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Thin-walled Box Beam Bending Distortion Analytical Analysis

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ABSTRACT

In this paper, an analytical method for determining the stress-deformation condition of thin-walled box beam subjected to bending condition was developed. Solution by Vlasov method from a stationary condition complementary energy in Stress-form was developed and its adaptation and application through computer software MAPLE was demonstrated. Quick results can be obtained for various geometries, material properties and loading conditions enhancing parametric studies in design. Distortion effect (or shear lag effect) due to flexural bending was also included.

Keywords: *Cross-Section Distortion, Thin-Walled Beam Bending, Shear Lag, Analytical Method*

Nomenclatures

E, G	=	modulus of elasticity and rigidity of thin walled panel, respectively
M	=	poisson ratio

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x, y, z	=	dimensional coordinates with respect to width, height and length of the thin panel beam
s	=	curvilinear coordinates with respect to contour of the thin panel
w_y	=	distributed load in y-direction
V_y	=	transversal shear force in the y-direction
M_x	=	flexural bending about axis x
M_z	=	torsional twisting about axis z
N_z, N_s, N_{zs}	=	stress resultants in thin panel, force per unit length
r	=	r-th panel
h_r	=	total thickness of the skin of the corresponding r-th panel
l_r	=	length along contour of the r-th panel
ρ	=	moment-arm of the shear flow q about axis z
ω	=	double of area of the analyzed contour

Introduction

In the preliminary design of thin-walled box beam it is desirable to have a reliable and simple analytical method for determining the stress-deformation condition, allowing direct link between parameters of design and the influence of these parameters on the stressed condition.

Engineering stress solutions of thin walled box beam have been developed by many researchers before. However, in this paper, solution by Vlasov method from a stationary condition complementary energy in Stress-form instead of Displacement-form was developed and its adaptation and application through computer software MAPLE was demonstrated. With the use of computer program, quick results can be obtained for various geometries, material properties and loading conditions enhancing parametric studies in the design. Distortion effect (or shear lag effect) due to flexural bending was also included.

Literature background

The theoretical formulation of linear elastic thin-walled beams was derived by Vlasov [1]. Many contemporary researchers are currently working to improve or apply this theory in their works to suit their applications. Some researchers developed fully closed form analytical method solutions while others mixed them with finite element method. In [3], [4] various complex calculation models of materials were described in detail.

Back and Will [5] presented a new finite element for the analysis of thin-walled open beams with an arbitrary cross section based on Timoshenko beam theory and Vlasov theory which included both flexural shear deformations and warping deformations.

Pavazza and Blagojevic [6] considered analytical approach to the problem of the cross-sectional distortion of prismatic beams with closed rectangular thin-walled cross-sections subjected to bending with influence of shear effect.

Obraztsov [2] presented analytical solution of prismatic moment-free (membrane) linear elastic homogeneous closed thin panel caisson using Vlasov method solved in term of displacement. The solution was derived from Lagrange variational principle using potential strain energy. Effects of warping due to torsion and flexural bending were accounted and also the caisson was subjected to variable transversal shear, flexural bending and torsional moment.

In this paper, an analytical method of prismatic closed section beam box made of thin isotropic materials was developed allowing analytical solution approach which is convenient for design analysis in engineering. The problem was solved in stress form like in [3] using Vlasov method from a condition of stationary complementary energy. Distortion effect (or shear lag effect as shown in Figure 1) due to flexural bending was also included. The beam box analytical model can be subjected to variable transversal distributed loads (w_y) with their consequential effects of shear load (V_y) and flexural bending moment (M_x) typical of many engineering loads on beam structure.

