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Lower bounds on Ricci curvature and quantitative behavior of singular sets

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Abstract Let Y^n denote the Gromov-Hausdorff limit $M_i^n \xrightarrow{d_{\mathrm{GH}}} Y^n$ of v-noncollapsed Riemannian manifolds with $\mathrm{Ric}_{M_i^n} \geq -(n-1)$. The singular set $\mathcal{S} \subset Y$ has a stratification $\mathcal{S}^0 \subset \mathcal{S}^1 \subset \cdots \subset \mathcal{S}$, where $y \in \mathcal{S}^k$ if no tangent cone at y splits off a factor $\mathbb{R}^{k+1}$ isometrically. Here, we define for all $\eta > 0$, $0 < r \leq 1$, the *k-th effective singular stratum* $\mathcal{S}^k_{\eta,r}$ satisfying $\bigcup_\eta \bigcap_r \mathcal{S}^k_{\eta,r} = \mathcal{S}^k$. Sharpening the known Hausdorff dimension bound $\dim \mathcal{S}^k \leq k$, we prove that for all y, the volume of the r-tubular neighborhood of $\mathcal{S}^k_{\eta,r}$ satisfies $\mathrm{Vol}(T_r(\mathcal{S}^k_{\eta,r}) \cap B_{\frac{1}{2}}(y)) \leq c(n, \mathrm{v}, \eta) r^{n-k-\eta}$. The proof involves a *quantitative differentiation* argument. This result has applications to Einstein manifolds. Let $\mathcal{B}_r$ denote the set of points at which the C^2-harmonic radius is $\leq r$. If also the M_i^n are Kähler-Einstein with L_2 curvature bound, $\|Rm\|_{L_2} \leq C$, then $\mathrm{Vol}(\mathcal{B}_r \cap B_{\frac{1}{2}}(y)) \leq c(n, \mathrm{v}, C) r^4$ for all y. In the Kähler-Einstein case, without assuming any integral curvature bound on the M_i^n, we obtain a slightly weaker volume bound on $\mathcal{B}_r$ which yields an a priori L_p curvature bound for all $p < 2$. The methodology developed in this paper is new and is applicable in many other contexts. These include harmonic maps, minimal hypersurfaces, mean curvature flow and critical sets of solutions to elliptic equations.

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1 Volume bounds for quantitative singular sets

Let (M^n, g) denote a Riemannian manifold whose Ricci curvature satisfies

$$\mathrm{Ric}_{M^n} \geq -(n-1)g. \tag{1.1}$$

Let $\mathrm{Vol}_{-1}(r)$ denote the volume of a ball of radius r in n-dimensional hyperbolic space of curvature $\equiv -1$. We will assume M^n is v-noncollapsed i.e. for all $x \in M^n$,

$$\frac{\mathrm{Vol}(B_1(x))}{\mathrm{Vol}_{-1}(1)} \geq \mathrm{v} > 0. \tag{1.2}$$

Let

$$M_i^n \xrightarrow{d_{\mathrm{GH}}} Y^n, \tag{1.3}$$

denote the Gromov-Hausdorff limit (possibly in the pointed sense) of a sequence of manifolds M_i^n satisfying (1.1), (1.2). In this case, the measured Gromov-Hausdorff limit of the Riemannian measures on the M_i^n is n-dimensional Hausdorff measure on Y^n. We will simply denote it by $\mathrm{Vol}(\,\cdot\,)$.

Relations (1.1)–(1.3) will be in force throughout the paper.

For $y \in Y^n$, every tangent cone Y_y is a metric cone $C(Z)$ with cross-section Z and vertex z^*. A point y is called *regular* if one (equivalently, every) tangent cone is isometric to $\mathbb{R}^n$. Otherwise y is called *singular*. The set of singular points is denoted by $\mathcal{S}$. The stratum $\mathcal{S}^k \subset \mathcal{S}$ is defined as the set of points for which no tangent cone splits off isometrically a factor $\mathbb{R}^{k+1}$. In fact, $\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2} = \emptyset$. Thus,

$$\mathcal{S}^0 \subset \mathcal{S}^1 \subset \cdots \subset \mathcal{S}^{n-2} = \mathcal{S}. \tag{1.4}$$

Moreover, in the sense of Hausdorff dimension, we have

$$\dim \mathcal{S}^k \leq k. \tag{1.5}$$

For the all of the above, see [3].

Given $\eta > 0$, $0 < r < 1$, we will define a quantitative version $\mathcal{S}^k_{\eta,r}$ of the singular stratum $\mathcal{S}^k$. The criterion for membership of $y \in Y^n$ in $\mathcal{S}^k_{\eta,r}$ involves the behavior of $B_s(y)$ for all $r \leq s \leq 1$. We will show that $\mathrm{Vol}(\mathcal{S}^k_{\eta,r}) \leq c(n, \mathrm{v}, \eta) r^{n-k-\eta}$; see Theorem 1.3.

Remark 1.1 In the special case, $Y^n = M^n$, with M^n smooth, the sets $\mathcal{S}^k$ are empty for all k. However, the sets $\mathcal{S}^k_{\eta,r}$ need not be empty. In fact, in the proofs of all of the estimates stated in this section, we can (and will) restrict attention to the case of smooth manifolds. Since the measure on Y^n is the limit of the Riemannian measures on the M_i^n, once proved for smooth manifolds, the estimates pass immediately to Gromov-Hausdorff limit spaces.

Denote by $(\underline{0}, z^*)$, a vertex of the metric cone with isometric splitting $\mathrm{R}^{k+1} \times C(Z)$.

Definition 1.2 For $\eta > 0$ and $0 < r < 1$, define the *kth effective singular stratum* $\mathcal{S}^k_{\eta,r} \subseteq Y^n$ by

$$\mathcal{S}^k_{\eta,r} := \big\{ y \mid d_{\mathrm{GH}}\big(B_s(y), B_s\big((\underline{0}, z^*)\big)\big) \geq \eta s, \text{ for all } \mathbb{R}^{k+1} \times C(Z) \\ \text{and all } r \leq s \leq 1 \big\}.$$

It follows directly from the definition that

$$\mathcal{S}^k_{\eta,r} \subset \mathcal{S}^{k'}_{\eta',r'} \quad \big(\text{if } k' \leq k,\ \eta' \leq \eta,\ r \leq r'\big). \tag{1.6}$$

Also, if $y \in \mathcal{S}^k$, then clearly, $y \in \bigcap_r \mathcal{S}^k_{\eta,r}$, for some $\eta > 0$, so

$$\mathcal{S}^k = \bigcup_\eta \bigcap_r \mathcal{S}^k_{\eta,r}. \tag{1.7}$$

Our first main result is a volume bound for $\mathcal{S}^k_{\eta,r}$. The proof will proceed by appropriately bounding the number of balls of radius r needed to cover $\mathcal{S}^k_{\eta,r} \cap B_1(\underline{x})$. Since by volume comparison, we have $\mathrm{Vol}(B_r(x)) \leq c(n)r^n$, so this will suffice.

Theorem 1.3 *There exists* $c(n, \mathrm{v}, \eta) > 0$ *such that if* $M^n_i \xrightarrow{d_{\mathrm{GH}}} Y^n$, *and the* (M^n_i, g_i) *satisfy the lower Ricci curvature bound* (1.1), *and* v*-noncollapsing condition* (1.2), *then for all* $y \in Y^n$ *and* $\eta > 0$,

$$\mathrm{Vol}\big(\mathcal{S}^k_{\eta,r} \cap B_{\frac{1}{2}}(y)\big) \leq c(n, \mathrm{v}, \eta) r^{n-k-\eta}. \tag{1.8}$$

Remark 1.4 It is an easy consequence of the definition of $\mathcal{S}^k_{\eta,r}$, that the bound in (1.8) actually implies (for a slightly different constant $c(n, \mathrm{v}, \eta)$)

$$\mathrm{Vol}\big(T_r\big(\mathcal{S}^k_{\eta,r}\big) \cap B_{\frac{1}{2}}(y)\big) \leq c(n, \mathrm{v}, \eta) r^{n-k-\eta}, \tag{1.9}$$

where $T_r(\mathcal{S}^k_{\eta,r})$ denotes the r-tubular neighborhood.

