Hierarchy Construction and Non-Abelian Families of Generic Topological Orders

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We generalize the hierarchy construction to generic $2 + 1D$ topological orders (which can be non-Abelian) by condensing Abelian anyons in one topological order to construct a new one. We show that such construction is reversible and leads to a new equivalence relation between topological orders. We refer to the corresponding equivalence class (the orbit of the hierarchy construction) as "the non-Abelian family." Each non-Abelian family has one or a few root topological orders with the smallest number of anyon types. All the Abelian topological orders belong to the trivial non-Abelian family whose root is the trivial topological order. We show that Abelian anyons in root topological orders must be bosons or fermions with trivial mutual statistics between them. The classification of topological orders is then greatly simplified, by focusing on the roots of each family: those roots are given by non-Abelian modular extensions of representation categories of Abelian groups.

Introduction.—The ultimate dream of classifying objects in nature may be creating a "table" for them. A classic example of such a classification result is the "periodic table" for chemical elements. As for the topological ordered [1,2] phases of matter, which has drawn more and more research interest recently, we are already able to create some tables for them [3–8], via the theory of (pre-)modular categories. However, efforts are needed to further understand the tables, for example, to reveal some "periodic" structures in the table.

In the periodic table, elements are divided into several families (the columns of the table), and those in the same family have similar chemical properties. The underlying reason for this is that elements in the same family have similar outer electron structures, and only differ by "noble gas cores." The last family consists of noble gas elements, which are chemically "inert," as they have no outer electrons besides the noble gas cores. Thus the family can be considered as the equivalence class up to the inert noble gas elements.

When it comes to topological orders, we also have inert ones: the Abelian topological orders are inert, for example, in the application of topological quantum computation [9,10]. Abelian anyons cannot support nonlocal topological degeneracy, which is an essential difference from non-Abelian anyons. Is it possible to define equivalence classes for topological orders, which are up to Abelian topological orders? In this Letter, we use the hierarchy construction to establish such equivalence classes, which we call the non-Abelian families. The hierarchy construction is well known in the study of Abelian fractional quantum Hall (FQH) states [11–14]. In this Letter we generalize it to arbitrary (potentially non-Abelian) topological orders.

We show that the generalized hierarchy construction is reversible. Thus, we can say that two topological orders belong to the same non-Abelian family if they are related by the hierarchy construction. Each non-Abelian family has special root topological orders (see Table I), with the following properties: (1) Root states have the smallest rank (number of anyon types) among the non-Abelian family. (2) Abelian anyons in a root state are all bosons or fermions, and have trivial mutual statistics with each other. Since any topological order in the same non-Abelian family can be reconstructed from a root state, our work simplifies the classification of generic topological orders to the classification of root states.

Our calculation is based on quantitative characterizations of topological orders. One way to do so is to use the S, T modular matrices obtained from the non-Abelian geometric phases of degenerate ground states on torus [1,2]. We show, starting from a topological order described by S, T, how to obtain another topological order described by new S', T' via a condensation of Abelian anyons. (For a less general approach based on wave functions, see Ref. [15].) The calculation uses the theory of fusion and braiding of quasiparticles (which we call anyons) in topological order. Such a theory is the so-called "unitary modular tensor category (UMTC) theory" (for a review and much more details on the UMTC, see Ref. [5]).

A UMTC $\mathcal{C}$ is simply a set of anyons (two anyons connected by a local operator are regarded as the same type), plus data to describe their fusion and braiding. Like the fusion of two spin-1 particles giving rise to a "direct sum" of spin-0,1,2 particles: $1 \otimes 1 = 0 \oplus 1 \oplus 2$, the fusion of two anyons i and j in general gives rise to a direct sum of several other anyons: $i \otimes j = \bigoplus_k N_k^{ij} k$. So the

TABLE I. The low-rank root topological orders for bosonic systems. We pick only one root for each non-Abelian family. The rank N is the number of anyon types and c the chiral central charge of the edge states. s_i and d_i are the topological spin and quantum dimension of the type-i anyon. The anyons not in parentheses have trivial mutual statistics with all Abelian anyons. Here $\zeta_n^m = \{\sin[\pi(m+1)/(n+2)]\}/\{\sin[\pi/(n+2)]\}$. Some roots are the stacking of simpler ones, such as $4_0 = 2_{\frac{14}{5}} \boxtimes 2_{\frac{26}{5}}$.

