

Super-Resolution in Phase Space

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Abstract: This work considers the problem of super-resolution. The goal is to resolve a Dirac distribution from knowledge of its discrete, low-pass, Fourier measurements. Classically, such problems have been dealt with parameter estimation methods. Recently, it has been shown that convex-optimization based formulations facilitate a continuous time solution to the super-resolution problem. Here we treat super-resolution from low-pass measurements in Phase Space. The Phase Space transformation parametrically generalizes a number of well known unitary mappings such as the Fractional Fourier, Fresnel, Laplace and Fourier transforms. Consequently, our work provides a general super-resolution strategy which is backward compatible with the usual Fourier domain result. We consider low-pass measurements of Dirac distributions in Phase Space and show that the super-resolution problem can be cast as Total Variation minimization. Remarkably, even though the setting is quite general, the bounds on the minimum separation distance of Dirac distributions is comparable to existing methods.

Index Terms

Fractional Fourier, low-pass, phase space, spike train, super-resolution, total variation.

CONTENTS

I	Introduction	1
II	Phase Space Representation of Signals	3
III	Sparse Signals in Phase Space	4
IV	Super-Resolution in Phase Space	5
IV-A	Problem Formulation	5
IV-B	Super-Resolution Via Convex Programming	6
IV-C	Remarks and Discussion	7
IV-D	An Application of Super-Resolution in Phase Space	8
V	Conclusion	9
	References	9

I. INTRODUCTION

Ernst Abbe's foundational work [1] in 1873 reported an observation regarding lack of resolvability of optical features beyond the diffraction limit. This problem is central to several areas of science and engineering such as optics [2], [3], imaging [4], geophysics [5], depth sensing [6] and astronomy [7].

In its abstract form, the problem can be stated as follows: How can we recover a K -sparse signal (spike train/Dirac impulses) with unknown locations and amplitudes from the knowledge of its low-pass measurements in the Fourier domain? The problem is challenging because it asks for recovery of a non-bandlimited signal from its projection onto the subspace of bandlimited functions. Two predominant approaches exist in the literature to super-resolution: optimization and parameter estimation based solutions.

Δ : Minimum separation f_c : Cut-off frequency			
Donoho [14]	Kahane [15]	Candès-G. [11]	Moitra [16]
$\Delta > \frac{1}{f_c}$	$\Delta > \frac{5}{f_c} \sqrt{\log\left(\frac{1}{2f_c}\right)}$	$\Delta > \frac{2}{f_c}$	$\Delta > \frac{1}{f_c-1}$

TABLE I: Exact Recovery Condition for Super-Resolution.

■ **Optimization Based Super-Resolution:** Early roots of ℓ_1 norm based optimization can be traced back to [8]. This approach has been revitalized due to the advent of *compressed sensing* [9]. A very recent idea in this context is that of *continuous sparse modeling* where the signal attributes are estimated on a continuously defined grid [10], [11] (instead of its discrete counterpart, as is the case for the usual compressive sensing setup).

Bredies and Pikkariainen [12] present an optimization method in the space of signed measures. In their setting, they consider discrete measurements while the solution space is infinite dimensional. In parallel, de Castro and Gamboa [13] and Candès and Fernandez-Granda [11] consider a Total Variation (TV) formulation for super-resolution.

■ **Parameter Estimation Based Super-Resolution:** The super-resolution problem has been studied in the signal processing context with the goal of resolving overlapping echoes/time-delay estimation [17], multi-path characterization [6] and deconvolution. Li and Speed [18] considered super-resolution in form of parametric deconvolution of spike trains. Vetterli, Blu and co-workers [19], [20] developed the idea of super-resolution as a sparse sampling problem. Eldar and co-workers developed super-resolution methods in the context of sub-Nyquist sampling and the Xampling framework [21]. Finally, since the super-resolution problem is closely linked with the spectral estimation problem [18], MUSIC, ESPRIT and matrix pencil [22], [23] based methods have also been used.

