Then

$$I_k(p) = \kappa^{-\beta-1} \left(\sum_{i=0}^{N-1} a_i \Gamma(\beta + i + 1) \kappa^{-1} + O(\kappa^{-N}) \right),$$

as $\kappa \to \infty$.

We apply this lemma to the integrals of the form $[f(\theta)]_\kappa = \int_0^\pi f(\theta) e^{\kappa \cos\theta} \sin^{n-2}\theta \, d\theta$. Performing the change of variable $t = 1 - \cos\theta$, we get

$$[f(\theta)]_\kappa = e^\kappa \int_0^2 f\big(\arccos(1-t)\big) e^{-\kappa t} (2t - t^2)^{\frac{n-3}{2}} dt.$$

Let $f(\theta) = 1 - \cos\theta$. We have

$$(2t - t^2)^{\frac{n-3}{2}} = 2^{\frac{n-3}{2}} t^{\frac{n-3}{2}} (1 - \tfrac{1}{2}t)^{\frac{n-3}{2}} = 2^{\frac{n-3}{2}} t^{\frac{n-3}{2}} \big(1 - \tfrac{n-3}{4} t + O(t^2)\big).$$

We call this function A, and apply Watson's Lemma to A and tA. This yields:

$$[1]_\kappa = \frac{2^{\frac{n-3}{2}}}{\kappa^{\frac{n-1}{2}}} \left(\Gamma(\tfrac{n-1}{2}) - \tfrac{n-3}{4} \Gamma(\tfrac{n+1}{2}) \tfrac{1}{\kappa} + O(\kappa^{-2}) \right),$$

$$[f(\theta)]_\kappa = \frac{2^{\frac{n-3}{2}}}{\kappa^{\frac{n+1}{2}}} \left(\Gamma(\tfrac{n+1}{2}) - \tfrac{n-3}{4} \Gamma(\tfrac{n+3}{2}) \tfrac{1}{\kappa} + O(\kappa^{-2}) \right).$$

Using $\Gamma(p+1) = p\,\Gamma(p)$, we get

$$\langle f(\theta) \rangle_{M\kappa} = \frac{[f(\theta)]_\kappa}{[1]_\kappa} = \frac{n-1}{2\kappa} - \frac{(n-1)(n-3)}{8\kappa^2} + O(\kappa^{-3}).$$

In particular, as $\kappa \to \infty$, we have the following expansion:

$$c(\kappa) = 1 - \frac{n-1}{2\kappa} + \frac{(n-1)(n-3)}{8\kappa^2} + O(\kappa^{-3}). \tag{B.1}$$

B.2 Asymptotics of $\langle f(\theta)\rangle_{\tilde{M}\kappa}$

We wish to decompose $\ell_\kappa(\cos\theta)$ as a polynomial in κ or κ^{-1}. In order to do so, we provide the following

Proposition B.2. Let L and D be two linear operators on the space of polynomials, defined by

$$L(P) = -(1-X^2)P'' + (n+1)XP' + (n-1)P$$
$$D(P) = -(1-X^2)P' + XP.$$

We then have the following expansions for ℓ_κ:

$$\ell_\kappa(\cos\theta) = \sum_{p=0}^{N} H_p(\cos\theta)\kappa^p + R^N_{\kappa,0}(\cos\theta), \quad \text{as } \kappa \to 0,$$
$$\ell\kappa(\cos\theta) = \sum_{p=1}^{N} G_p(\cos\theta)\kappa^{-p} + R^N_{\kappa,\infty}(\cos\theta), \quad \text{as } \kappa \to \infty,$$

where H_p and G_p^N are polynomials of degree p and at most $N-p$, respectively, given by the following induction relations:

$$\begin{cases} L(H_0) = 1 \\ L(H_{p+1}) = -D(H_p) \end{cases} \quad \text{and} \quad \begin{cases} D(G_1^N)(\cos\theta) = 1 + O_0(\theta^{2N}) \\ \left(D(G_{p+1}^N) + L(G_p^N)\right)(\cos\theta) = O_0(\theta^{2(N-p)}), \end{cases} \tag{B.2}$$

and where the remainders $R^N_{\kappa,0}$ and $R^N_{\kappa,\infty}$ satisfy, for any function f such that $\theta \mapsto f(\theta)\sin^{\frac{n}{2}}\theta$ belongs to $L^2(0,\pi)$ and $|f(\theta)| = O(\theta^{2\beta})$ in a neighbourhood of 0, the following estimates, for $\kappa \to 0$ and $\kappa \to \infty$, respectively:

$$\langle f(\theta)R^N_{\kappa,0}(\cos\theta)\sin^2\theta\rangle_{M_\kappa} = O_0(\kappa^{N+1})$$
$$\langle f(\theta)R^N_{\kappa,\infty}(\cos\theta)\sin^2\theta\rangle_{M_\kappa} = O_\infty(\kappa^{-\beta-N-2}).$$