N_c	D^2	$s_1, s_2, \cdots$	$d_1, d_2, \cdots$
1_0	1	0	1
$2_{\frac{14}{5}}$	3.618	$0, \frac{2}{5}$	$1, \zeta_3^1$
$2_{\frac{26}{5}}$	3.618	$0, \frac{3}{5}$	$1, \zeta_3^1$
$3_{\frac{5}{2}}$	4	$0, \frac{1}{2}, (\frac{5}{16})$	$1, 1, (\zeta_2^1)$
$3_{\frac{8}{7}}$	9.295	$0, \frac{6}{7}, \frac{2}{7}$	$1, \zeta_5^1, \zeta_5^2$
$3_{\frac{48}{7}}$	9.295	$0, \frac{1}{7}, \frac{5}{7}$	$1, \zeta_5^1, \zeta_5^2$
4_0	13.09	$0, \frac{2}{5}, \frac{3}{5}, 0$	$1, \zeta_3^1, \zeta_3^1, \zeta_8^2$
$4_{\frac{12}{5}}$	13.09	$0, \frac{3}{5}, \frac{3}{5}, \frac{1}{5}$	$1, \zeta_3^1, \zeta_3^1, \zeta_8^2$
$4_{\frac{28}{5}}$	13.09	$0, \frac{2}{5}, \frac{2}{5}, \frac{4}{5}$	$1, \zeta_3^1, \zeta_3^1, \zeta_8^2$
$4_{\frac{10}{3}}$	19.23	$0, \frac{1}{3}, \frac{2}{9}, \frac{2}{3}$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$
$4_{\frac{14}{3}}$	19.23	$0, \frac{2}{3}, \frac{7}{9}, \frac{1}{3}$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$
5_2	12	$0, 0, (\frac{1}{8}), (\frac{5}{8}), \frac{1}{3}$	$1, 1, (\zeta_4^1), (\zeta_4^1), 2$
5_6	12	$0, 0, (\frac{7}{8}), (\frac{3}{8}), \frac{2}{3}$	$1, 1, (\zeta_4^1), (\zeta_4^1), 2$
$5_{\frac{16}{11}}$	34.64	$0, \frac{9}{11}, \frac{2}{11}, \frac{1}{11}, \frac{6}{11}$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$
$5_{\frac{72}{11}}$	34.64	$0, \frac{2}{11}, \frac{9}{11}, \frac{10}{11}, \frac{5}{11}$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$
$5_{\frac{18}{7}}$	35.34	$0, \frac{6}{7}, \frac{6}{7}, \frac{1}{7}, \frac{3}{7}$	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^1, \zeta_{12}^4$
$5_{\frac{38}{7}}$	35.34	$0, \frac{1}{7}, \frac{1}{7}, \frac{6}{7}, \frac{4}{7}$	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$
$6_{\frac{3}{10}}$	14.47	$0, \frac{1}{2}, (\frac{11}{16}), \frac{9}{10}, \frac{2}{5}, (\frac{7}{80})$	$1, 1, (\zeta_2^1), \zeta_3^1, \zeta_3^1, (\zeta_2^1\zeta_3^1)$
$6_{\frac{77}{10}}$	14.47	$0, \frac{1}{2}, (\frac{5}{16}), \frac{1}{10}, \frac{3}{5}, (\frac{73}{80})$	$1, 1, (\zeta_2^1), \zeta_3^1, \zeta_3^1, (\zeta_2^1\zeta_3^1)$
6_0	20	$0, 0, \frac{1}{5}, \frac{4}{5}, (0), (\frac{1}{2})$	$1, 1, 2, 2, (\sqrt{5}), (\sqrt{5})$
6_4	20	$0, 0, \frac{2}{5}, \frac{3}{5}, (\frac{1}{4}), (\frac{3}{4})$	$1, 1, 2, 2, (\sqrt{5}), (\sqrt{5})$
$6_{\frac{58}{35}}$	33.63	$0, \frac{2}{5}, \frac{1}{7}, \frac{5}{7}, \frac{19}{35}, \frac{4}{35}$	$1, \zeta_3^1, \zeta_5^1, \zeta_5^2, \zeta_3^1\zeta_5^1, \zeta_5^2\zeta_3^1$
$6_{\frac{138}{35}}$	33.63	$0, \frac{2}{5}, \frac{6}{7}, \frac{2}{7}, \frac{9}{35}, \frac{24}{35}$	$1, \zeta_3^1, \zeta_5^1, \zeta_5^2, \zeta_3^1\zeta_5^1, \zeta_5^2\zeta_3^1$
$6_{\frac{142}{35}}$	33.63	$0, \frac{3}{5}, \frac{1}{7}, \frac{5}{7}, \frac{26}{35}, \frac{11}{35}$	$1, \zeta_3^1, \zeta_5^1, \zeta_5^2, \zeta_3^1\zeta_5^1, \zeta_5^2\zeta_3^1$
$6_{\frac{222}{35}}$	33.63	$0, \frac{3}{5}, \frac{6}{7}, \frac{2}{7}, \frac{16}{35}, \frac{31}{35}$	$1, \zeta_3^1, \zeta_5^1, \zeta_5^2, \zeta_3^1\zeta_5^1, \zeta_5^2\zeta_3^1$
$6_{\frac{46}{13}}$	56.74	$0, \frac{4}{13}, \frac{2}{13}, \frac{7}{13}, \frac{6}{13}, \frac{12}{13}$	$1, \zeta_{11}^1, \zeta_{11}^2, \zeta_{11}^3, \zeta_{11}^4, \zeta_{11}^5$
$6_{\frac{58}{13}}$	56.74	$0, \frac{9}{13}, \frac{11}{13}, \frac{6}{13}, \frac{7}{13}, \frac{1}{13}$	$1, \zeta_{11}^1, \zeta_{11}^2, \zeta_{11}^3, \zeta_{11}^4, \zeta_{11}^5$
$6_{\frac{8}{3}}$	74.61	$0, \frac{1}{9}, \frac{1}{9}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}$	$1, \zeta_7^3, \zeta_7^3, \zeta_7^3, \zeta_{16}^4, \zeta_{16}^6$
$6_{\frac{16}{3}}$	74.61	$0, \frac{8}{9}, \frac{8}{9}, \frac{8}{9}, \frac{2}{3}, \frac{1}{3}$	$1, \zeta_7^3, \zeta_7^3, \zeta_7^3, \zeta_{16}^4, \zeta_{16}^6$
6_2	100.6	$0, \frac{6}{7}, \frac{5}{7}, \frac{3}{7}, 0, \frac{1}{3}$	$1, \frac{3+\sqrt{21}}{2} \times 3, \frac{5+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}$
6_6	100.6	$0, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}, 0, \frac{2}{3}$	$1, \frac{3+\sqrt{21}}{2} \times 3, \frac{5+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}$