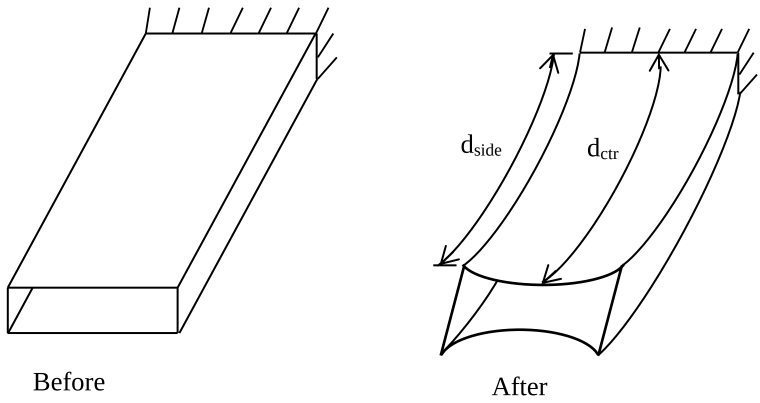

Figure 1: Shear lag effect of box beam after bend-up by load, contraction d_{side} differs from d_{ctr}

Analytical Method Formulation

Let consider a doubly-symmetric prismatic thin-walled box beam made of 4 membrane structural panels and on it operate only normal stress resultants N_z, N_s and shear stress resultant N_{zs} (Figure 2). Origin of coordinate axes x, y and z is at the centroid of the section and also x and y axes are parallel to the sides of the box section. Curvilinear coordinate s is also introduced for use with the contour integral as applied in [11, 12]. Note that s can be defined as a function of x and y.

Then the equations of equilibrium can be written like equations 3.90a and 3.90b in [7] as Equation 1.

$$\frac{\partial N_z}{\partial z} + \frac{\partial N_{zs}}{\partial s} = 0 \quad , \quad \frac{\partial N_{zs}}{\partial z} + \frac{\partial N_s}{\partial s} = 0 \tag{1}$$

All stresses will be the result of loads due to, w_y with its consequential effects of shear V_y and flexural bending M_x (Figure 2).

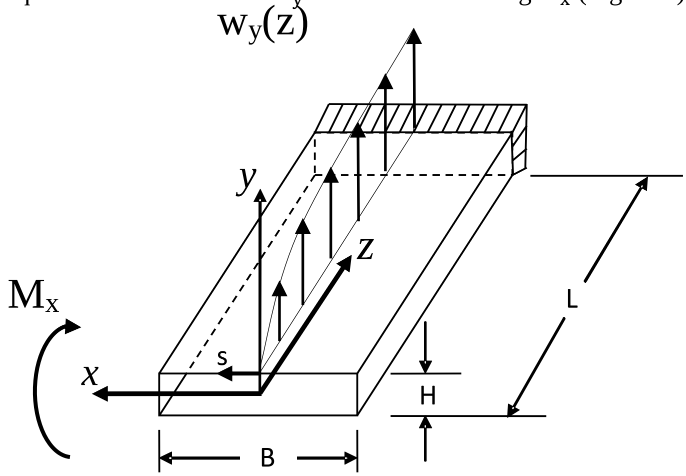

Figure 2: Box beam under transversal distributed load

Using equation (1), it is possible to express stress resultants N_{zs} and N_s in term of longitudinal stress resultant N_z as Equation 2.

$$N_{zs} = -\int \frac{\partial N_z}{\partial z} ds + q_0(z)$$

$$N_s = \iint \frac{\partial^2 N_z}{\partial z^2} ds^2 - [q_0'(z) \int ds] + n(z) \tag{2}$$

Herein after the symbol prime means a derivative with respect to coordinate z from the functions which depend only on z; $q_0(z)$ and $n(z)$ are functions of integration with respect to shear flow and contour stress resultants and provide displacement compatibility in the contour of the prismatic thin panel.