Remark 1.5 There is a possibility that on the right-hand side of (1.8), the factor $r^{n-k-\eta}$ can be replaced by one of the form $r^{n-k}(\log r)^{c(n,\mathrm{v},\eta)}$. However, it seems unlikely that in general, the appearance of η on the right-hand side of (1.8) can be entirely removed. In the application to Kähler-Einstein manifolds given in Theorem 1.14, this is of no consequence since the bound in (1.8) controls a lower order term; compare (1.18).

Remark 1.6 As will be indicated in Sects. 2 and 3, the proof of Theorem 1.3 employs an instance of *quantitative differentiation* in the sense of Sect. 14 of [8].

Definition 1.7 If $y \in Y^n$ and the metric is not C^2 in some neighborhood of y, then $r_{har}(y) = 0$. Otherwise, $r_{har}(x)$ is the supremum of those r such that the ball $B_r(y)$ is contained in the domain of a harmonic coordinate system such that $g_{ij}(0) = \delta_{ij}$ and

$$|g_{ij}|_{C^1} \le r^{-1}, \qquad |g_{ij}|_{C^2} \le r^{-2}.$$

Recall that by elliptic regularity, there exist constants, $c(n,k)$ such that if M^n is Einstein, then for g_{ij} as above, we have

$$|g_{ij}|_{C^k} \le c(n,k) r_{har}^{-k}.$$

Also, the curvature tensor Rm satisfies

$$\sup_{B_{r_{har}(y)}(y)} |Rm| \le c(n) r_{har}^{-2}. \tag{1.10}$$

Put

$$\mathcal{B}_r = \big\{ y \mid r_{har}(y) \le r \big\}. \tag{1.11}$$

Remark 1.8 Let $\widetilde{\mathcal{B}}_r \subset Y^n$ denote the set of points such that either $r_0(y) = 0$ or $|Rm(y)| \ge c(n) r^{-2}$. In particular, $\widetilde{\mathcal{B}}_r \subset \mathcal{B}_r$. Since r_{har} is 1-Lipschitz, it follows that

$$T_r(\widetilde{\mathcal{B}}_r) \subseteq T_r(\mathcal{B}_r) \subseteq \mathcal{B}_{2r}\,. \tag{1.12}$$

Hence estimates on $\mathrm{Vol}(\mathcal{B}_r)$ imply estimates on $\mathrm{Vol}(T_r(\widetilde{\mathcal{B}}_r))$. Equivalently, in view of the noncollapsing assumption (1.2), estimates on $\mathrm{Vol}(\mathcal{B}_r)$ imply on the covering number of $\widetilde{\mathcal{B}}_r$.

Under the additional assumption that the M_i^n are Einstein and (for some of our results) satisfy an integral curvature bound, we will apply Theorem 1.3 in combination with ϵ-regularity theorems to control the volume of the set $\mathcal{B}_r$. In this case, we replace (1.1) by the 2-sided bound

$$|\mathrm{Ric}_{M_i^n}| \le n - 1. \tag{1.13}$$

Our first result of this type follows by combining Theorem 1.3 with the ϵ-regularity theorems, Theorem 6.2 of [3] and Theorem 5.2 of [7];[1] the detailed argument is given in Sect. 5.

Theorem 1.9 *There exists $\eta_0 = \eta_0(n, \mathrm{v}) > 0$ such that if $M_i^n \xrightarrow{d_{\mathrm{GH}}} Y^n$, and the M_i^n are Einstein manifolds satisfying the* v*-noncollapsing condition* (1.2), *and the Ricci curvature bound* (1.13), *then for every* $0 < r < 1$:

[1]The latter is due independently to G. Tian.

1. *If $\eta < \eta_0$, then we have $\mathcal{B}_r \subset \mathcal{S}^{n-2}_{\eta,r}$. In particular, for all $y \in Y^n$,*

$$\mathrm{Vol}\big(\mathcal{B}_r \cap B_{\frac{1}{2}}(y)\big) \le c(n, \mathrm{v}, \eta) r^{2-\eta}. \tag{1.14}$$

2. *If in addition, the M^n_i are Kähler, then we have $\mathcal{B}_r \subset \mathcal{S}^{n-4}_{\eta,r}$. In particular, for all $y \in Y^n$,*

$$\mathrm{Vol}\big(\mathcal{B}_r \cap B_{\frac{1}{2}}(y)\big) \le c(n, \mathrm{v}, \eta) r^{4-\eta}. \tag{1.15}$$

Remark 1.10 Conjecturally, in item 2. above, the Kähler assumption can be dropped.

Corollary 1.11 *Let Y^n be as in Theorem 1.9. Then:*

1. *In case* 1. *of Theorem* 1.9, *for every* $0 < p < 1$,

$$⨍_{B_{\frac{1}{2}}(y)} |Rm|^p \le c(n) \cdot ⨍_{B_{\frac{1}{2}}(y)} (r_{har})^{-2p} < c(n, \mathrm{v}, p) \quad (\textit{for all } p < 1).$$

2. *In case* 2. *of Theorem* 1.9, *for every* $0 < p < 2$,

$$⨍_{B_{\frac{1}{2}}(y)} |Rm|^p \le c(n) \cdot ⨍_{B_{\frac{1}{2}}(y)} (r_{har})^{-2p} < c(n, \mathrm{v}, p) \quad (\textit{for all } p < 2).$$

Remark 1.12 Theorem 1.9 and Corollary 1.11 remain true assuming the Ricci curvature bound (1.13) and a bound on $|\nabla \mathrm{Ric}^i_{M^n}|$. Alternatively, If the C^2-harmonic radius is replaced by the $C^{1,\alpha}$-harmonic radius, then Theorem 1.9 and Corollary 1.11 hold with only the Ricci curvature bound, $|\mathrm{Ric}_{M^n_i}| \le n-1$.

Remark 1.13 Even if $\mathcal{B}_r$ were replaced by the smaller set $\widetilde{\mathcal{B}}_r$, the assertions of Corollary 1.11 would be new.

In our next result (whose proof will be given in Sect. 6) we assume in addition, the L_p curvature bound

$$⨍_{B_1(x)} |Rm|^p \le C. \tag{1.16}$$

Recall in this connection, that for Kähler-Einstein manifolds, we have the topological L_2 curvature bound

$$\int_{M^n} |Rm|^2 \le c(n) \cdot \Big(\big|\big(c_1^2 \cup [\omega]^{(n/2)-2}\big)\big(M^n\big)\big| + \big|\big(c_2 \cup [\omega]^{(n/2)-2}\big)\big(M^n\big)\big|\Big), \tag{1.17}$$

where c_1, c_2 denote the first and second Chern classes and $[\omega]$ denotes the Kähler class; see e.g. [4] and compare also the L_p bound ($p < 2$) in item 2. of Corollary 1.11, which holds without assuming a bound on the right-hand side of (1.17).