fusion of anyons is quantitatively described by a rank-3 integer tensor N_k^{ij}. From N_k^{ij}, we can determine the number of the internal degrees of freedom of an anyon, which is the so-called quantum dimension. For example, the quantum dimension of a spin-S particle is $d = 2S + 1$. For an anyon i, its quantum dimension d_i, which can be noninteger, is the largest eigenvalue of matrix N_i with $(N_i)_{kj} = N_k^{ij}$.

After knowing the fusion, the braiding of anyons can be fully determined by the fractional part of their angular

momentum L^z: $s_i = \text{mod}(L_i^z, 1)$. s_i is called the topological spin (or simply spin) of the anyon i. The last piece of data to characterize topological orders is the chiral central charge c, which is the number of right-moving edge modes minus the number of left-moving edge modes.

It turns out that two sets of data (S, T) and (N_k^{ij}, s_i) can fully determine each other,

$$T_{ij} = e^{2\pi i s_i}\delta_{ij}, \qquad S_{ij} = \sum_k e^{2\pi i(s_i+s_j-s_k)} \frac{N_k^{ij}d_k}{D},$$

$$e^{2\pi i s_i} = T_{ii}, \qquad N_k^{ij} = \sum_l \frac{S_{li}S_{lj}\overline{S_{lk}}}{S_{l1}}, \qquad (1)$$

where $D = \sqrt{\sum d_i^2}$ is the total quantum dimension.