The longitudinal stress resultant N_z is defined as the products of functions $X_i(z)$ and $\varphi_i(s)$ to determine the stress distribution in the prismatic thin membrane panel. The first three functions $X_1(z)$, $X_2(z)$and $X_3(z)$ take into account the simple beam theory components of the elastic stress, i.e. $N_{z,SB} = (X_1 + X_2 y + X_3 x)\, h_i$. They are assigned as multipliers for obtaining the traditional geometrical parameters of the section and an additional function $X_4(z)$ takes into account of distortion section in the clamping area of the structure due to bending about axis x. Then the general representation of N_z and N_{zs} can be written down as Equation 3.

$$N_z = N_{z,SB} + X_4\varphi_4$$

$$N_{zs} = -\int (N'_{z,SB} + X'_4\varphi_4)\, ds + q_0(z) \qquad (3)$$

where

$$\varphi_4 = \overline{\varphi_4} + C_1 N_{z,SB} \qquad , \qquad \overline{\varphi_4} = x^2 y$$

$$C_1 \equiv \text{ orthogonaliztion coefficient}$$

Interesting to note that in case of simple beam (denoted by subscript '$_{,SB}$') where distortion is disregarded then Equation 3 becomes:

$$N_{zs,SB} = -\int N'_{z,SB}\, ds + q_0(z)$$

The values of the first three functions $X_1(z)$, $X_2(z)$ and $X_3(z)$ of $N_{z,SB}$, have the usual appearance of the solution of simple beam theory problem. As in [3] the normal resultant stresses (N_z, N_s), and shear resultant stress (N_{zs}) are derived based on transversal shear V_y and bending moment M_x.

$$V_y = \int w_y\, dz \qquad \& \qquad M_x = \int V_y\, dz$$

From simple thin-walled closed section panel beam subjected to transverse shear and torsion, normal and shear stress resultant from [2], [4] give Equation 4 and 5.

$$N_{z,SB} = -\frac{M_x}{I_x} y h_i \tag{4}$$

$$N_{zs,SB} = q_V(z) + q_0(z) \tag{5}$$

where

$$M_x = \int_0^z V_y(z)dz \quad , \quad V_y = \int_0^z w_y(z)dz$$

$$q_V = \frac{V_y Q_x^{rel}}{I_x} \quad , \quad Q_x^{rel} = \int_0^s \bar{y} ds$$

$$I_x = \text{area moment of inertia taken at the centroid}$$

and comparing with Equation 3 also note that

$$q_V(z) = -\int N'_{z,SB}\, ds$$

Substitute into Equation 3 become

$$N_{zs} = q_V(z) + q_0(z) - \int (X'_4 \varphi_4)\, ds$$

Furthermore, by using Equation 5 yields

$$N_{zs} = N_{zs,SB} - \int (X'_4 \varphi_4)\, ds$$

Any two nontrivial functions u(x) and v(x) are said to be orthogonal if [8].

$$\langle u,v \rangle = \int u \cdot v\, dx = 0$$

As was done in [2], orthogonalization of function $\overline{\varphi_4}$ with expression $N_{z,SB}$, is carried out such that the function is self-equilibrium (self-balanced) in the section of the thin panel and only influenced by redistribution of stress resultants due to distortional constraints of the contour in the clamped (or built-in) area. Then new functions can be written down as Equation 6.

$$\varphi_4 = \overline{\varphi_4} + C_1 N_{z,SB} \tag{6}$$

Coefficient of orthogonalization, C_1 is obtained by satisfying the condition:

$$\oint N_{z,SB}\varphi_4\,ds = 0$$

In the closed section thin membrane panel that is loaded with the given moment and force, stress resultant N_s is one order less than the other stress resultants. Therefore in the calculations only stress resultants N_z and N_{zs} are considered. Next the unknown function $X_4(z)$ is found using the Variational Principle of the Least Work. In this case Potential Energy is written down in the form:

$$U = \int_0^L \left[\oint \left(\frac{N_z^2}{2E_zh} + \frac{N_{zs}^2}{2G_{zs}h}\right) ds\right] dz$$

$$U = \frac{1}{2}\int_0^L \left[\oint \left(\frac{N_z^2}{E_zh} + \frac{N_{zs}^2}{G_{zs}h}\right) ds\right] dz$$