The proof of this result can be found in Appendix B.1 of [9]. We can then expand

$$\langle f(\theta)\rangle_{\tilde{M}\kappa} = \begin{cases} \dfrac{\sum_{p=0}^N \langle f(\theta)H_p(\cos\theta)\sin^2\theta\rangle_{M\kappa}\kappa^p}{\sum_{p=0}^N \langle H_p(\cos\theta)\sin^2\theta\rangle_{M\kappa}\kappa^p} + O_0(\kappa^{N+1}) \\ \dfrac{\sum_{p=1}^N \langle f(\theta)G_p^N(\cos\theta)\sin^2\theta\rangle_{M\kappa}\kappa^{-p}}{\sum_{p=1}^N \langle G_p^N(\cos\theta)\sin^2\theta\rangle_{M\kappa}\kappa^{-p}} + O_\infty(\kappa^{-\beta-N}). \end{cases}$$

Since we have

$$H_0 = \frac{1}{n-1}, \qquad H_1 = -X\frac{X}{2n(n-1)}, \qquad G_1^2 = \frac{4-X}{3}, \qquad G_2^2 = \frac{2(n-2)}{3},$$

we have

$$\langle \cos\theta \rangle_{\tilde{M}\kappa} = \frac{\frac{1}{n-1}\langle \cos\theta \sin^2\theta \rangle_{M\kappa} - \frac{\kappa}{2n(n-1)}\langle \cos^2\theta \sin^2\theta \rangle_{M\kappa}}{\frac{1}{n-1}\langle \sin^2\theta \rangle_{M\kappa} - \frac{\kappa}{2n(n-1)}\langle \cos\theta \sin^2\theta \rangle_{M\kappa}} + O_0(\kappa^2),$$
$$\langle \cos\theta - 1 \rangle_{\tilde{M}\kappa} = \frac{\frac{1}{3\kappa}\langle \cos\theta(4-\cos\theta)\sin^2\theta \rangle_{M\kappa} - \frac{2(n-2)}{3\kappa^2}\langle \cos\theta \sin^2\theta \rangle_{M\kappa}}{\frac{1}{3\kappa}\langle (4-\cos\theta)\sin^2\theta \rangle_{M\kappa} - \frac{2(n-2)}{3\kappa^2}\langle \sin^2\theta \rangle_{M\kappa}} + O_\infty(\kappa^{-3}).$$

The last step consists of computing the terms of the form $\langle \cos^i\theta \sin^2\theta \rangle_{M\kappa}$. We do so by expressing them in terms of $c(\kappa)$. Integrating by parts, we have

$$\langle \sin^2\theta \rangle_{M\kappa} = \tfrac{n-1}{\kappa}c,$$
$$\langle \cos\theta \sin^2\theta \rangle_{M\kappa} = \tfrac{n-1}{\kappa}(1 - \tfrac{n}{\kappa}c),$$
$$\langle \cos^2\theta \sin^2\theta \rangle_{M\kappa} = \langle \sin^2\theta \rangle_{M\kappa} - \langle \sin^4\theta \rangle_{M\kappa} = \tfrac{n-1}{\kappa}\left(c - \tfrac{n+1}{\kappa}(1 - \tfrac{n}{\kappa}c)\right).$$

Finally, we get the following expansion:

$$\tilde{c}(\kappa) = \langle \cos\theta \rangle_{\tilde{M}\kappa} = \begin{cases} \frac{2n-1}{2n(n+2)}\kappa + O_0(\kappa^2), \\ 1 - \frac{n+1}{2\kappa} + \frac{(n+1)(3n-7)}{24\kappa^2} + O_\infty(\kappa^{-3}). \end{cases} \tag{B.3}$$