In contrast to parameter estimation techniques where the recovery condition is based on the number of spikes, non-parametric methods provide a bound in the form of a minimum separation condition. More precisely, let K be the number of spikes to be super-resolved, f_c be the cut-off frequency in Fourier domain and let Δ denote the minimum spacing between any two spikes. While parameter estimation methods require $f_c \geq 2K + 1$ assuming that K is known a priori, non-parametric methods super-resolve the spikes provided that $f_c > f_{\text{SR}}(\Delta)$ where f_{SR} is some function. The works of Donoho [14], Kahane [15], Candès and Fernandez-Granda [11] and Moitra [16] provide a theoretical guarantee for the super-resolution problem in terms of f_{SR} . We summarize the recovery guarantees in terms of f_{SR} in Table I.

The Fourier transform is well suited for examining signals which are linear combinations of sinusoids. In many practical applications such as radar, sonar, holography, wave-physics and quantum optics, the basic building blocks of the signals are not sinusoidal. Often polynomial phase, Fourier-like transformations of form $e^{j\phi(t)}$ are well suited for analysis of such signals. For this purpose, tools such the Fresnel transform [24], Fractional Fourier transform [25] and the Chirp transform [26], [27] have been developed in the literature.

Our goal here is to extend existing super-resolution results to integral transformations other than the Fourier transform. To this end, we cast the super-resolution problem in phase domain which parametrically generalizes

some of the well-known unitary transformations. Some examples are listed in Table II. We show that exact recovery of spike trains from low-pass measurements in phase space is possible by minimizing the signal's TV-norm. This is accomplished using a convex program. Remarkably, even with the general construct of the problem, our theoretical guarantee for exact recovery is the same as in [11].

Throughout the paper, \mathbb{R} , \mathbb{C} and \mathbb{Z} denote sets of real, complex and integers. We use z^* to denote the complex conjugate of $z \in \mathbb{C}$, $\Re z = \frac{1}{2}(z + z^*)$ is the real part of z and $j = \sqrt{-1}$. Discrete sequences are represented by $s[m]$, $m \in \mathbb{Z}$ while their continuous counterparts are represented by $s(t)$, $t \in \mathbb{R}$. The L_2 inner product between functions f and g is denoted by $\langle f, g \rangle = \int f g^* dt$. Function composition is denoted by $(f \circ g)(x) = f(g(x))$. Convolution between functions f and g is defined as $(f * g)(t) = \int f(z) g(t - z) dz$. We use \mathbf{s} to represent the row-vector of a discrete sequence $s[m]$, $m \in [0, \dots, M - 1]$. We use bold capitals to represent matrices, for example \mathbf{D} and \mathbf{D}^\dagger is the pseudo-inverse of \mathbf{D} . Calligraphic letters such as \mathcal{L} are used to denote operators. Sets are denoted by capitalized roman font, S . The estimate of function/sequence μ is represented by $\tilde{\mu}$.

II. PHASE SPACE REPRESENTATION OF SIGNALS

The term *Phase Space* naturally arises in areas of optics [28] and mathematical physics [29]. The Fractional Fourier transform (FrFT) [25], [30]–[32] and the Linear Canonical Transform (LCT) [28], [33], [34] are instantiations of phase space transformations. These transformations have found a number of applications in signal processing [25] and communication problems such as shift-invariant sampling/approximation theory [30], [34], operator theory, multi-carrier communications [32], array processing [31] and optical signal processing [28]. In this paper, we will work with the *Linear Canonical Transform* (LCT) [29].

Definition 1 (LCT): Let $\mathbf{\Lambda} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $ad - bc = 1$. The LCT of a function $f(t)$, $t \in \mathbb{R}$ is a parametric, unitary, integral mapping, $\mathcal{L}_{\mathbf{\Lambda}} : f \rightarrow \hat{f}$ with respect to the time-frequency kernel $k_{\mathbf{\Lambda}}$,

$$\hat{f}(\omega) = \underbrace{\mathcal{L}_{\mathbf{\Lambda}}[f]}_{\text{LCT}}(\omega) \stackrel{\text{def}}{=} \begin{cases} \langle f, k_{\mathbf{\Lambda}}(\cdot, \omega) \rangle & b \neq 0, \\ \sqrt{d} e^{-j\frac{1}{2}cd\omega^2} f(d\omega) & b = 0, \end{cases} \quad (1)$$

where the transformation kernel is defined by

$$k_{\mathbf{\Lambda}}(t, \omega) = \frac{1}{\sqrt{-j2\pi b}} \exp\left(-j\frac{1}{2b}(at^2 + d\omega^2 - 2\omega t)\right). \quad (2)$$