Theorem 1.14 [2] *Let the assumptions be as in Theorem 1.9 and assume in addition that the M_i^n are Kähler-Einstein and satisfy the L_p curvature bound* (1.16)*, for some integer p, with $2 \le p \le \frac{n}{2}$. Then for every $0 < r < 1$,*

$$\operatorname{Vol}\big(\mathcal{B}_r \cap B_{\frac{1}{2}}(y)\big) \le c(n, \mathrm{v}, C) r^{2p}. \tag{1.18}$$

In particular, if the right-hand side of (1.17) *is bounded by C, then*

$$\operatorname{Vol}\big(\mathcal{B}_r \cap B_{\frac{1}{2}}(y)\big) \le c(n, \mathrm{v}, C) r^{4}. \tag{1.19}$$

It is of key importance that η does not appear on the right-hand side of (1.18), (1.19); compare (1.5), (1.8), (1.14), (1.15). Let us indicate how this comes about.

Note that the estimates in (1.18), (1.19), strengthen the known bounds on the Hausdorff measure $\mathcal{H}^{n-2p}(\mathcal{S}^{n-2p})$ which in particular is finite; see [4, 7]. Those bounds are obtained by combining standard maximal function estimates for the L_p norm of the curvature with the certain ϵ-regularity theorems to estimate $\mathcal{H}^{n-2p}(\mathcal{S}^{n-2p} \setminus \mathcal{S}^{n-2p-1})$, and then using (1.5), $\dim \mathcal{S}^{n-2p-1} \le n-2p-1$, which implies $\mathcal{H}^{n-2p}(\mathcal{S}^{n-2p-1}) = 0$.

In fact, a slight modification of the first part of the argument gives the leading term on the right-hand side of (1.19), whereas the terms controlled by Theorem 1.3, which are lower order, can be (and are) suppressed. The bound on these terms (which requires the hypothesis of Theorem 1.14) can be viewed as strengthened version of the estimate $\mathcal{H}^{n-2p}(\mathcal{S}^{n-2p-1}) = 0$.

More specifically, Theorem 1.3 is only used to control the volumes of certain subsets of $\mathcal{S}^{n-2p-1}_{\eta_0,\gamma^{-i}}$, where $r \le \gamma^{-i} \le 1$, $1 > \eta_0 = \eta_0(n) > 0$ is sufficiently small and $\gamma = \gamma(\eta_0)$. (The precise meaning of "sufficiently small" is

[2]The results of the present paper arose in the course of our ongoing investigations concerning the structure of Gromov-Hausdorff limit spaces with Ricci curvature bounded below and in particular, on the structure of the singular set for limits of Einstein manifolds. On the other hand, it has come to our attention that Theorem 1.14 and Remark 1.17 below are stated as conjectures (Hypothesis V and Supplements) in an informal document "Discussion of the Kähler-Einstein problem" written by S. Donaldson in the summer of 2009, available at http://www2.imperial.ac.uk/~skdona/KENOTES.PDF. It was announced there that the complex dimension 3 case of Theorem 1.14 would be treated in a forthcoming paper of Donaldson and X. Chen; see [6]. The general case is treated in [5]. Their work, like ours, makes use of [2–4]. Unlike ours, it utilizes essentially a rigidity result for almost complex structures; see [10].

dictated by the constant in the ϵ-regularity theorem of Sect. 5 of [7].) For instance, in the extreme case in which $\gamma^{-(i+1)} \le r$, we have $\mathrm{Vol}(\mathcal{S}^{n-2p-1}_{\eta_0,r}) \le c(n, \mathrm{v}, \eta_0) r^{2p+1-\eta_0}$ and the sum of the remaining terms satisfies a bound of the same form. Since $2p + 1 - \eta_0 > 2p$, the volume bound on these terms can be suppressed.

Remark 1.15 In the proof of Theorem 1.9 by contrast, Theorem 1.3 is used to control the highest order term.

Remark 1.16 It is possible that Theorem 1.14 holds for Einstein manifolds which are not necessarily Kähler. In any case, if p is an even integer, then apart from some exceptional cases, the ϵ-regularity theorems of Sect. 8 can be used to show that (1.18) holds. For p an integer, using Sect. 4 of [7], one gets (with no exceptional cases and all $\eta > 0$) the less sharp estimate

$$\mathrm{Vol}\big(\mathcal{B}_r \cap B_{\frac{1}{2}}(y)\big) \le c(n, \mathrm{v}, \eta, C) r^{2p-\eta}.$$

Remark 1.17 Among the connected components of $B_{\frac{1}{2}}(y) \setminus \mathcal{B}_r$, there is a component $\hat{A}_r$, such that

$$\mathrm{Vol}\big(B_{\frac{1}{2}}(y) \setminus \hat{A}_r\big) \le c(n, \mathrm{v}, C) r^{\frac{(2p-1)n}{n-1}}. \tag{1.20}$$

To see this note that as previously mentioned, $B_{\frac{1}{2}}(y) \setminus \mathcal{B}_r \subset \mathcal{C}_r$, for some subset $\mathcal{C}_r$ which is the union of at most $c(n, \mathrm{v}, C) r^{-2p}$ balls of radius r. Moreover, for $r = r_0(n, \mathrm{v}, \eta)$ sufficiently small, there exists $B_{r_0}(y') \subset (B_{\frac{1}{2}}(y) \setminus \mathcal{C}_{r_0})$. For $r \le r_0$, let $A_r \subset (B_{\frac{1}{2}}(y) \setminus \mathcal{C}_r)$ denote the component containing $B_{r_0}(y')$. Clearly, $\mathrm{Vol}_{n-1}(\partial \mathcal{C}_r) \le c(n, \mathrm{v}, C) r^{2p-1}$ and in particular, $\mathrm{Vol}_{n-1}(\partial A_r) \le c(n, \mathrm{v}, C) r^{2p-1}$. Since $\mathrm{Vol}(A_r) \ge \mathrm{Vol}(B_{r_0}(y'))$ (a definite lower bound) the isoperimetric inequality for manifolds satisfying (1.1), (1.2), gives $\mathrm{Vol}(B_{\frac{1}{2}}(y) \setminus A_r) \le c(n, \mathrm{v}, C) r^{\frac{(2p-1)n}{n-1}}$. This implies (1.20).

Remark 1.18 As briefly indicated in the discussion following the statement of Theorem 1.14, in proving that theorem, the ϵ-regularity theorem must be applied on all scales between 1 and r. Here, the fact that the hypothesis of the relevant ϵ-regularity requires that two distinct conditions must be satisfied *simultaneously* raises an issue that does not arise in the proof of Theorem 1.9; for details, see Sect. 6.

2 Outline of the proof of Theorem 1.3

As mentioned in abstract, the methodology developed in this paper is new and is applicable in many other contexts. These include harmonic maps and minimal hypersurfaces ([9]), mean curvature flow and critical sets of solutions to elliptic equations. In the present section, we will give an informal explanation of the main ideas.

To prove Theorem 1.3, we exhibit $\mathcal{S}^k_{\eta,r}$ as a generalized Cantor set. In particular, we show that at most locations and scales $\geq r$, there exists $\ell \leq k$, such that $\mathcal{S}^k_{\eta,r}$ lies very close to a k-dimensional subset of the form $\mathrm{R}^\ell \times \{z^*\}$, where R^ℓ is a factor of an approximate local isometric splitting. Once this has been done, the volume computation is an essentially standard induction argument based on iterated ball coverings.

The following toy example illustrates our approach in highly simplified situation corresponding to the case $\mathcal{S}^0$. Notably, a significant issue which must be addressed in the actual situation is not present in the toy example; see the subsection below entitled "Implementation of cone-splitting".

Start with the interval $[0, 1]$ (so in effect, we are pretending that $n = 1$, although this plays no essential role). Remove a subinterval from the center, then remove central subintervals from each of the two remaining subintervals, etc. Fix $\eta > 0$. We chose the lengths of the 2^i distinct subintervals which remain at the i-th stage to be $r_i = t_1 \cdots t_i$, where we assume that for some $i(\eta) < \infty$, we have $t_i^\eta \leq \frac{1}{2}$ for all $j > i(\eta)$. Denote the generalized Cantor set which is intersection of this sequence of subsets by C. The following volume estimate strengthens the Hausdorff dimension estimate $\dim C \leq \eta$.