We are now ready to provide the following

Proposition B.3. We have the following expansions:

1. For $\rho \to n$,

$$c = \tfrac{\sqrt{n+2}}{n}\sqrt{\rho - n} + O(\rho - n),$$
$$\tilde{c} = \tfrac{2n-1}{2n\sqrt{n+2}}\sqrt{\rho - n} + O(\rho - n),$$
$$\lambda = \tfrac{-1}{4\sqrt{n+2}}\tfrac{1}{\sqrt{\rho-n}} + O(1),$$
$$\theta_c = \tfrac{\pi}{2} - \tfrac{2}{\sqrt{n(n+2)}}\sqrt{\rho - n} + O(\rho - n).$$

2. For $\rho \to \infty$,

$$c = 1 - \tfrac{n-1}{2}\rho^{-1} + \tfrac{(n-1)(n+1)}{8}\rho^{-2} + O(\rho^{-3}),$$
$$\tilde{c} = 1 - \tfrac{n+1}{2}\rho^{-1} + \tfrac{(n+1)(3n+1)}{24}\rho^{-2} + O(\rho^{-3}),$$
$$\lambda = -\tfrac{n+1}{6} + O(\rho^{-3}),$$
$$\theta_c = \arctan\left(\tfrac{\sqrt{6(n+1)}}{4}\right) + O(\rho^{-1}).$$

Proof. First we compute an expansion of $\rho = \dfrac{\kappa}{c}$ (compatibility condition):

$$\rho = \begin{cases} n + \frac{1}{n+2}\kappa^2 + O_0(\kappa^4), \\ \kappa + \frac{n-1}{2} + \frac{(n-1)(n+1)}{8\kappa} + O_\infty(\kappa^{-2}), \end{cases}$$

which we can reverse to find

$$\kappa = \begin{cases} \sqrt{n+2}\sqrt{\rho - n} + O_n(\rho - n), \\ \rho - \frac{n-1}{2} - \frac{(n-1)(n+1)}{8\rho} + O_\infty(\rho^{-2}). \end{cases}$$

Now, inserting this expansion into

$$\lambda = \begin{cases} -\frac{1}{4\kappa} + O_0(1) \\ -\frac{n+1}{6\kappa^2} + O_\infty(\kappa^{-3}) \end{cases}$$

and

$$\theta_c = \begin{cases} \frac{\pi}{2} - \frac{2}{(n+2)\sqrt{n}}\kappa + O_0(\kappa^2) \\ \arctan\left(\frac{\sqrt{n+1}\sqrt{6}}{4}\right) + O_\infty(\kappa^{-1}), \end{cases}$$

the claim follows. ☐

C

The codes

C.1 Simulation of the Vicsek model

```
% SIMULATION OF A VICSEK-TYPE FLOCK IN A SQUARE
clear all
rng('shuffle')
size=25.; % size of the box
v=0.03; % speed module
dt=1.; % time step
r=1; % interaction radius
totaltime=10000;
eta=0.1; % noise intensity
nflock=300; % flock dimension

figure
axis([0 size 0 size])
axis('square')
box on
hold on
set(gca,'xtick',[],'ytick',[])

id=ones(nflock,1);
x=rand(nflock,1).*size;
y=rand(nflock,1).*size;
theta=2.*pi.*randn(nflock,1);
m=mean(theta);
vx=v.*cos(theta);
vy=v.*sin(theta);
meandir=theta;
% BEGIN INTERACTIONS
```

```
        for nsteps=1:totaltime
            nsteps
            for indiv=1:nflock
                % periodicity
                if (x(indiv)<0);x(indiv)=x(indiv)+size;end
                if (y(indiv)<0);y(indiv)=y(indiv)+size;end
                if (x(indiv)>size);x(indiv)=x(indiv)-size;end
                if (y(indiv)>size);y(indiv)=y(indiv)-size;end

                % 1. CALCULATE THE DISTANCE
                dist(1:nflock)=(id.*x(indiv)-x(1:nflock)).^2+(id.*y(indiv)...
                    ↪ -y(1:nflock)).^2;
                % 2. COMPUTE THE NEW DIRECTION,
                % CONSIDER ONLY THE INDIVIDUALS WHICH ARE 'CLOSE'
                % these steps could be made more efficient
                index=find(dist<r);
                thetanear=theta(index);
                c=mean(cos(thetanear));
                s=mean(sin(thetanear));
                meandir(indiv)=atan2(s,c); % four-quadrant inverse tangent
            end
            % 3. AND INTRODUCING SOME UNIFORMLY DISTRIBUTED NOISE
            theta=meandir+eta.*(rand(nflock,1))-.5*eta*id;
            vx=v.*cos(theta);
            vy=v.*sin(theta);
            x=x+vx.*dt;
            y=y+vy.*dt;
            cla % clears the plot
            set(gcf,'doublebuffer','on') % avoids flickering
            quiver(x,y,vx,vy,0.5)
            quiver(12.5,12.5,1*cos(mean(theta)),1*sin(mean(theta)),2,'...
                ↪ LineWidth',2,'MaxHeadSize',1,'Color','black')
            circles(x(1),y(1),r,'facecolor','r','edgecolor','r','FaceAlpha...
                ↪ ',.2,'EdgeAlpha',.3)
            drawnow
        end
```