Let

$$C_{11} = {}^{1}\!/_{E_zh} \qquad \text{and} \qquad C_{33} = {}^{1}\!/_{G_{zs}h}$$

$$U = \frac{1}{2}\int_0^L \oint (C_{11}N_z^2 + C_{33}N_{zs}^2)dsdz$$

Let stress function,

$$\Phi = \frac{1}{2}\oint [C_{11}N_z^2 + C_{33}N_{zs}^2]ds$$

$$U = \int_0^L \Phi dz$$

Recalling

$$N_z = N_{z,SB} + X_4\varphi_4$$

$$N_{zs} = q_V - X_4' \int \varphi_4 ds + q_0(z)$$

$q_0(z)$ is written using the following familiar torsional moment equilibrium relationship,

$$M_z = \oint N_{zs}\rho ds$$

But, since torsional moment M_z is not considered, therefore:

$$0 = \oint N_{zs}\rho ds$$
$$0 = \oint \left(q_V - X_4' \int \varphi_4 ds + q_0(z) \right) \rho ds$$
$$0 = \oint q_V \rho ds - X_4' \oint \left(\int \varphi_4 ds \right) \rho ds + \oint q_0(z) \rho ds$$
$$\oint q_0(z) \rho ds = -\oint q_V \rho ds + X_4' \oint \left(\int \varphi_4 ds \right) \rho ds$$
$$q_0(z)\omega = \oint q_0(z) \rho ds$$
$$q_0(z) = \frac{1}{\omega}\left[-\oint q_V \rho ds + X_4' \oint \left(\int \varphi_4 ds \right) \rho ds \right]$$
$$q_0(z) = -\frac{1}{\omega}\oint q_V \rho ds + X_4' \frac{1}{\omega}\oint \left(\int \varphi_4 ds \right) \rho ds$$

and let

$$q_{Mzv} = q_V - \frac{1}{\omega}\oint q_V \rho ds$$
$$b_4 = \int \varphi_4 ds - \frac{1}{\omega}\oint \left(\int \varphi_4 ds \right) \rho ds$$

therefore

$$N_{zs} = q_{Mzv} - X_4' b_4$$

As a result,

$$U = \frac{1}{2}\int_0^L \oint \left\{ C_{11}\left[N_{z,SB} + X_4\varphi_4\right]^2 + C_{33}[q_{Mzv} - X_4' b_4]^2 \right\} dsdz$$
$$\Phi = \frac{1}{2}\oint \left\{ C_{11}\left[N_{z,SB} + X_4\varphi_4\right]^2 + C_{33}[q_{Mzv} - X_4' b_4]^2 \right\} ds$$

A necessary condition that X_4 minimize $U(X_4)$ is that the Euler equation:

$$\frac{d}{dz}\frac{\partial \Phi}{\partial X_4'} - \frac{\partial \Phi}{\partial X_4} = 0$$

is satisfied. So differentiate Φ accordingly and substitute into the necessary condition,

$$\frac{d}{dz}\left\{ \oint [-C_{33}(q_{Mzv} - X_4' b_4) b_4] ds \right\} - \oint \left[C_{11}\left(N_{z,SB} + X_4\varphi_4\right)\varphi_4 \right] ds = 0$$

Continue to differentiate with respect to z,

$$\oint\left[-C_{33}\left(b_4 q'_{Mzv} - b_4^2 X''_4 - 2b_4 b'_4 X'_4 + b'_4 q_{Mzv}\right)\right]ds - \oint\left[C_{11}\left(N_{z,SB} + X_4(z)\varphi_4\right)\varphi_4\right]ds = 0$$

And expanding

$$\oint C_{33} b_4^2 X''_4 ds + \oint C_{33} 2b_4 b'_4 X'_4 ds + \oint C_{11} X_4 \varphi_4^2 ds - = \oint C_{11} N_{z,SB}\varphi_4 ds + \oint C_{33} b_4 q'_{Mzv} ds + \oint C_{33} b'_4 q_{Mzv} ds$$

As b_4 is independent of z, therefore $b'_4 = 0$ and also since X_4 and X''_4 are independent of s, they can be taken out of the integrations.