Hierarchy construction in generic topological orders.— Let us consider an Abelian anyon condensation in a generic topological order, described by a UMTC $\mathcal{C}$. (Such a condensation in an Abelian topological order is discussed in Supplemental Material [16].) The anyons in $\mathcal{C}$ are labeled by $i, j, k, \cdots$. Let a_c be an Abelian anyon in $\mathcal{C}$ with spin s_c. We condense a_c into the Laughlin state $\Psi = \prod(z_i - z_j)^{m_c - 2s_c}$, where $m_c = $ even and $m_c - 2s_c \neq 0$ [21]. The resulting topological order is described by UMTC $\mathcal{D}$, determined by $\mathcal{C}$, a_c, and m_c.

To calculate $\mathcal{D}$, we note that the anyons in $\mathcal{D}$ are the anyons i in $\mathcal{C}$ dressed with the vortices of the Laughlin state of a_c. The vorticity is given by $m - t_i$, where m is an integer, and $2\pi t_i$ is the mutual statistics angle between anyon i and the condensing anyon a_c in the original topological order $\mathcal{C}$, which can be extracted from the S matrix $e^{-2\pi i t_i} = S_{ia_c}^{\mathcal{C}}/S_{i1}^{\mathcal{C}}$, or $t_i = s_{i \otimes a_c} - s_i - s_{a_c}$. Thus anyons in $\mathcal{D}$ are labeled by pairs $I = (i, m)$. We like to ask the following: What are the spin and fusion rules of $I = (i, m)$?

The spin of (i, m) is given by the spin of i plus the spin of the $m - t_i$ flux in the Laughlin state,

$$s_{(i,m)} = s_i + \frac{1}{2}\frac{(m - t_i)^2}{m_c - 2s_c}. \qquad (2)$$

Fusing i with $m - t_i$ flux and j with $n - t_j$ flux gives us $i \otimes j$ with $m - t_i + n - t_j$ flux,

$$(i, m) \otimes (j, n) \sim \oplus_k N_k^{ij}(k, m - t_i + n - t_j + t_k), \qquad (3)$$

where N_k^{ij} is the fusion coefficient in $\mathcal{C}$. Since a_c with $m_c - t_{a_c} = m_c - 2s_c$ flux is condensed, fusing with the (a_c, m_c) anyon does not change the anyon type in $\mathcal{D}$. So, we have an equivalence relation,

$$(i, m) \sim (i \otimes a_c, m - t_i + m_c - 2s_c + t_{i \otimes a_c}). \qquad (4)$$

The above three relations fully determine the topological order $\mathcal{D}$ [5,6].

It is important to fix a "gauge" for t_i, say by choosing $t_i \in [0, 1)$. The same label (i, m) may label different anyons under different gauge choices of t_i. Similarly, we have fixed a gauge for s_c that fixed the meaning of m_c. Note that t_{a_c} is automatically fixed when s_c is fixed: $t_{a_c} = 2s_c$, while other t_i can be freely chosen. This ensures that the equivalence relation (4) is compatible with fusion (3), where (4) is generated by fusing with the trivial anyon (a_c, m_c). The combinations $m - t_i$, $m_c - 2s_c$ determine the final spins and fusion rules; they are gauge-invariant quantities. Thus, if we change the gauge of t_i, s_c, i.e., modify them by some integers, m, m_c should be modified by the same integers to ensure that the construction remains the same.

Below we study the properties of $\mathcal{D}$ in detail. Let $M_c = m_c - 2s_c$. Applying the equivalence relation (4) q times, we obtain

$$(i, m) \sim (i \otimes a_c^{\otimes q}, m - t_i + qM_c + t_{i \otimes a_c^{\otimes q}}). \qquad (5)$$

Let q_c be the period of a_c, i.e., the smallest positive integer such that $a_c^{\otimes q_c} = \mathbf{1}$. We see that $(i, m) \sim (i, m + q_c M_c)$. Thus, we can focus on the reduced range of $m \in \{0, 1, 2, \ldots, q_c |M_c| - 1\}$. Let $|\mathcal{C}|, |\mathcal{D}|$ denote the rank of $\mathcal{C}$, $\mathcal{D}$, respectively. Now within the reduced range of m, we have $q_c |M_c||\mathcal{C}|$ different labels, and we want to show that all the orbits generated by the equivalence relation (5) have the same length, which is q_c. To see this, just note that for $0 < q < q_c$, either $i \neq i \otimes a_c^{\otimes q}$, or if $i = i \otimes a_c^{\otimes q}$, $m \neq m - t_i + qM_c + t_{i \otimes a_c^{\otimes q}} = m + qM_c$; in other words, the labels (i, m) are all different within q_c steps. It follows that the rank of $\mathcal{D}$ is $|\mathcal{D}| = |M_c||\mathcal{C}|$.