C.2 Computation of λ

```
        % MAIN: WE COMPUTE LAMBDA AND PLOT IT AS A FUNCTION OF RHO
        clear all
        figure
        axis([0 8 -3 0])
        set(gca,'xtick',[0,2,4,6,8],'ytick',[-3,-2.5,-2,-1.5,-1,-0.5,0])
        box on
        hold on

```

```
    for n=2:4
        i=0;
        for k=0.1:0.1:8
            i=i+1;
            c(i)=Cfunction(k,n);
            ctilde(i)=Ctildefunction(k,n);
            r(i)=k/c(i);
            lambda(i)=(r(i)-n-k*ctilde(i))/(k*(r(i)-n-k*c(i)));
        end

        plot(r,lambda,'LineWidth',0.8)
    end

    xlabel('Density \rho') % x-axis label
    ylabel('\lambda') % y-axis label
    legend({'n=2','n=3','n=4'},'Position',[0.7 0.7 0.12 0.1])
```

```
function [A]=ffunction(k,n)
% WE CONSIDER f SUCH THAT f_k(x)=(sinx)^(n/2-1)g_k(x)
% WE WISH TO COMPUTE IT USING A FINITE DIFFERENCE APPROACH
    N=3000;
    for i=1:N-1
        theta(i)=i*pi/N;
    end
    for i=1:N-1
        b(i)=-N^2/pi^2*exp(k*cos((i+1/2)*pi/N))/exp(k*cos((i)*pi/N...
            ↪ ));
        btilde(i)=-N^2/pi^2*exp(k*cos((i-1/2)*pi/N))/exp(k*cos((i)...
            ↪ *pi/N));
        d(i)=(n-2)/(2*sin(theta(i)).^2)*(1+(n-2)/2*cos(theta(i))....
            ↪ ^2)-k*cos(theta(i))+N^2/pi^2*(exp(k*cos((i-1/2)*pi/...
            ↪ N))+exp(k*cos((i+1/2)*pi/N)))/exp(k*cos((i)*pi/N));
    end
    btilde=btilde(2:end);
    b=b(1:end-1);
    A=gallery('tridiag',N-1,btilde,d,b);
end
```

```
function [c]=Cfunction(k,n)
    N=3000;
    for i=1:N-1
        theta(i)=i*pi/N;
    end
    c1=cos(theta).*exp(k*cos(theta)).*sin(theta).^(n-2);
    c2=exp(k*cos(theta)).*sin(theta).^(n-2);
    C1=trapz(c1);
    C2=trapz(c2);
```

```
        c=C1/C2;
    end
```

```
    function[ctilde]=Ctildefunction(k,n)
        N=3000;
        for i=1:N-1
            theta(i)=i*pi/N;
        end
        A=ffunction(k,n);
        S=sin(theta).^(n/2);
        S=S';
        F=A\S; %vector of functions f^i_k, solution of AF=S
        G=(sin(theta)'.^(1-n/2)).*F; %vector of functions g^i_k
        H=G.*((sin(theta)).^(-1));
        theta=theta';
        c1=cos(theta).*H.*exp(k*cos(theta)).*sin(theta).^n;
        c2=H.*exp(k*cos(theta)).*sin(theta).^n;
        C1=trapz(c1);
        C2=trapz(c2);
        ctilde=C1/C2;
    end
```

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