Set $\max_{i \leq i(\eta)} 2^i r_i^\eta = c(\eta)$. Then $2^j r_j^\eta \leq c(\eta)$ for any $j \geq i(\eta)$. For all j, we have

$$\begin{aligned}\mathrm{Vol}\big(T_{r_j}(C)\big) &\leq 2^j \cdot r_j \\ &\leq \big(2^j \cdot r_j^\eta\big) \cdot r_j^{1-\eta} \\ &\leq c(\eta) r_j^{1-\eta},\end{aligned}$$

which easily implies the same estimate with r_j replaced by any $r \leq 1$ and $c(\eta)$ replaced by $2 \cdot c(\eta)$.

The inequality $\dim \mathcal{S}^k \leq k$**.**

Next we recall from [3], the proof of the inequality $\dim \mathcal{S}^k \leq k$. The proof relies on an iterated blow up argument. The following geometric facts are used. (i) For limit spaces satisfying (1.1), (1.3), every tangent cone Y_y is a metric cone. (ii) The splitting theorem holds for such tangent cones.

Consider first the case $k = 0$. By a density argument, if $\dim \mathcal{S}^0 = 0$ were to fail, it would already fail for some tangent cone Y_y for which the vertex y^*_∞, is a density point of $\mathcal{S}^0(Y_y)$. Thus, there would exist $y'_\infty \in \mathcal{S}^0(Y_y)$, a density point of $\mathcal{S}^0(Y_y)$, with $y'_\infty \neq y^*_\infty$. Moreover, by the same reasoning, the assertion would fail in the same way for some tangent cone $(Y_y)_{y'_\infty}$ at

y'_∞. But since $y'_\infty \neq y^*_\infty$, by (i), y'_∞ is an interior point of a ray emanating from y^*_∞. After blow up at y'_∞, we obtain a line in $(Y_y)_{y'_\infty}$ and a density point $(y'_\infty)_\infty$ of $\mathcal{S}_0$ lying on this line. By (ii), this line splits off isometrically, which contradicts $(y'_\infty)_\infty \in \mathcal{S}_0$. Similarly, by employing additional blow ups, one gets dim $\mathcal{S}^k \leq k$ for all k; for further details, see [3].

An issue involving multiple scales.

Proving Theorem 1.3 requires either finding a quantitative version of the preceding (noneffective) blow up argument, or finding a different argument which can in fact be made quantitative. It is natural to investigate the following idea for quantifying the blow up argument: Rather than doing multiple blow ups to split off additional lines as isometric factors, do an "appropriate" sequence of rescalings which stop short of going to the blow up the limit. The difficulty is that this leads to a sequence balls whose radii decrease very rapidly and the resulting issue of having to work simultaneously on a sequence of different scales. In fact, it is not clear to us how to resolve the quantitative issues which arise from this approach.

Instead of blow up we use a different principle, the "cone-splitting principle". When its hypotheses are satisfied, the cone-splitting gives rise to an "additional splitting" of a single cone on a fixed scale. We show that in our context, the hypotheses are indeed satisfied at most locations and scales. In particular, this gives a new proof that dim $\mathcal{S}^k \leq k$ (though of course, the quantitative version that we actually prove is much stronger).

Cone-splitting, a replacement for blow up.

In its nonquantitative form, the cone-splitting principle gives a criterion which guarantees that a metric cone $\mathrm{R}^\ell \times C(Z)$, which splits off a Euclidean factor R^ℓ, actually splits off a factor of $\mathrm{R}^{\ell+1}$. (Here and in the next paragraph, all splittings are isometric and $C(Z)$ denotes a metric cone with vertex z^*.)

Cone-splitting Suppose that for some $C(\underline{Z})$ with vertex $\underline{z}^*$, there is an isometry $I : \mathrm{R}^\ell \times C(Z) \to C(\underline{Z})$ such that $\underline{z}^* \notin I(\mathrm{R}^\ell \times \{z^*\})$. Then for some W, $\mathrm{R}^\ell \times C(Z)$ is isometric to $R^{\ell+1} \times C(W)$.[3]

To see the relevance, note that in the proof of dim $\mathcal{S}^0 = 0$ which was recalled above, if we knew that $y'_\infty \neq y$ was the vertex of some other cone structure on Y_y, then $Y_y \equiv \mathbb{R} \times Y'_y$. Thus, we would obtain the required "additional splitting" without the necessity of passing to a blow up. In actuality, we

[3]For our purposes, we only need the cone-splitting principle for tangent cones, which case it follows from the splitting theorem of [2]. In fact, by an elementary argument (which we omit) the cone-splitting principle holds for arbitrary metric cones. We do not know an explicit reference for this fact.

need the following quantitative version, which is stated somewhat informally; for the precise statement, see Lemma 4.1.

Quantitative version of the cone-splitting principle Consider a metric ball $B_r(p)$ and for $\delta = \delta(\eta)$ sufficiently small, a δr-Gromov-Hausdorff equivalence $J_\delta : B_r(p) \to B_r((\underline{0}, z^*) \subset \mathrm{R}^\ell \times C(Z)$. Also assume for some $q \in B_r(p)$, that $J_\delta(q)$ does not lie too close to $J_\delta(\mathrm{R}^\ell \times \{z^*\})$, Finally, assume that there is a δr-Gromov-Hausdorff equivalence $J'_\delta : B_r(q) \to B_r(\underline{z}^*) \subset C(\underline{Z})$. Then $B_r(p)$ is ηr-Gromov-Hausdorff close to a ball $B_r((\underline{0}, w^*)) \subset \mathrm{R}^{\ell+1} \times C(W)$, for some cone $\mathrm{R}^{\ell+1} \times C(W)$.

Implementation of cone-splitting.

As noted above, if we knew that y'_∞ was the vertex of some (other) cone structure on Y_y, then we would obtain the required "additional splitting" without the necessity of passing to a blow up. Roughly speaking, to implement the quantitative version of cone-splitting, we need to know that this holds approximately at most locations and scales.

In fact, given a suitable notion of scale, $\gamma < 1$, then for each x, the balls $B_{\gamma^i}(x)$ $(i = 0, 1, \ldots)$ look as conical as we like (with x playing the role of the vertex) on all but a definite number of scales γ^i. This statement, which is close to being implicit in [3], is a quantitative version of the fact that tangent cones are metric cones. It constitutes a "quantitative differentiation" theorem in the sense of Sect. 14 of [8].

Were it not for the fact that the collection of excluded scales (those scales γ^i for which $B_{\gamma^i}(x)$ is not sufficiently close to looking conical) might depend on the point x, we could use the cone-splitting principle to show that $\mathcal{S}^k_{\eta,\gamma^j}$ "looks as k-dimensional as we like" on all but a definite number of scales. Since there is a bound on the number of excluded scales this easily suffices to complete the Cantor type volume computation. This amounts to inductively bounding the number of balls of radius γ^j needed to cover $\mathcal{S}^k_{\eta,\gamma^j}$. The general volume bound for $\mathcal{S}^k_{\eta,r}$, $0 < r \leq 1$, follows directly from the case $r = \gamma^j$.

In order to deal with the above mentioned difficulty, we decompose the space into subsets, each of which consists of those points with precisely the same collection of excluded scales. The bound on the number of excluded scales has the additional consequence that there are "not too many" of these subsets. To each such set, we can apply the argument based on cone-splitting. Since there "not too many" such sets, we can simply add the resulting estimates. This finishes the proof. (Without bringing in this decomposition, we do not know how to complete the argument.)