Strictly speaking, anyons in $\mathcal{D}$ should one-to-one correspond to the equivalence classes of (i, m). However, as the orbits have the same length, it would be more convenient to use (i, m) directly (as we show in Supplemental Material [16], this is the same as working in a premodular category $\tilde{\mathcal{D}}$). For example, when we need to sum over anyons in $\mathcal{D}$, we can instead do $\sum_{I \in \mathcal{D}} \rightarrow (1/q_c) \sum_{i \in \mathcal{C}} \sum_{m=0}^{q_c |M_c| - 1}$. Now we are ready to calculate other quantities of the new topological order $\mathcal{D}$. First, it is easy to see that the quantum dimensions remain the same $d_{(i,m)} = d_i$. The total quantum dimension is then

$$D_{\mathcal{D}}^2 = \frac{1}{q_c} \sum_{i \in \mathcal{C}} \sum_{m=0}^{q_c |M_c| - 1} d_{(i,m)}^2 = |M_c| D_{\mathcal{C}}^2. \qquad (6)$$

The S matrix is

$$S_{(i,m),(j,n)}^{\mathcal{D}} = \sum_k \frac{N_k^{ij}}{D_{\mathcal{D}}} d_k e^{2\pi i [s_{(i,m)} + s_{(j,n)} - s_{(k,m+n+t_k - t_i - t_j)}]}$$

$$= \frac{1}{\sqrt{|M_c|}} S_{ij}^{\mathcal{C}} e^{-2\pi i \frac{(m - t_i)(n - t_j)}{M_c}}. \qquad (7)$$

It is straightforward to check that $S_{(i,m),(j,n)}^{\mathcal{D}}$ is unitary [with respect to equivalence classes of (i, m)]. Moreover, this formula for S can recover the equivalence relation (5) and fusion rules (3) via unitarity and the Verlinde formula.

The new $S^{\mathcal{D}}$, $T^{\mathcal{D}}$ matrices [$T^{\mathcal{D}}$ matrix is determined by the spin of anyons $s_{(i,m)}^{\mathcal{D}}$ in (2)], as well as $S^{\mathcal{C}}$, $T^{\mathcal{C}}$, should obey the modular relations $STS = e^{2\pi i \frac{c}{8}} T^\dagger S T^\dagger$, from which we can extract the central charge of $\mathcal{D}$. The new central charge is found to be (see Supplemental Material [16])

$$c^{\mathcal{D}} = c^{\mathcal{C}} + \mathrm{sgn}(M_c). \qquad (8)$$

Clearly, the one-step hierarchy construction described by (2), (7), and (8) is fully determined by an Abelian anyon a_c and M_c, where $M_c + 2s_c$ is an even integer. In Supplemental Material [16], we discuss the above hierarchy construction more rigorously at the full categorical level.

As an application, let us explain the "eightfold way" observed in the table of topological orders [5,6]: whenever there is a fermionic quasiparticle, the topological order has eight companions with the same rank and quantum dimensions but different spins and central charges. If we apply the one-step condensation with a_c being a fermion, and $M_c = \pm 1$, a new topological order of the same rank is obtained [22]. The spins of the anyons carrying fermion parity flux (having nontrivial mutual statistics with the fermion a_c) are shifted by $\pm 1/8$, and the central charge is shifted by ± 1, while all the quantum dimensions remain the same. If we repeat it eight times, we go back to the original state (up to an E_8 state), generating the eightfold way.