$$X''_4 \oint C_{33} b_4^2 ds - X_4 \oint C_{11}\varphi_4^2 ds = \oint C_{11} N_{z,SB}\varphi_4 ds + \oint C_{33} b_4 q'_{Mzv} ds$$

And simplifying,

$$A_{11} X''_4 - A_{12} X_4 = B_1 + B_2$$

where

$$A_{11} = \oint C_{33} b_4^2 ds \quad , \quad A_{12} = \oint C_{11}\varphi_4^2 ds,$$

$$B_1 = \oint C_{11} N_{z,SB}\varphi_4 ds \quad , \quad B_2 = \oint C_{33} b_4 q'_{Mzv} ds$$

In short, after substitution in the expression of energy the stresses as in Equation 3 and performing integration on the contour, the energy function is obtained as $U = \int_0^L \Phi(X_4, X'_4, z)dz$ and minimization of this function produces a differential equation of displacement compatibility for determination of unknown function $X_4(z)$. And the natural boundary condition (condition of clamping) for determining constants of the solution at $z = L$ derived from calculus of variations is as Equation 7.

$$\frac{\partial\Phi}{\partial X'_4} = 0 \quad \rightarrow \quad \oint C_{33}\, q_{Mzv} b_4 ds - X'_4 \oint C_{33} b_4^2 ds = 0 \qquad (7)$$

And at $z = 0$, static equilibrium boundary condition is imposed.

$$\oint N_z y ds = M_x' = 0$$

Thus after solving the differential equation and applying the boundary conditions the solution for N_z and N_{zs} can be obtained.

Illustrative examples

To demonstrate the validity of the analytical Stress-form method developed, an exemplary structure was analyzed using it and was compared to two other methods, i.e. Displacement-form Method and Finite Element Method.

Consider a rectangular prismatic box beam loaded by transversal shear and flexural bending as follows: $w_0 = 0.01\ \mathrm{N/m}$, $V_y = 0.01z\ \mathrm{N}$, $M_x = 0.005z^2\ \mathrm{N \cdot m}$. The dimensions of the rectangular prismatic beam box are equal to: length = 5.0 m, width = 1.0 m, height = 0.2 m. The panels have uniform thickness of, therefore let $h_i = h = 0.01$ m.

Characteristics of material: elastic modulus, $E = 200.0\ \mathrm{GPa}$, shear modulus $G = 79.3$ GPa and Poisson ratio, $\mu = 0.3$.

Currently developed method using MAPLE

For analytical solution in finding constants of the problem, at $z = 0$ static equilibrium boundary conditions are imposed and at $z = L$ natural boundary conditions in the form (7) are used.

Solving using MAPLE the following stress along z –coordinate is obtained at $x = 0.5$ and $y = -0.1$.

$$\sigma_z = {}^{N_z}/_{h} = 2.3437\, z^2 + 1.1526 \cdot 10^{-7}\ e^{3.7693\, z} - 0.76769\, e^{-3.7693\, z} - 0.93663$$

This equation is plotted in Figure 4.The solution of the stress at the clamped end (at $z = 5$) and along the horizontal (x-coordinate) of the bottom panel (at $y = -0.1$) is:

$$\sigma_z = {}^{N_z}/_{h} = 48.564 + 106.98\, x^2$$

and its related curve is shown in Figure 5. Furthermore, solution of the stress at the clamped end (at $z = 5$) and along vertical (y-coordinate) of on the side panel (at $x = 0.5$) is:

$$\sigma_z = {}^{N_z}/_{h} = -753.10\, y$$

and its related curve is shown in Figure 6. For verification of these results, two methods were used, which are Displacement-form Method as in [2] and Finite Element Method.