3 Reduction to the covering lemma

As noted at the beginning of Sect. 1, in proving Theorem 1.3, we can (and will) restrict attention to the case of smooth manifolds. Suppose for some convenient choice $\gamma = \gamma(\eta) < 1$, we can prove (1.8) with some constant $\tilde{c}(n, \mathrm{v}, \eta)$ and all r of the form γ^j. Given r arbitrary, by choosing j such that $\gamma^{j+1} < r \le \gamma^j$, we obtain (1.8) for this r with constant $c(n, \mathrm{v}, \eta) = \tilde{c}(n, \mathrm{v}, \eta)(\gamma(\eta))^{-(n-k-\eta)}$. Thus, in proving (1.8), we can (and will) consider only r of the form γ^j.

An appropriate choice of γ is given in (3.1). Lemma 3.1 below (the covering lemma) asserts that the set $\mathcal{S}^k_{\eta,\gamma^j}$ can be covered by a collection of sets, $\{\mathcal{C}^k_{\eta,\gamma^j}\}$, each of which consists of a not too large collection of balls of radius γ^j. The cardinality of the collection $\{\mathcal{C}^k_{\eta,\gamma^j}\}$ goes to infinity $\mathcal{C}^k_{\eta,\gamma^j}$ as $j \to \infty$. However, by Lemma 3.1, the growth rate is $\le j^{K(\eta,\mathrm{v},n)}$, which is slow enough to be negligible for our purposes. This estimate follows from a quantitative differentiation argument.

The criterion for membership in each particular set $\mathcal{C}^k_{\eta,\gamma^j}$ represents one of the possible behaviors on the scales $1, \gamma, \gamma^2, \dots, \gamma^j$, which could cause a point to lie in $\mathcal{S}^k_{\eta,\gamma^j}$, for $i \le j$. A priori, the number of such different behaviors is 2^j. However, as explained above, for any fixed M^n, k, η, only a small fraction $\le j^{K(\eta,\mathrm{v},n)} \cdot 2^{-j}$ of these can actually occur.

Proof of Theorem 1.3 Let $\underline{x} \in M^n$ and consider $\mathcal{S}^k_{\eta,r} \cap B_{\frac{1}{2}}(\underline{x})$ for some fixed $\eta > 0$ as in (1.8). For $c_0 = c_0(n) > 1$ to be specified below, put

$$\gamma = \gamma(\eta) = c_0^{-\frac{2}{\eta}}. \tag{3.1}$$

Lemma 3.1 *There exists $c_1 = c_1(n) \ge c_0$, $K = K(n, \mathrm{v}, \gamma)$, $Q = Q(n, \mathrm{v}, \gamma) = K + n$, such that for every $j \in \mathbb{Z}_+$:*

1. *The set $\mathcal{S}^k_{\eta,\gamma^j} \cap B_1(\underline{x})$ is contained in the union of at most j^K nonempty sets $\mathcal{C}^k_{\eta,\gamma^j}$.*
2. *Each set $\mathcal{C}^k_{\eta,\gamma^j}$ is the union of at most $(c_1\gamma^{-n})^Q \cdot (c_0\gamma^{-k})^{j-Q}$ balls of radius γ^j.*

Let us provisionally assume Lemma 3.1. Then by volume comparison, we have $\mathrm{Vol}(B_{\gamma^j}(x)) \le c_2(n)\gamma^{jn}$, which together with

$$c_0^j = \left(\gamma^j\right)^{-\frac{\eta}{2}},$$

$$j^K \le c(n, \mathrm{v}, \gamma)\left(\gamma^j\right)^{-\frac{\eta}{2}},$$

gives

$$\begin{aligned}\mathrm{Vol}\big(\mathcal{S}^k_{\eta,\gamma^j}\cap B_1(\underline{x})\big) &\le j^K\cdot\big[\big(c_1\gamma^{-n}\big)^Q\cdot\big(c_0\gamma^{-k}\big)^{j-Q}\big]\cdot c_2\cdot\big(\gamma^j\big)^n\\ &\le \underline{c}(n,\mathrm{v},\gamma)\cdot j^K\cdot c_0^j\cdot\big(\gamma^j\big)^{n-k}\\ &\le \underline{c}(n,\mathrm{v},\gamma)\cdot\big(\gamma^j\big)^{n-k-\eta},\end{aligned}\tag{3.2}$$

where $\underline{c}(n,\mathrm{v},\gamma)=(c_1(n)/c_0(n))^Q\cdot c_2(n)\cdot\gamma^{-(n-k)Q}$. From the above, for all $r\le 1$, we get (1.8) i.e.

$$\begin{aligned}\mathrm{Vol}\big(\mathcal{S}^k_{\eta,r}\cap B_1(\underline{x})\big) &\le \gamma^{-1}\cdot\underline{c}(n,\mathrm{v},\gamma)\cdot r^{n-k-\eta}\\ &\le c(n,\mathrm{v},\eta)r^{n-k-\eta}.\end{aligned}$$

Therefore, modulo the proof of Lemma 3.1, we get Theorem 1.3. □

Proof of Lemma 3.1 The sets $\mathcal{C}^k_{\eta,\gamma^j}$ will be indexed as follows. Consider the set of j-tuples T^j whose each of whose entries is either 0, 1. Denote the number of entries equal to 1 by $|T^j|$. We are going to show the existence of $K=K(n,\mathrm{v},\gamma)\in\mathbb{Z}_+$ (as above) such that every $\mathcal{C}^k_{\eta,\gamma^j}$ corresponds to some unique T^j with $|T^j|\le K$. We denote this set by $\mathcal{C}^k_{\eta,\gamma^j}(T^j)$. Since the number of T^j with $|T^j|\le K$ is at most

$$\binom{j}{K}\le j^K,\tag{3.3}$$

the cardinality of $\{\mathcal{C}^k_{\eta,\gamma^j}(T^j)\}$ is at most j^K.

In order to specify the correspondence $T^j\to\mathcal{C}^k_{\eta,j}(T^j)$, we need a quantity we call the *t-metric nonconicality* $\mathcal{N}_t(B_r(x))\ge 0$ of a ball $B_r(x)$. As in Sect. 1, let $C(Z)$ denote the metric cone on Z with vertex z^*. Let $t\ge 1$, then we say $\mathcal{N}_t(B_r(x))\le\epsilon$ if there exists $C(Z)$ such that

$$d_{\mathrm{GH}}(B_{tr}(x),B_{tr}(z^*))\le\epsilon r.\tag{3.4}$$

We put

$$\begin{aligned}H_{t,r,\epsilon}&=\big\{x\in B_1(\underline{x})\mid\mathcal{N}_t\big(B_r(x)\big)\ge\epsilon\big\},\\ L_{t,r,\epsilon}&=\big\{x\in B_1(\underline{x})\mid\mathcal{N}_t\big(B_r(x)\big)<\epsilon\big\}.\end{aligned}\tag{3.5}$$

Eventually, we will fix $\epsilon=\epsilon(n,\gamma)$, the value in Lemma 3.2 below.

To each x we associate a j-tuple $T^j(x)$. For all $i\le j$, by definition, the i-th entry of $T^j(x)$ is 1 if $x\in H_{\gamma^{-n},\gamma^i,\epsilon}$ and 0 if $x\in L_{\gamma^{-n},\gamma^i,\epsilon}$. Then for each j-tuple T^j define

$$E_{T^j}=\big\{x\in B_1(\underline{x})\mid T^j(x)=T^j\big\}.$$

Below we will show that if E_{T^j} is nonempty then

$$\big|T^j\big|<K(n,\mathrm{v},\epsilon)\quad(\text{if } E_{T^j}\ne\emptyset).\tag{3.6}$$

Because the sets $\mathcal{C}^k_{\eta,\gamma^{j-1}}(T^{j-1})$ are indexed by such tuples, (3.6), together with (3.3), finishes item 1. of Lemma 3.1.