Reverse construction and non-Abelian families.—The one-step condensation from $\mathcal{C}$ to $\mathcal{D}$ is always reversible. In $\mathcal{D}$, choosing $a_c' = (\mathbf{1}, 1)$, $s_c' = 1/2M_c$, $m_c' = 0$, $M_c' = -1/M_c$, and repeating the construction, we go back to $\mathcal{C}$. One may first perform the construction for a premodular $\tilde{\mathcal{D}}$ and then reduce the resulting category to a modular category. We find that the mutual statistics between (i, m) and $a_c' = (\mathbf{1}, 1)$ is $t_{(i,m)}' = (m - t_i)/M_c$. Let (i, m, p), (j, n, q) label the anyons after the above one-step condensation; the new S matrix is

$$S_{(i,m,p),(j,n,q)} = S_{ij}^{\mathcal{C}} e^{-[2\pi i (m - t_i)(n - t_j)/M_c]} e^{-[2\pi i (p - t_{(i,m)}')(q - t_{(j,n)}')/M_c']}$$

$$= S_{ij}^{\mathcal{C}} e^{2\pi i (t_i q + t_j p - t_{a_c} pq)} = S_{i \otimes \bar{a}_c^{\otimes p}, j \otimes \bar{a}_c^{\otimes q}}^{\mathcal{C}}, \qquad (9)$$

which means that we can identify (i, m, p) with $i \otimes \bar{a}_c^{\otimes p}$ ($\bar{a}_c$ denotes the antiparticle of a_c). It is easy to check that they have the same spin $s_{(i,m,p)} = s_{i \otimes \bar{a}_c^{\otimes p}}$. Therefore, $i \sim (i \otimes a_c^{\otimes p}, m, p)$, $\forall m, p$; we have come back to the original state $\mathcal{C}$. Therefore, generic hierarchy constructions are reversible, which defines an equivalence relation between topological orders. We call the corresponding equivalence classes the non-Abelian families.

Now we examine the important quantity $M_c = m_c - 2s_c$, which relates the ranks before and after the one-step condensation, $|\mathcal{D}| = |M_c||\mathcal{C}|$. Since m_c is a freely chosen even integer, when a_c is not a boson or fermion ($s_c \neq 0$ or $1/2$ mod 1), we can always make $0 < |M_c| < 1$, which means that the rank is reduced after one-step condensation. We then have the first important conclusion: Each non-Abelian family has root topological orders with the smallest rank; the Abelian anyons in the root states are all bosons or fermions.

We can further show that the Abelian bosons or fermions in the root states have trivial mutual statistics among them. To see this, assuming that a, b are Abelian anyons in a root state, since the mutual statistics is given by $DS_{ab} = \exp[2\pi i(s_a + s_b - s_{a\otimes b})]$, and $a, b, a \otimes b$ are all bosons or fermions, nontrivial mutual statistics can only be $DS_{ab} = -1$. Now consider two cases: (1) one of a, b, say a, is a fermion, then by condensing a (choosing $a_c = a$, $m_c = 2$, $s_c = 1/2$, $t_b = 1/2$), in the new topological order, the rank remains the same but $s_{(b,0)} = s_b + (t_b^2/2M_c) = s_b + 1/8$, which means $(b,0)$ is an Abelian anyon but neither a boson nor a fermion. By condensing $(b,0)$ again we can reduce the rank, which conflicts with the root state assumption. (2) a, b are all bosons. Still we condense a with $m_c = 2$, $s_c = 0$, $t_b = 1/2$. In the new topological order the rank is doubled but $s_{(b,0)} = s_b + (t_b^2/2M_c) = 1/16$, which means further condensing $(b,0)$ with $m_c' = 0$ the rank is reduced to $1/8$, which is again, smaller than the rank of the beginning root state, thus contradictory.

Therefore, in the root states, Abelian anyons are bosons or fermions with trivial mutual statistics. We also have a straightforward corollary: all Abelian topological orders have the same unique root state, which is the trivial topological order. In other words, all Abelian topological orders are in the same trivial non-Abelian family, which resembles the noble gas family in the periodic table. Thus, the non-Abelian families are indeed equivalence classes up to Abelian topological orders.