Displacement-form Method

The following Equation 8 was derived by method of Displacement-form shown in [2].

$$\sigma_z = -\frac{w_0 z^2}{2I_x}\varphi_2 + \frac{c w_0}{k_1^2 I_{1\varphi}}\left[1 - \cosh(k_1 z) + \begin{pmatrix}\tanh(k_1 L) \\ -\frac{k_1 L}{\cosh(k_1 L)}\end{pmatrix}\sinh(k_1 z)\right]\varphi_3 \tag{8}$$

where

$$I_x = H^2\left(\frac{A_1}{6} - \frac{A_2}{2}\right) \quad , \quad A_1 = Hh \quad , \quad A_2 = Bh$$

$$c = \frac{HB^2 A_2}{6I_x} \quad , \quad I_{1\varphi} = \frac{1}{15}A_2 B^4 - c^2 I_x$$

$$k_1^2 = \frac{2B^2 G A_2}{3EI_{1\varphi}}$$

$$\varphi_2 = -\frac{H}{2} \quad , \quad \varphi_3 = \frac{B^2}{4} - \left(\frac{B}{2}\right)^2 - c\frac{H}{2}$$

The curve representing this Displacement-form equation is shown in Figure 4.

Finite Element Method

The Finite Element (FE) model as shown in Figure 3a was developed and analyzed using SOLIDWORKS software [10]. The results of FE analysis are as shown in Figure 4, 5 and 6.

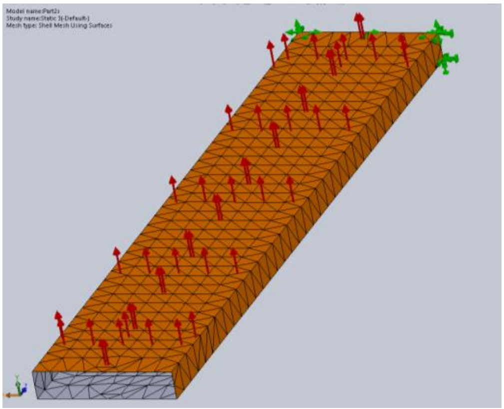

Figure 3a: Finite element model

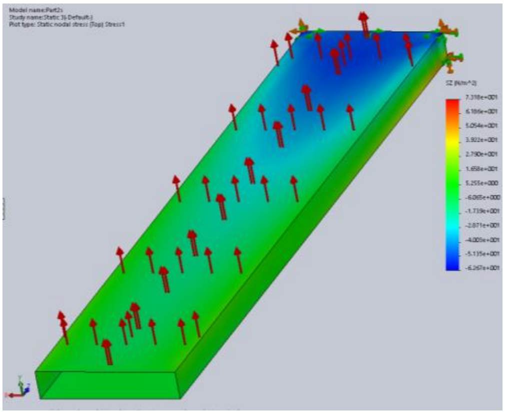

Figure 3b: Finite element analysis result

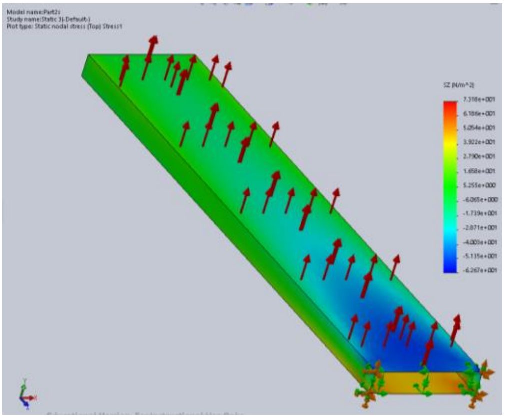

Figure 3c: Finite element analysis result

Discussion

From the results of calculation presented in Figure 4, it is visible that the suggested Stress-form analytical approach developed here gives good agreement with respect to the values of stress in the thin panel in comparison with established Displacement-form analytical approach from [2]. However, when compared to the FE method approach in Figure 4, 5 and 6, the Stress-form approach shows slightly conservative results at the intermediate stress area but are very close matching at the high stress area.