Let T^{j-1} be obtained from T^j by dropping the last entry. Assume that the nonempty subset $\mathcal{C}^k_{\eta,\gamma^{j-1}}(T^{j-1})$ (which is a union of balls of radius γ^{j-1}) has been defined and satisfies item 2. of the Claim and $\mathcal{C}^k_{\eta,\gamma^{j-1}}(T^{j-1}) \supset \mathcal{S}^k_{\eta,\gamma^j} \cap E_{T^j}$. For each ball $B_{\gamma^{j-1}}(x)$ of $\mathcal{C}^k_{\eta,\gamma^{j-1}}(T^{j-1})$, take a minimal covering of $B_{\gamma^{j-1}}(x) \cap \mathcal{S}^k_{\eta,\gamma^j} \cap E_{T^j}$ by balls of radius γ^j with centers in $B_{\gamma^{j-1}}(x) \cap \mathcal{S}^k_{\eta,\gamma^j} \cap E_{T^j}$. Define the union of all balls so obtained to be $\mathcal{C}^k_{\eta,\gamma^j}(T^j)$, *provided it is nonempty.*

Since $\gamma^j/\gamma^{j-1} = \gamma$, from the lower Ricci curvature bound (1.1) and relative volume comparison, it is clear that for each $B_{\gamma^{j-1}}(x)$ as above, the associated minimal covering has at most $c_1(n)\gamma^{-n}$ balls. (This is the $c_1 = c_1(n)$ appearing in (3.2).) However, when $j > n$ and the j-entry of T^j is 0 we use instead the following lemma, whose proof will be given in Sect. 4.

Lemma 3.2 (Covering lemma) *There exists $\epsilon = \epsilon(n,\gamma)$, such that if $\mathcal{N}_{\gamma^{-n}}(B_{\gamma^{j-1}}(x)) \le \epsilon$ and $B_{\gamma^{j-1}}(x)$ is a ball of $\mathcal{C}^k_{\eta,\gamma^{j-1}}(T^{j-1})$, then the number of balls in the minimal covering of $B_{\gamma^{j-1}}(x) \cap \mathcal{S}^k_{\eta,\gamma^j} \cap L_{\gamma^{-n},\gamma^j,\epsilon}$ is $\le c_0\gamma^{-k}$.*

Remark 3.3 In order to apply Lemma 3.2, we need $j > n$. This explains the appearance of the quantity, $Q = K + n$ in the statement of Lemma 3.1.

Remark 3.4 Lemma 3.2 can be viewed as the quantitative analog of the density argument in the proof that $\dim \mathcal{S}^k \le k$. Its proof is a direct consequence of Corollary 4.2 of Lemma 4.1 (the cone-splitting lemma). Corollary 4.2 provides the quantitative analog of the application of the splitting theorem in the proof that $\dim \mathcal{S}^k \le k$; see Sect. 4.

Assuming Lemma 3.2, an obvious induction argument yields the bound on the number of balls of $\mathcal{C}^k_{\eta,\gamma^j}$ appearing in item 2. of Lemma 3.1. The factor with exponent Q in item 2. arises from the (at most Q) scales on which the hypothesis of Lemma 3.2 is not satisfied and we are forced to use the standard covering by at most $c_1\gamma^{-n}$ balls. The factor with exponent $j - Q$ arises from the remaining scales on which we can cover by at most $c_0\gamma^k$ balls as guaranteed by Lemma 3.2.

We close this section by verifying (3.6) which, as previously noted, suffices to verify item 1. of Lemma 3.1.

Let the notation be as in (1.2). For $r > 0$, we consider the volume ratio

$$\mathcal{V}_r(x) = \frac{\mathrm{Vol}(B_r(x))}{\mathrm{Vol}_{-1}(r)} \downarrow . \tag{3.7}$$

The fact that $\mathcal{V}_r(x)$ is a nonincreasing function of r is just the Bishop-Gromov inequality.

For $t > s$, define the (t, s)-*volume energy* $\mathcal{W}_{t,s}(x)$ by

$$\mathcal{W}_{t,s}(x) = \log \frac{\mathcal{V}_s(x)}{\mathcal{V}_t(x)} \geq 0.$$

Note that if $s_1 \geq t_2$, then

$$\mathcal{W}_{t_1,s_2}(x) \geq \mathcal{W}_{t_1,s_1}(x) + \mathcal{W}_{t_2,s_2}(x), \tag{3.8}$$

with equality if $t_2 = s_1$. Let (s_i, t_i) denote a possibly infinite sequence of intervals with $s_i \geq t_{i+1}$ and $t_1 = 1$.

Since $\lim_{r\to 0} \log \mathcal{V}_r(x) = 0$ and the v-noncollapsing assumption (1.2) holds, by using (3.8) together with induction and passing to the limit, we get

$$\log \frac{1}{\mathrm{v}} \geq \log \frac{1}{\mathcal{V}_1(x)} \geq \mathcal{W}_{t_1,s_1} + \mathcal{W}_{t_2,s_2} + \cdots, \tag{3.9}$$

where the terms on the right-hand side are all nonnegative.

Fix $\delta > 0$ and let N denote the number of i such that

$$\mathcal{W}_{\gamma^{i-n},\gamma^i} > \delta.$$

Then

$$N \leq (n+1) \cdot \delta^{-1} \cdot \log \frac{1}{\mathrm{v}}. \tag{3.10}$$

Otherwise, there would be at least $\delta^{-1} \cdot \log \frac{1}{\mathrm{v}}$ *disjoint* closed intervals of the form $[\gamma^i, \gamma^{i-n}]$ with $\mathcal{W}_{\gamma^{i-n},\gamma^i} > \delta$, contradicting (3.9).

Let $\epsilon = \epsilon(n, \gamma)$ be as in Lemma 3.2. The "almost volume cone implies almost metric cone" theorem of [2] implies the existence of $\delta = \delta(\epsilon)$ such that if $\mathcal{W}_{\gamma^{i-n},\gamma^i} \leq \delta$ then $\mathcal{N}_{\gamma^{-n}}(B_{\gamma^i}(x)) \leq \epsilon\gamma^i$, i.e. $x \in L_{\gamma^{-n},\gamma^i,\epsilon}$. This gives (3.6), which completes the proof of Lemma 3.2, modulo that of Lemma 3.1. □

Remark 3.5 Clearly, (3.6) is the quantitative version of the fact that for noncollapsed limit spaces tangent cones are metric cones; compare the proof of the inequality, $\dim \mathcal{S}^k$, which was recalled at the beginning of this section. As previously indicated, relation (3.6) and its proof provide an instance of *quantitative differentiation* in the sense of Sect. 14 of [8].

4 Proof of the covering lemma via the cone-splitting lemma

Assume that the cone $\mathbb{R}^\ell \times C(\overline{Z})$ is a Gromov-Hausdorff limit space with the lower bound on Ricci curvature tending to zero. Suppose in addition that there exists $y' \notin \mathbb{R}^\ell \times \{\overline{z}^*\}$, a cone $C(\hat{Z})$ and an isometry $I : \mathbb{R}^\ell \times C(\overline{Z}) \to C(\hat{Z})$ with $I(y') = \hat{z}^*$. Then $\mathbb{R}^\ell \times C(\overline{Z})$ is isometric to a cone $\mathbb{R}^{\ell+1} \times C(\tilde{Z})$. This follows because if both $\overline{z}^*$ and y' are vertices of cone structures then it is virtually immediate that there must be a line which passes through these points. Therefore, the result follows from the splitting theorem; compare the discussion of cone-splitting in Sect. 2.