To easily determine if two states belong to the same non-Abelian family, it is very helpful to introduce some non-Abelian invariants. One is the fractional part of the central charge. Since the one-step condensation changes the central charge by $\text{sgn}(M_c)$ [see (8)], we know that central charges in the same non-Abelian family can only differ by integers. Another invariant is the quantum dimension. It is not hard to check that, in the one-step condensation, the number of anyons with the same quantum dimension d, denoted by $N(d_i = d)$, is also multiplied by $|M_c|$. So the ratio of $N(d_i = d)$ to the rank is also a constant. The third invariant is a bit involved. Note that in the one-step condensation, if i has trivial mutual statistics with a_c, $t_i = 0$, then $(i,0)$ in $\mathcal{D}$ have the same spin as i in $\mathcal{C}$ and the same mutual statistics with $(j,m), \forall m$ as that between i and j in $\mathcal{C}$. Therefore, the centralizer of Abelian anyons, namely, the subset of anyons that have trivial mutual statistics with all Abelian anyons

(the anyons not in parentheses in Table I), is the same within one non-Abelian family. These facts enable us to quickly tell that two states are not in the same non-Abelian family.

Examples: Realizations of non-Abelian FQH states were first proposed in Refs. [17] and [18]. One of them is [17,19] $\Psi_{\nu=1}(\{z_i\}) = [\chi_2(\{z_i\})]^2$, where $\chi_k(\{z_i\})$ is the many-fermion wave function with k filled Landau levels. The bulk effective theory is the $SU(2)^f_{-2}$ Chern-Simons (CS) theory with three types of anyons and the edge has $c = 5/2$ (see Supplemental Material [16]). So the state is one of the root states $N_c = 3_{\frac{5}{2}}$ in Table I. Another bosonic non-Abelian FQH liquid at $\nu = 1$ is [18] $\Psi_{\nu=1} = \mathcal{A}[(1/z_1 - z_2)(1/z_2 - z_3)\cdots]\prod(z_i - z_j)$, whose edge has $c = 3/2$. It is the state described by $N_c = 3_{\frac{3}{2}}$, which belongs to the same non-Abelian family as the $3_{\frac{5}{2}}$ state above. The experimentally realized $\nu = 5/2$ FQH state is likely to belong to this non-Abelian family [23–25].

A more interesting non-Abelian state (which can perform universal topological quantum computation [26]) is $\Psi_{\nu=\frac{3}{2}}(\{z_i\}) = [\chi_3(\{z_i\})]^2$, whose edge has $c = \frac{21}{5}$. The bulk effective theory is the $SU(2)^f_{-3}$ CS theory with four types of anyons [17,19]. So the state is $N_c = 4_{\frac{21}{5}}$, which belongs to the same non-Abelian family as the state $2_{\frac{26}{5}}$ in Table I (see Supplemental Material [16], which contains more examples of non-Abelian states and non-Abelian families).

We remark that the topological orders studied in this Letter do not require and do not have any symmetry. However, some $c = 0$ topological orders with a Z_2 automorphism $i \rightarrow i'$ that changes the sign of spins $s_i = -s_{i'}$ can be realized by time-reversal symmetric states [27].

Conclusion and outlook.—In this Letter we introduced the hierarchy construction in generic topological orders, which established a new equivalence relation: Two topological orders related by the hierarchy construction belong to the same non-Abelian family. This reveals intriguing new structures in the classification of topological orders. Non-Abelian families are equivalence classes up to Abelian topological orders. Topological orders in the same non-Abelian family share some properties, such as quantum dimensions and the fractional part of central charges.

In particular, we studied the root states, the states in a non-Abelian family with the smallest rank. Other states can be constructed from the root states via the hierarchy construction. Thus, classifying all topological orders is the same as classifying all root states, namely, all states such that their Abelian anyons have trivial mutual statistics. In other words, we can try to generate all possible topological orders by constructing all the root states, which can be obtained by starting with an Abelian group G, extending its representation category $\text{Rep}(G)$ or $s\text{Rep}(G^f)$ to a modular category [7,8] while requiring all the extra anyons to be non-Abelian (which is referred to as a non-Abelian modular extension). This is a promising

future problem and may be an efficient way to produce tables of topological orders.

Although in this Letter we focused on bosonic topological orders with no symmetry (described by modular categories), the construction also applies to bosonic or fermionic topological orders with any finite on-site symmetry (described by certain premodular categories) [7,8]. The same argument goes for non-Abelian families and root states with symmetries.

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