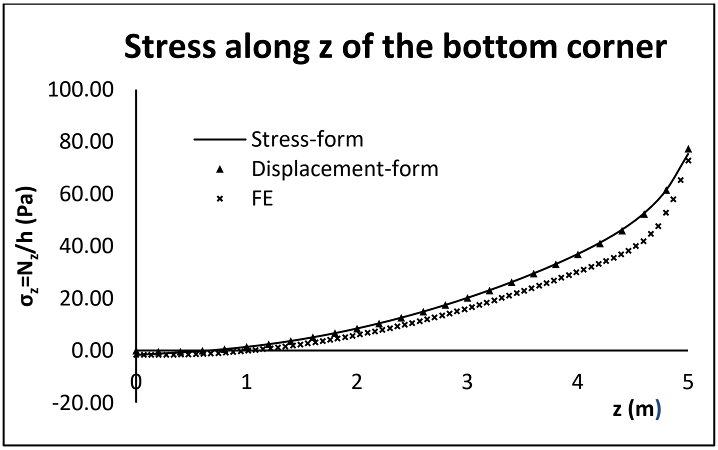

Figure 4: Stress distribution along z vs. distance z

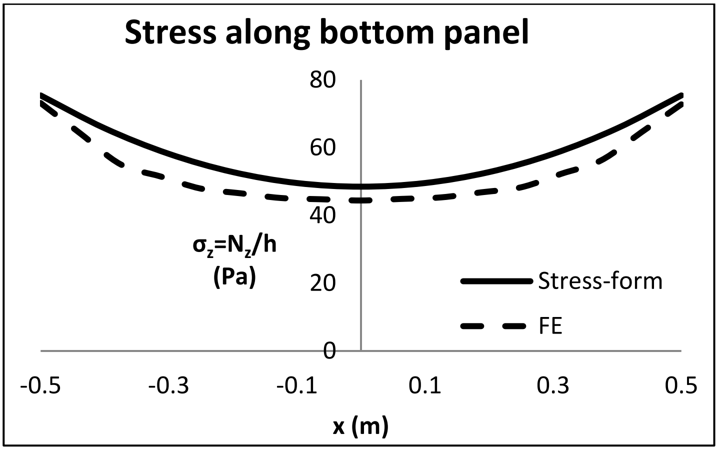

Figure 5: Stress distribution along bottom panel vs. distance x at clamping area

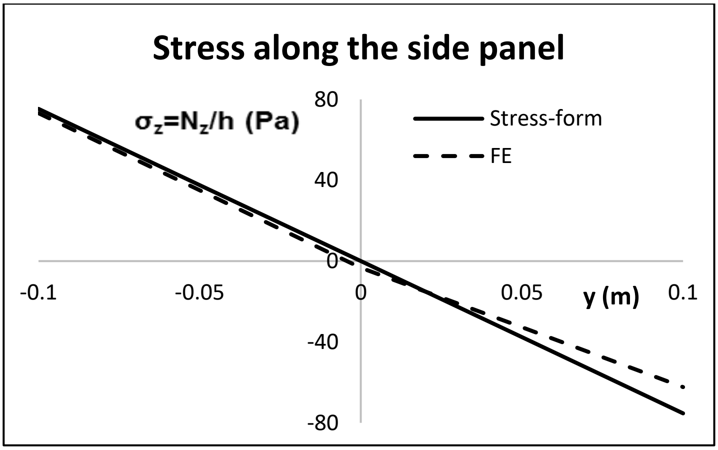

Figure 6: Stress distribution alongside panel versus distance y at the clamping area

Conclusion

Analytical method of prismatic closed section box beam that is made of thin-walled isotropic material panels subjected to transverse load was developed. This paper uses convenient approach to determine the stresses involved in engineering design analysis. Distortion effect (or shear lag effect) due to flexural bending was also included. The problem was solved in Stress-form using Vlasov method from a condition of stationary complementary energy. Sample calculation performed using the developed method here and method of Displacement-form shown in [2] shows that the results of both methods coincided well but slightly conservative in comparison to Finite Element Method.

Acknowledgements

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