We continue to denote by $T_s(\cdot)$ the s-tubular neighborhood. Recall that $L_{t,r,\epsilon}$ is defined in (3.5). The above, together with an obvious compactness argument (and rescaling) yields the following.

Lemma 4.1 (Cone-splitting lemma) *For all $\gamma, \tau, \psi > 0$ there exists $0 < \epsilon = \epsilon(n, \gamma, \tau, \psi) < \psi$, $0 < \theta = \theta(n, \gamma, \tau, \psi)$, such that the following holds. Let $r \le \theta$ and assume that for some cone $\mathbb{R}^\ell \times C(Z)$ there is ϵr-Gromov-Hausdorff equivalence*

$$F : B_{\gamma^{-1}r}\big((\underline{0}, z^*)\big) \to B_{\gamma^{-1}r}(x).$$

If there exists

$$x' \in B_r(x) \cap L_{\gamma^{-1},r,\epsilon},$$

with

$$x' \notin T_{\tau r}\big(F\big(\mathbb{R}^\ell \times \{z^*\}\big)\big) \cap B_r(x),$$

then for some cone $\mathbb{R}^{\ell+1} \times C(\tilde{Z})$,

$$d_{\mathrm{GH}}\big(B_r(x), B_r\big((\underline{0}, \tilde{z}^*)\big)\big) < \psi r.$$

Corollary 4.2 *For all $\gamma, \tau, \psi > 0$ there exists $0 < \delta(n, \gamma, \tau, \psi)$ and $0 < \theta(n, \gamma, \tau, \psi)$ such that the following holds. Let $r \le \theta$ and $x \in L_{\gamma^{-n},\delta,r}$. Then there exists a cone $\mathbb{R}^\ell \times C(\tilde{Z})$ with a ψr-Gromov-Hausdorff equivalence*

$$F : B_r\big((\underline{0}, \tilde{z}^*)\big) \to B_r(x),$$

such that

$$L_{\gamma^{-n},\delta,r} \cap B_r(x) \subseteq T_{\tau r}\big(F\big(\mathbb{R}^\ell \times \{\tilde{z}^*\}\big)\big).$$

Proof For $\epsilon(n, \gamma, \tau, \psi)$ as in Lemma 4.1, inductively define $\epsilon^{[n-i]} = \epsilon \circ \epsilon \circ \cdots \circ \epsilon(n, \gamma^{-n}, \tau, \psi)$ (i factors in the composition). Then $\epsilon^{[0]} < \epsilon^{[1]} < \cdots < \epsilon$. Put $\delta = \epsilon^{[0]}$. Since by assumption, $x \in L_{\gamma^{-n},\delta,r}$, there exists a largest ℓ such that for some cone $\mathbb{R}^\ell \times C(\tilde{Z})$, there is an $\epsilon^{[n-\ell]}r$-Gromov-Hausdorff equivalence $F : B_{\gamma^{-(n-\ell)}r}((\underline{0}, z^*) \to B_{\gamma^{-(n-\ell)}r}(x)$. To see that the conclusion holds for this value of ℓ, apply Lemma 4.1 with the replacements: $r \to \gamma^{-(n-\ell-1)}r$, $\tau = \gamma^{-(n-\ell-1)}\tau$, $\epsilon \to \epsilon^{[\ell]}$, $\psi \to \epsilon^{[\ell+1]}$. □

Proof of Lemma 3.2 Let $B_{\gamma^{j-1}}(x)$ be as in Lemma 3.2. Since by assumption, $x \in \mathcal{S}^k_{\eta,\gamma^i} \cap L_{\gamma^{-n},\gamma^j,\epsilon}$ no cone as in (3.4) with $t = \gamma^{-n} \cdot \gamma^{j-1}$ can split off a factor R^{k+1} isometrically. By applying Corollary 4.2 with $r = \gamma^{j-1}$, $\psi = \frac{1}{10}\gamma$ it follows that for some $\ell \leq k$ and F as in the corollary, we have

$$B_{\gamma^{j-1}}(x) \cap \mathcal{S}^k_{\eta,\gamma^i} \cap L_{\gamma^{-n},\gamma^j,\epsilon} \subset F\left(T_{\frac{1}{10}\gamma^j}\left(\mathbb{R}^\ell \times \left\{\tilde{z}^*\right\}\right)\right) \cap B_{\gamma^{j-1}}(x).$$

Clearly, this suffices to complete the proof. □

5 Curvature estimates absent a priori integral bounds

In this short section we prove Theorem 1.9. Recall that the assumptions are that (M^n, g) is an Einstein manifold which satisfies the v-noncollapsing condition (1.2) and the bound (1.13) on the Einstein constant. item 1. pertains to the real case and item 2. to the Kähler case. The curvature estimates of Theorem 1.9 follow by combining the geometric ϵ-regularity theorems of [3] and [7] with Theorem 1.3. The proofs of these theorems rely on an ϵ-regularity theorem of Anderson; see [1]. We now recall the statements.

Let $(\underline{0}, z^*)$ denote the vertex of the cone $\mathbb{R}^\ell \times C(Z)$. Assume that (M^n, g) is an Einstein manifold which satisfies the v-noncollapsing condition (1.2) and the bound (1.13) on the Einstein constant. In our language, the ϵ-regularity theorem of [3], which does not assume the Kähler condition, asserts that there exists $\epsilon_0(n, \mathrm{v}) > 0$ such that if

$$d_{\mathrm{GH}}\left(B_r(x), B_r\left((\underline{0}, z^*)\right)\right) \leq \epsilon_0 r \quad (\ell > n-2),$$

then on $B_{\frac{1}{2}r}(x)$ there exists a harmonic coordinate system in which the g_{ij} and g^{ij} have definite C^k bounds, for all k. In particular, the C^2-harmonic radius satisfies $r_{har}(x) \geq c(n)r$; see Definition 1.7.

By [7], in the Kähler-Einstein case, the same conclusion holds if $\ell > n-4$. (Conjecturally, the Kähler condition can be dropped.)

Proof of Theorem 1.9 Since the arguments are mutadis mutandis the same for items 1. and 2. of Theorem 1.9, we will just prove item 1. In this case, by the ϵ-regularity theorem, for all $\eta \leq \epsilon_0$,

$$\mathcal{B}_r \cap B_{\frac{1}{2}}(\underline{x}) \subseteq T_r\left(\mathcal{S}^{n-2}_{\eta,Cr}\right) \cap B_{\frac{1}{2}}(\underline{x}).$$

Thus, by Theorem 1.3, we have

$$\mathrm{Vol}\big(\mathcal{B}_r \cap B_{\frac{1}{2}}(x)\big) \leq \mathrm{Vol}\big(T_{Cr}\big(\mathcal{S}^{n-2}_{\eta,Cr}\big) \cap B_{\frac{1}{2}}(x)\big)$$
$$\leq C(n, \mathrm{v}, \eta) r^{2-\eta},$$

which completes the proof. □

6 Curvature estimates given a priori integral bounds

In this section we prove Theorem 1.14.