Now since $b = 0$ amounts to dilating the function $\sqrt{d} e^{-j\frac{1}{2}cd\omega^2} f(d\omega)$ (see (1)), we will develop our results for the case when $b \neq 0$.

The LCT satisfies a useful operator composition property: $\mathcal{L}_{\mathbf{\Lambda}_1} \circ \mathcal{L}_{\mathbf{\Lambda}_2} = \mathcal{L}_{\mathbf{\Lambda}_3}$ with $\mathbf{\Lambda}_3 = \mathbf{\Lambda}_2 \mathbf{\Lambda}_1$. This can be used to show that,

$$f(t) = \underbrace{\mathcal{L}_{\mathbf{\Lambda}^{-1}}[\hat{f}]}_{\text{Inverse-LCT}}(t) \stackrel{\text{def}}{=} \begin{cases} \langle \hat{f}, k_{\mathbf{\Lambda}^{-1}}(t, \cdot) \rangle & b \neq 0 \\ \sqrt{a} e^{-j\frac{1}{2}cat^2} f(at) & b = 0. \end{cases} \quad (3)$$

The LCT parametrizes a number of well-known, unitary transformations, some of which are listed in Table II. For $\mathbf{\Lambda} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \mathbf{\Lambda}_{\text{FT}}$, the LCT amounts to the Fourier transform (upto a constant). (FT). Similarly, with the 2×2 rotation matrix $\mathbf{\Lambda}_{\theta}$ (see Table II), we obtain the Fractional Fourier transform (FrFT). Matrix factorization

TABLE II: Parametric Representation of Unitary Transformations

Parameter Matrix (Λ)	Corresponding Transform
$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \Lambda_\theta$	Fractional Fourier Transform
$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \Lambda_{\text{FT}}$	Fourier Transform (FT)
$\begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix} = \Lambda_{\text{LT}}$	Laplace Transform (LT)
$\begin{bmatrix} j \cos \theta & j \sin \theta \\ j \sin \theta & -j \cos \theta \end{bmatrix}$	Fractional Laplace Transform
$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$	Fresnel Transform
$\begin{bmatrix} 1 & jb \\ j & 1 \end{bmatrix}$	Bilateral Laplace Transform
$\begin{bmatrix} 1 & -jb \\ 0 & 1 \end{bmatrix}, b \geq 0$	Gauss–Weierstrass Transform
$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & e^{-j\pi/2} \\ -e^{-j\pi/2} & 1 \end{bmatrix}$	Bargmann Transform

shows that the LCT is related to the FT and the FrFT. The Fourier matrix factorization is simple:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & b^{-1} \end{bmatrix} \Lambda_{\text{FT}} \begin{bmatrix} 1 & 0 \\ a/b & 1 \end{bmatrix} \Leftrightarrow \Lambda = \mathbf{M}_1 \Lambda_{\text{FT}} \mathbf{M}_2,$$

and leads to the implementation $\mathcal{L}_\Lambda = \mathcal{L}_{\mathbf{M}_1} \circ \mathcal{L}_{\Lambda_{\text{FT}}} \circ \mathcal{L}_{\mathbf{M}_2}$. The Fractional Fourier matrix factorization is more involved. Relying on the *Iwasawa Decomposition* [35],

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \Lambda_\theta \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma^{-1} \end{bmatrix} \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \Leftrightarrow \Lambda = \Lambda_\theta \mathbf{D} \mathbf{U},$$

where \mathbf{D} is a diagonal matrix with $\Gamma = \sqrt{a^2 + c^2}$ and \mathbf{U} is an upper-triangular matrix with $u = (ab + cd) / \Gamma^2$.

