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Algorithms for solving discrete control problems on networks*

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Abstract

We consider the discrete optimal problem on networks with integral-time cost criterion by a trajectory when the starting and final states of the system are fixed. A polynomial-time algorithm for solving this problem is proposed.

1 Introduction

In this paper we consider discrete control problems on networks from [1,2]. The dynamics of the system in such problems is described by a directed graph of passages [2]. The vertices of the graph in such problems correspond to the states of system and its edges signify the possibility of the system passage from one state to another. Moreover, on the edges of the graph the cost functions are defined, which depend on time and express the expenditure or the income when the dynamic system passes from one state to another.

We study the discrete control problem on networks in the case when the cost functions on edges of the graph are positive and non-decreasing. A new polynomial-time algorithm for solving the problem is proposed. The algorithm can be used for checking the optimization principle for dynamic networks[3].

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2 Problem formulation and dynamical programming method

We shall consider the optimal control problem on networks from [3,4]. Let L be a dynamical system with finite set of states X, $|X|=N$ and at every discrete moment of time $t = 0, 1, 2, \ldots$ the state of the system L is $x(t) \in X$. Note that here we associate $x(t)$ with an abstract element. Two states x_s *and* x_f are chosen in X, where x_s is a starting state of the system L, $x_s = x(0)$ *and* x_f is the final state of the system, i.e. x_f is the state into which the system should be brought. The dynamics of the system is described by a directed graph of passages $G=(X,E)$, $|E| = m$, an edge $e=(x,y)$ which signifies the possibility of passage of the system L from the state $x=x(t)$ to the state $y=x(t+1)$ at any moment of time $t = 0, 1, 2, \ldots$. That means that the edges $e=(x,y) \in E$ can be regarded as the possible values of the control parameter $u(t)$ when the state of the system is $x=x(t)$, $t = 0, 1, 2, \ldots$. The next state $y=x(t+1)$ of the system L is determined uniquely by $x=x(t)$ at the moment of time t and an edge $e=(x,y) \in E(x)$, where

$$E(x) = \{(x,y) \in E \mid y \in X\}$$

So $E(x)=E(x(t))$ correspond to the admissible set $U_t(x(t))$ for the control parameter $u(t)$ at every moment of time t. To each edge $e=(x,y)$ a function $c_e(t)$ is assigned, which reflects the cost of system's passage from the state $x(t) = x \in X$ to the state $x(t+1) = y \in X$ at any moment of time $t = 0, 1, 2, \ldots$.

We consider the discrete optimal problem on networks [1,2,3] for which the sequence of system's passages

$$(x(0), x(1)), (x(1), x(2)), \ldots, (x(T-1), x(T)) \in E$$

which transfers the system L from the state $x_s = x(0)$ to the state $x_f = x(T)$ with minimal integral-time cost of the passages by a trajectory $x_s = x(0), x(1), \ldots, x(T) = x_f$ must be found.

Here may be two variants of the problem:

- the number of the stages (time T) is fixed

- T is unknown and it must be found

Let us consider that T is fixed. Denote by

$$F_{x_s x_f}(T) = \min_{x_s=x(0),x(1),\dots,x(T)=x_f} \sum_{t=0}^{T-1} c_{x(t),x(t+1)}(t)$$

the minimal integral-time cost of system's passages from x_s to x_f. If the state x_f couldn't be reached by using T stages, i.e. in G there exist no path $P^*(x_s, x_f)$ from x_s to x_f which contains exactly T edges, then we put $F_{x_s x_f}(T) = \infty$.

It is easy to observe that using the dynamical programming method we could tabulate the values $F_{x_s x(t)}(t),\ t = 0, 1, 2, \dots, T$ ($F_{x_s x(0)}(0) = 0$) because for $F_{x_s x(t)}(t)$ the following recurrent formula can be written

$$F_{x_s x(t)}(t) = \min_{x(t-1)\in X_G^-(x(t))} \{F_{x_s x(t)}(t-1) + c_{x(t-1),x(t)}(t-1)\},$$

where

$$X_G^-(y) = \{x \in X \mid e = (x, y) \in E\}$$

So, if T is fixed, then problem can be solved in time $O(N^2T)$ (here we do not take into consideration the number of operations for calculations the value of functions $c_e(t)$ for given t).