The proof uses the following corollary of Theorem 1.3. For $r_1 < r_2$, put

$$\mathcal{S}^k_{\eta,r_1,r_2} := \big\{x \mid d_{\mathrm{GH}}\big(B_s(x), B_s\big((\underline{0}, z^*)\big)\big) \geq \eta s, \text{ for all } \mathbb{R}^{k+1} \times C(Z) \text{ and all } r_1 \leq s \leq r_2\big\}. \tag{6.1}$$

Corollary 6.1

$$\mathrm{Vol}\big(\mathcal{S}^k_{\eta,r_1,r_2} \cap B_{r_1}(x)\big) \leq c(n, \mathrm{v}, \eta)\big(r_2^{-1} r_1\big)^{-(k+\eta)} \cdot r_1^n \tag{6.2}$$
$$= c(n, \mathrm{v}, \eta) r_1^{n-k-\eta} \cdot r_2^{(k+\eta)}. \tag{6.3}$$

Proof Let $\hat{B}_{r_2}(x)$ denote the ball of radius $\frac{1}{2}$ obtained by rescaling the metric on $B_{r_2}(x)$ by a factor $\frac{1}{2} \cdot r_2^{-1}$ and let $\hat{\mathcal{S}}^k_{\eta,r}$ denote $\mathcal{S}^k_{\eta,r}$ for the rescaled metric. Then

$$\mathcal{S}^k_{\eta,r_1,r_2} \cap B_{r_2}(x) = \hat{\mathcal{S}}^k_{\eta,\frac{1}{2}r_2^{-1}r_1} \cap \hat{B}_{\frac{1}{2}r_2^{-1}r_1}(x).$$

If we apply Theorem 1.3 in the rescaled situation and interpret the conclusion for the original metric, we get (6.2). □

Recall that in addition to (1.2), (1.13), and the assumption that (M^n, g) is Einstein, we assume the L_p curvature bound (1.16).

The proof of Theorem 1.9 also uses the ϵ-regularity theorems of [4] ($p = 2$), [7] ($p \geq 2$) and Theorem 1.3 for the case $k = n - 2p - 1$. We now recall the statement from [7].

As usual, $(\underline{0}, z^*)$ denotes the vertex of the cone $\mathbb{R}^\ell \times C(Z)$. Assume that (M^n, g) is a Kähler-Einstein manifold which satisfies the v-noncollapsing condition (1.2) and the bound (1.13) on the Einstein constant. Then there exists $\epsilon_0(n, \mathrm{v}, p) > 0$, $\eta_0(n, \mathrm{v}, p) > 0$ such that if

$$d_{\mathrm{GH}}\big(B_r(x), B_r\big((\underline{0}, z^*)\big)\big) < \eta_0 r \quad (\ell \geq n - 2p), \tag{6.4}$$

$$r^{2p} \fint_{B_r(x)} |Rm|^p \leq \epsilon_0, \tag{6.5}$$

then on $B_{\frac{1}{2}r}(x)$ there exists a harmonic coordinate system in which the g_{ij} and g^{ij} have definite C^k bounds, for all k. In particular, $r_{har}(x) \geq c(n)r$.

Proof of Theorem 1.14 Note that since the ϵ-regularity theorem requires that two independent conditions hold simultaneously, we must control the collection of balls on which either one of them fails to hold.

Fix ϵ_0 as above and let $\mathcal{D}_{\epsilon_0,r}$ denote the union of the balls $B_r(x)$ with $x \in B_{\frac{1}{2}}(\underline{x})$, for which (6.5) fails to hold. By a standard covering argument it follows from the L_p curvature bound (1.16) that $\mathcal{D}_{\epsilon_0,r} \cap B_{\frac{1}{2}}(\underline{x})$ can be covered by a collection of balls $\{B_r(x_i)\}$ such that we have

$$\mathrm{Vol}\big(\mathcal{D}_{\epsilon_0,r} \cap B_{\frac{1}{2}}(\underline{x})\big) \leq \sum_i \mathrm{Vol}\big(B_r(x_i)\big) \leq c(n)C\epsilon_0^{-1} \cdot r^{2p}. \tag{6.6}$$

In particular, for γ as in Sect. 3, $\eta = \eta_0 < 1$ and $k = n - 2p - 1$, by applying Corollary 6.1 to each the balls $B_r(x_i)$ whose union covers $\mathcal{D}_{\epsilon_0,\gamma^i}$ and summing the resulting estimates we get for all $1 \leq i < j$,

$$\mathrm{Vol}\big(\mathcal{S}^{n-2p-1}_{\eta,\gamma^j,\gamma^i} \cap \mathcal{D}_{\epsilon_0,\gamma^{i-1}}\big) \leq c(n, \mathrm{v}, \eta_0, C)\big(\gamma^j\big)^{2p+1-\eta_0} \cdot \big(\gamma^i\big)^{1+\eta_0}. \tag{6.7}$$

Summing (6.7) over $1 \leq i \leq j$ and bounding the right-hand side in terms of the geometric series with ratio $\gamma^{1+\eta_0}$ gives

$$\sum_{1\leq i\leq j} \mathrm{Vol}\big(\mathcal{S}^{n-2p-1}_{\eta_0,\gamma^j,\gamma^i} \cap \mathcal{D}_{\epsilon_0,\gamma^{i-1}}\big) \leq c(n, \mathrm{v}, \eta_0, C, p)\big(\gamma^j\big)^{2p+1-\eta_0}. \tag{6.8}$$

We claim that

$$\begin{aligned}\mathcal{B}_{\gamma^j} \cap B_{\frac{1}{2}}(\underline{x}) \subset \big(&\mathcal{S}^{n-2p-1}_{\eta_0,\gamma^j} \cap B_{\frac{1}{2}}(\underline{x})\big) \\ &\cup \left(\bigcup_{1\leq i\leq j} \mathcal{S}^{n-2p-1}_{\eta_0,\gamma^j,\gamma^i} \cap \mathcal{D}_{\epsilon_0,\gamma^{i-1}}\right) \cup \mathcal{D}_{\epsilon_0,\gamma^j}.\end{aligned} \tag{6.9}$$

This will suffice to complete the proof of Theorem 1.14 for the case $r = \gamma^j$, since by (6.8), together with (6.6) for $r = \gamma^j$ and Theorem 1.3 for $r = \gamma^j$, it follows that the volume of the set on the right-hand side is $\leq c(n, tv, \eta_0, \epsilon_0, p, C)(\gamma^j)^{2p}$. As in the proof of Theorem 1.3 the general case follows directly from the case $r = \gamma^j$.

Let A' denote the complement of A. To establish the claim, we note that the complement of the set on right-hand side of (6.9) is equal to

$$\big((\mathcal{S}^{n-2p-1}_{\eta_0,\gamma^j})' \cup B_{\frac{1}{2}}(\underline{x})'\big) \cap \left(\bigcap_{0\leq i\leq j} (\mathcal{S}^{n-2p-1}_{\eta_0,\gamma^j,\gamma^i})' \cup (\mathcal{D}_{\epsilon_0,\gamma^{i-1}})'\right) \cap (\mathcal{D}_{\epsilon_0,\gamma^j})'.$$

By expanding out and dropping the terms which start with $B_{\frac{1}{2}}(\underline{x})'$, we obtain an expression that is a union of terms, each of which is of the form

$$\big(\mathcal{S}^{n-2p-1}_{\eta_0,\gamma^j}\big)' \cap \cdots \cap \big(\mathcal{S}^{n-2p-1}_{\eta_0,\gamma^j,\gamma^i}\big)' \cap (\mathcal{D}_{\epsilon_0,\gamma^i})' \cap \left(\bigcap_{i<\ell\leq j} (\mathcal{D}_{\epsilon_0,\gamma^\ell})'\right) \cap (\mathcal{D}_{\epsilon_0,\gamma^j})'$$

$$\subset \big(\mathcal{S}^{n-2p-1}_{\eta_0,\gamma^j,\gamma^i}\big)' \cap (\mathcal{D}_{\epsilon_0,\gamma^i})' \cap (\mathcal{D}_{\epsilon_0,\gamma^{i+1}})' \cap \cdots \cap (\mathcal{D}_{\epsilon_0,\gamma^j})' \tag{6.10}$$

for some i with $1 \leq i \leq j$. (The terms represented by the dots can be either $\mathcal{S}$'s or $\mathcal{D}$'s.) By (6.4), (6.5), the set on the second line of (6.10) satisfies the hypothesis of the ϵ-regularity theorem of [7], so this completes the proof. □

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