A useful operation that is linked with phase space is the convolution/filtering operator. For the Fourier domain, we have the convolution–multiplication property: $(f * g) = \mathcal{L}_{\Lambda^{-1}} [\mathcal{L}_\Lambda [f] \mathcal{L}_\Lambda [g]]$, with $\Lambda = \Lambda_{\text{FT}}$. Unfortunately, this property is not preserved in phase space. To circumvent this problem, we use a version of the FrFT convolution operator [30].

Definition 2 (LCT Convolution/Filtering): Let $*_\Lambda$ denote the convolution/filtering operation in LCT domain and $*$ be the usual Fourier domain convolution operator. Convolution of functions f and g in the LCT domain is defined by

$$(f *_\Lambda g)(t) = \frac{e^{-j\frac{at^2}{2b}}}{\sqrt{j2\pi b}} \underbrace{\left(f(t) e^{+j\frac{at^2}{2b}} * g(t) e^{+j\frac{at^2}{2b}} \right)}_{\text{Convolution of Modulated Functions}}. \quad (4)$$

By following the steps in [30], it is easy to verify that the operation in (4) admits a convolution–multiplication property:

$$\mathcal{L}_\Lambda [f *_\Lambda g](\omega) = e^{-j\frac{d\omega^2}{2b}} \hat{f}(\omega) \hat{g}(\omega). \quad (5)$$

III. SPARSE SIGNALS IN PHASE SPACE

Consider a K -sparse object/spike train modeled by,

$$s(t) = \sum_{k=0}^{K-1} c_k \delta(t - t_k), \quad t \in \mathbb{R} \quad (6)$$

where δ denotes the Dirac mass with weights $\{c_k\}_k \in \mathbb{C}$ that is activated on locations $\{t_k\} \in [0, \tau)$, $k = 0, \dots, K-1$. Since we are dealing with a finite length signal s that lives on the interval $[0, \tau)$, we investigate

its representation in phase space using the Fractional Fourier series [36] analog of the LCT.

Definition 3 (Linear Canonical Series): Let f be a compactly supported function such that $f \neq 0, \forall t \in [0, \tau]$ and zero elsewhere and let $\kappa_{b,\tau} = \sqrt{2\pi b/\tau}$, $b \neq 0$. The Linear Canonical Series (LCS) expansion of the function f is given by

$$f(t) = \kappa_{b,\tau} \sum_{n=-\infty}^{n=+\infty} \hat{f}[n] k_{\Lambda}(t, n\omega_0 b), \quad \omega_0 = \frac{2\pi}{\tau}. \quad (7)$$

The LCS coefficients \hat{f} are evaluated at $\omega = n\omega_0 b, n \in \mathbb{Z}$,

$$\hat{f}[n] = \kappa_{b,\tau} \mathcal{L}_{\Lambda}[f](n\omega_0 b) \equiv \kappa_{b,\tau} \langle f, k_{\Lambda}(\cdot, n\omega_0 b) \rangle. \quad (8)$$

Note that by appropriately parameterizing Λ , one can easily design the basis functions for any of the transformations listed in Table II. For example, with $\Lambda = \Lambda_{\text{FT}}$ in (8), we obtain the Fourier Series expansion of f .

We now compute the series coefficients for the sparse signal $s(t)$ in (6). We use (8), to compute the LCS coefficients

$$\begin{aligned} \hat{s}[n] &= \kappa_{b,\tau} \mathcal{L}_{\Lambda}[f](n\omega_0 b) \stackrel{(8)}{=} \kappa_{b,\tau} \int_{\tau} s(t) k_{\Lambda}^*(t, n\omega_0 b) dt \\ &= \kappa_{b,\tau} \sum_{k=0}^{K-1} c_k k_{\Lambda}^*(t_k, n\omega_0 b). \end{aligned} \quad (9)$$

Plugging the coefficients into the LCS representation (7), we obtain the phase space representation of $s(t)$

$$\begin{aligned} s(t) &\stackrel{(7)}{=} \kappa_{b,\tau} \sum_{m=-\infty}^{m=+\infty} \hat{s}[m] k_{\Lambda}(t, m\omega_0 b), \quad \omega_0 = \frac{2\pi}{\tau} \\ &= \frac{e^{-j\frac{at^2}{2b}}}{\tau} \sum_{m=-\infty}^{m=+\infty} \underbrace{\left(\sum_{k=0}^{K-1} c_k e^{j\frac{a}{2b}t_k^2} e^{-jm\omega_0 t_k} \right)}_{\hat{y}[m]} e^{jm\omega_0 t} \end{aligned} \quad (10)$$

$$= \frac{e^{-j\frac{at^2}{2b}}}{\tau} \sum_{m=-\infty}^{m=+\infty} \hat{y}[m] e^{jm\omega_0 t}. \quad (11)$$

From [11], we see that $\hat{y}[m]$ are precisely the Fourier series coefficients of $\tau s(t) e^{j\frac{at^2}{2b}}$. Thus, even though we are dealing with the phase space, the linear frequency modulated, sparse signal s is completely characterized by its Fourier series coefficients \hat{y} .

IV. SUPER-RESOLUTION IN PHASE SPACE

A. Problem Formulation

Let $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$. Consider the frequency modulated function,

$$\phi_{\text{LP}}(t) = (\Omega/b) e^{-j\frac{at^2}{2b}} \text{sinc}((\Omega/b)t). \quad (12)$$

Note that this function is $(\Omega\pi)$ -bandlimited because in phase space the function $\phi_{\text{LP}}(t)$ is compactly supported, or,

$$\hat{\phi}_{\text{LP}}(\omega) = \mathcal{L}_{\Lambda}[\phi_{\text{LP}}](\omega) = \frac{e^{+j\frac{d\omega^2}{2b}}}{\sqrt{j2\pi b}} \Pi\left(\frac{\omega}{2\pi\Omega}\right)$$

where $\Pi(\omega) = 1, |\omega| \leq 1/2$ and zero, elsewhere.

With s defined in (11), its low-pass version is,

$$h(t) = \underbrace{(s * \Lambda \phi_{\text{LP}})(t)}_{\text{Low-Pass Filtering}} \stackrel{(4)}{=} \frac{e^{-j\frac{at^2}{2b}}}{\sqrt{j2\pi b}} \frac{1}{\tau} \sum_{|m| \leq \lfloor \Omega\tau/2b \rfloor} \hat{y}[m] e^{j\omega_0 m t}, \quad (13)$$

where $\lfloor \cdot \rfloor$ is the floor operation.

Suppose we observe N discrete, low-pass measurements sampled with sampling rate $T = b/\Omega$,

$$h[n] = h(t)|_{t=nT}, \quad T = b/\Omega, \quad n = 0, \dots, N-1. \quad (14)$$

By modulating h , we obtain measurements of the form,

$$y[n] = \underbrace{\sqrt{j2\pi b\tau} e^{+j\frac{a(nT)^2}{2b}}}_{\text{Modulated Measurements}} h[n] = \sum_{|m| \leq f_c} \hat{y}[m] e^{j\omega_0 m(nT)} \quad (15)$$

where $f_c = \lfloor \Omega\tau/2b \rfloor$. In vector-matrix notation, we have low-pass measurements, $\mathbf{y} = \mathbf{V}_{\text{IDFT}} \hat{\mathbf{y}}$, where $\mathbf{V}_{\text{IDFT}} \in \mathbb{C}^{N \times (2f_c+1)}$ is the usual inverse-DFT matrix with elements $[\mathbf{V}_{\text{IDFT}}]_{n,m} = e^{jm\omega_0(nT)}$ and we assume that $N \geq 2f_c + 1$ so that \mathbf{V}_{IDFT} is a full-rank matrix.

Having obtained $\hat{\mathbf{y}}$, the question then is: how many samples of $\hat{\mathbf{y}}$ are sufficient for complete characterization of $\hat{y}[m] \stackrel{(10)}{=} \sum_{k=0}^{K-1} \left(c_k e^{+j\frac{a}{2b} t_k^2} \right) e^{-jm\omega_0 t_k}$? Indeed, from spectral estimation theory [22], we know that we need at least $2K + 1$ values of \hat{y} to solve for $\{c_k, t_k\}_k$. Consequently, whenever

$$f_c = \lfloor \frac{\Omega\tau}{2b} \rfloor \geq K \Leftrightarrow \lfloor \frac{\tau}{2T} \rfloor \geq K, \quad (16)$$

the system of equations is complete in \hat{y} meaning that we have at least $2K + 1$ values of \hat{y} and we can solve for $\{c_k, t_k\}$. Thus (16) provides a bound on the minimum sampling density in phase space.

B. Super-Resolution Via Convex Programming

Given N measurements \mathbf{y} , we obtain $\hat{\mathbf{y}}$ using $\hat{\mathbf{y}} = \mathbf{V}_{\text{IDFT}}^\dagger \mathbf{y}$ (15). From the phase space development of the problem, we know that

$$\begin{aligned} \hat{y}[m] &\stackrel{(10)}{=} \sum_{k=0}^{K-1} \left(c_k e^{+j\frac{a}{2b} t_k^2} \right) e^{-jm\omega_0 t_k} \\ &= \int_0^\tau \mu(t) e^{-jm\omega_0 t} dt, \quad |m| \leq f_c = \lfloor \Omega\tau/2b \rfloor \end{aligned} \quad (17)$$

where

$$\mu(t) = \sum_{k=0}^{K-1} \underbrace{c_k e^{+j\frac{a}{2b} t_k^2}}_{\rho_k} \delta(t - t_k) \quad (18)$$

and $\rho_k \stackrel{\text{def}}{=} c_k e^{+j\frac{a}{2b} t_k^2}$ are the new weights for $s(t)$.

We are now left to solve the standard super-resolution problem [11] where one has access to the low-pass measurements $\hat{y}[m]$ and the signal to be super-resolved is prescribed in (18). Hence, the problem of recovering

μ from $\hat{\mathbf{y}}$ can now be solved by using,

Super-Resolution in Phase Space (Primal Problem)

$$\min_{\tilde{\mu}} \|\tilde{\mu}\|_{\text{TV}} \quad \text{subject to} \quad \left\{ \hat{\mathbf{y}}[m] \stackrel{(17)}{=} \int_0^\tau \tilde{\mu}(t) e^{-jm\omega_0 t} dt \right\}_{|m| \leq f_c}$$

(19)

where, for the model assumed in (18), the TV-norm amounts to, $\|\mu\|_{\text{TV}} = \sum_k |\rho_k| \stackrel{(18)}{=} \sum_k |c_k|$. We note that in principle $\|\tilde{\mu}\|_{\text{TV}} = \|s\|_{\text{TV}}$, however, due to the nature of phase space measurements, the real/complex weights $\{c_k\}_k$ need to be demodulated using the linear frequency modulation term $e^{+j\frac{a}{2b}t_k^2}$ which depends on Λ .

The problem in (19) seeks to recover the infinite-dimensional variable $\hat{\mu}$ from finitely many constraints set up in (17). This continuous optimization problem has a tractable dual problem. As was shown in [11], a semidefinite program (SDP) can be used to recover $\tilde{\mu}$ by computing $\{t_k\}_k$ first and then recovering $\{c_k\}_k$ using a least squares fit. The SDP equivalent [11] of the convex dual of (19) is,

Semidefinite Program (Dual Problem)

$$\begin{aligned} \max_{\mathbf{u}, \mathbf{M}} \quad & \Re \langle \hat{\mathbf{y}}, \mathbf{u} \rangle \quad \text{subject to,} \\ \begin{bmatrix} \mathbf{M} & \mathbf{u} \\ \mathbf{u}^* & 1 \end{bmatrix} \succ 0, \quad & \sum_{j \in S_2, m \in S_1} [\mathbf{M}]_{m, m+j} = \delta_j \end{aligned}$$

where $S_1 = [1, 2f_c + 1 - j]$, $S_2 = [0, 2f_c]$, $\mathbf{M} \in \mathbb{C}^{(2f_c+1) \times (2f_c+1)}$ is some Hermitian matrix and $\mathbf{u} \in \mathbb{C}^{2f_c+1}$ is a complex vector.

The SDP input $\hat{\mathbf{y}}$ results in a vector \mathbf{u} . In order to recover the locations $\{t_k\}_k$, we construct the polynomial of degree $N_0 = 4f_c$,

$$p_{N_0}(z) = 1 - \sum_{|k| \leq 2f_c} u_k z^k, \quad z \in \mathbb{C}. \quad (20)$$

The roots of $p_{4f_c}(z)$, $z = e^{j\omega_0 t}$ lead to the locations $\{t_k\}_k$. Knowing $\hat{\mathbf{y}}$ together with the estimates, $\{\tilde{t}_k\}_k$, we use the constraints in (17) to set up a system of equations which leads to amplitude estimates $\tilde{c}_k = \tilde{\rho}_k e^{-j\frac{a}{2b}\tilde{t}_k^2}$. (see (18)). Finally, we recover our super-resolved signal, $\tilde{s}(t) = \sum_{k=0}^{K-1} \tilde{c}_k \delta(t - \tilde{t}_k)$. Stepwise procedure for super-resolution in phase space is outlined in Algorithm 1.

In view of [11], let us invoke the definition of minimum distance $\Delta = \inf_{\{t_k\}_k, t_k \neq t_l} |t_k - t_l|$. With $f_c = \lfloor \Omega\tau/2b \rfloor$, the exact recovery requirement for phase space is as follows:

Theorem 1 (Exact Recovery in Phase Space): Let the support set of $s(t)$ in (6) be $S = \{\tilde{t}_k\}_k$. If the minimum distance obeys the bound $\Delta(S) f_c \geq 2$, then $s(t)$ is a unique solution to (19).

The proof of this theorem is a straight-forward consequence of [11]. Moreover, due to inherent Fourier structure of the phase space problem, our work may benefit from the ideas discussed in [13], [14], [16], [23].

C. Remarks and Discussion

■ **Backward Compatibility** With the choice of parameter matrix $\Lambda = \Lambda_{\text{FT}}$ (cf. Table II), our result coincides with the usual, Fourier domain case of super-resolution [11], [13]. Furthermore, $\Lambda = \Lambda_\theta$ relates to the case of Fractional Fourier domain for which our result generalizes a previous known result [37].

Algorithm 1: Super-Resolution in Phase Space.

Input: Low-pass samples $h[n] \stackrel{(13)}{=} (s *_{\Lambda} \phi_{\text{LP}})(nT)$
Modulate Samples $h[n] \rightarrow \sqrt{j2\pi b|\tau|} e^{+j\frac{a(nT)^2}{2b}} h[n] = y[n]$
Data: $\hat{\mathbf{y}} = \mathbf{V}_{\text{IDFT}}^{\dagger} \mathbf{y}, \hat{\mathbf{y}} \in \mathbb{C}^{2f_c+1}$
Solve SDP : $\max_{\mathbf{u}, \mathbf{M}} \Re \langle \hat{\mathbf{y}}, \mathbf{u} \rangle$ subject to $\begin{bmatrix} \mathbf{M} & \mathbf{u} \\ \mathbf{u}^* & 1 \end{bmatrix} \succ 0$
Construct Polynomial: $p_{N_0}(z) = 1 - \sum_{|k| \leq 2f_c} u_k z^k$
Support: $p_{N_0}(e^{j\omega_0 t}) = 0 \rightarrow \{\tilde{t}_k\}_k$
Weights: $\min_{\tilde{\rho}_k} \left| \hat{y}[m] - \sum_{k=0}^{K-1} \tilde{\rho}_k e^{-j\omega_0 \tilde{t}_k} \right|^2 \rightarrow \{\tilde{\rho}_k\}_k$
Output: $\tilde{s}(t) = \sum_{k=0}^{K-1} \tilde{c}_k \delta(t - \tilde{t}_k), \{\tilde{c}_k = \tilde{\rho}_k e^{-j\frac{a}{2b}\tilde{t}_k^2}\}_k$

Exact Recovery Condition Even though our super-resolution naturally extends to a number of well known unitary transformations, the exact recovery condition remains unchanged. Hence re-formulating the super-resolution problem in context of phase space comes at no extra cost in the sense of recovery requirement.

D. An Application of Super-Resolution in Phase Space

Bandlimited signals are compactly supported in the Fourier domain. When a bandlimited signal is corrupted by additive impulsive noise or AIN, the holes/zeros in the spectrum are filled by the spectral components that characterize the impulsive noise which is essentially non-bandlimited. Wolf [38] used the idea of curve-fitting the out-of-band components for identification of impulsive noise components. Here, we formulate the problem of denoising linear frequency modulated (LFM) signals that are corrupted by AIN. Since LFM signals are the basis functions of phase space transformations, it is clear that such signals are bandlimited in the LCT domain.

Consider a bandlimited LFM signal

$$r_{\text{BL}}(t) = \kappa_{b,\tau} \sum_{|m| \leq M} \hat{r}_{\text{BL}}[m] k_{\Lambda}(t, m\omega_0 b),$$

with $\hat{r}_{\text{BL}}[m] = 0, |m| > M$ and let $r(t) = r_{\text{BL}}(t) + s(t)$ be the signal corrupted by AIN. Clearly, $r(t)$ is non-bandlimited in phase space due to $s(t)$. Suppose we observe low-pass filtered samples of $r(t)$, that is,

$$\underbrace{(r *_{\Lambda} \phi_{\text{LP}})(nT)}_{h[n]} = e^{-j\frac{a(nT)^2}{2b}} \sum_{|m| \leq f_c} \underbrace{(c_1 \hat{y}_1[m] + c_2 \hat{y}_2[m])}_{\hat{y}_r[m]} e^{j\omega_0 m n T}$$

where c_1, c_2 are known constants, $\hat{y}_1[m] = \hat{r}_{\text{BL}}[m] e^{-j\frac{d(m\omega_0 b)^2}{2b}}$ and $\hat{y}_2[m] = \hat{y}[m]$ (as in (17)) where $f_c = \lfloor \Omega\tau/2b \rfloor \equiv \lfloor \tau/2T \rfloor$. Again, let us define $y[n] = h[n] e^{j\frac{a(nT)^2}{2b}}, n = 0, \dots, N-1, N \geq 2f_c + 1$. Provided that $f_c \geq M + 2K + 1$, we have,

$$\hat{y}_r[m] = \begin{cases} c_1 \hat{y}_1[m] + c_2 \hat{y}_2[m] & |m| \leq M \\ c_2 \hat{y}_2[m] & |m| > M \end{cases},$$

which leads to complete characterization of $s(t)$ since the $2K + 1$ values of $\hat{y}_2[m]$ can be used with (19) to solve for $s(t)$. With $\hat{\mathbf{y}} = \hat{\mathbf{y}}_r, m > M$ we can use Algorithm 1 for exact denoising of $r(t)$.

V. CONCLUSION

We develop a method for super-resolution in phase space. The phase space transformation generalizes a number of well known transforms (see Table II). More precisely, we are concerned with recovery of spike trains from their low-pass samples. For this purpose, we filter the spike train with a kernel which is bandlimited in phase space. We show that even though we are dealing with a general class of parametric transformations, the low-pass samples are completely characterized by chirp-modulated Fourier series. Having made this link, we show that the recovery of spikes from their low-rate measurements can be cast as a total-variation minimization—a problem that can be tackled by convex programming. In closing, our work extends the recent results of [11] without altering the exact recovery condition. That said, the cut-off frequency is a function of the transform being used for investigation. Our work warrants future research, specially for the case of additive noise.

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