In the case when T is unknown we shall consider $T \in [T_1, T_2]$ where T_1 and T_2 are given. The problem in this case can be reduced to $T_2 - T_1 + 1$ problems with $T = T_1,\ T = T_1 + 1, \dots,\ T = T_2$ respectively. Obviously, for positive and non-decreasing functions on edges the problem can be solved in time $O(N^3)$.

Further we shall study the problem when T is not fixed. We denote by T^* the optimal time for the following discrete control problem.

3 The main results and algorithm

We have formulated the problem for positive and non-decreasing functions $c_e(t)$ on edges $e \in E$ which coincides with the discrete optimal control problem on G with starting state x_s and final state x_f. In this case the optimal trajectory $x_s = x(0), x(1), \dots, x(T) = x_f$ corresponds in G to the directed path $P^*(x_s, x_f)$ from x_s to x_f. We call this path the optimal path for the dynamic network. For the path $P^*(x_s, x_f)$ contains no more than *N-1* edges, the problem can be solved in finite time by using dynamical programming techniques.

Here we propose a more simple algorithm for solving this problem.

3.1 An algorithm for solving the problem with given optimal value of stages T^*

Algorithm 1

Preliminary step(Step 0)

Set $X^0 = x_f, E^0 = \emptyset$. Assign to every vertex $x \in X$ two labels $t(x)$ and $R(x)$ as follows:

$$R(x_f) = 0, \quad t(x_f) = T,$$

$$R(x) = \infty, \quad t(x) = \infty \ \forall x \in X \backslash \{x_f\}$$

General step(Step k)

Find the set

$$X^k = \{x \in X \backslash X^{k-1} \mid (x, y) \in E, y \in X^{k-1}\},$$

$$E^{'} = \{(x^*, y^*) \in E(X^{k-1}) \mid c_{(x^*,y^*)}(t(y^*) - 1) + R(y^*)\} = \min_{y \in X^{k-1}} \min_{x \in V^+(y) \backslash X^{k-1}} \{c_{(x,y)}(t(y) - 1) + R(y)\}$$

where

$$E(X^{k-1}) = \{(x, y) \in E \mid y \in X^{k-1}, x \in X \backslash X^{k-1}\},$$

$$V^+(y) = \{x \in X \mid (x, y) \in E\}$$

Find the set of vertices

$$X^{'} = \{x \in X^k \mid \exists y \ , \ (x, y) \in E^{'}\}$$

For every $x^{'} \in X^{'}$ select one edge $(x^{'}, y^{'}) \in E^{'}$ and build the union $E_k^{'}$ of such edges. After that change the labels $t(x^{'})$ and $R(x^{'})$ for every vertex $x^{'} \in X^{'}$ are as follows

$$R(x^{'}) = R(y^{'}) + c_{(x^{'},\ y^{'})}(t(y^{'}) - 1),$$

$$t(x^{'}) = t(y^{'}) - 1, \quad \forall (x^{'}, y^{'}) \in E_k^{'}$$

Replace the set X^k by $X^{k-1} \bigcup X^{'}$ and E^k by $E^{k-1} \bigcup E_k^{'}$. Fix sets $Y = X^k$, $E^* = E^k$ and the tree $H = (Y, E^*)$. If ($x_s \in X^k$ or $k = T$) then STOP, otherwise go to the next step $k+1$.

Note that the tree $H = (Y, E^*)$ contains paths from every $x \in Y$ to x_f. The labels $R(x), x \in Y$, indicate the cost of optimal path from $x \in Y$ to x_f of the optimal path $P^*(x_s, x_f)$ which contains x and $t(x)$ represents the time moment at which this path passes through vertex x.

In [3,4] have been shown that if the optimization principle for dynamical network is satisfied then a more effective algorithm for solving the optimal control problem on network can be elaborated. Note that the optimization principle on dynamic network is satisfied if the following condition holds: Let us consider that the cost functions $c_e(t), \quad e \in E$, in the dynamic network have the property that if $P^*(x_s, x)$ is an arbitrary optimal path from x_s to x which can be represented as $P^*(x_s, x) = P_1^*(x_s, y) \bigcup P_2^*(y, x)$, where $P_1^*(x_s, y)$ and $P_2^*(y, x)$ have no common edges, than a leading part $P_1^*(x_s, y)$ of the path $P^*(x_s, x)$ is also an optimal path of the problem in G with given starting state x_s and final state y.

Theorem 1 *Let $(G, c(t), x_s, x_f)$ be a dynamic network, where the vector-function $c(t) = (c_{e_1}(t), c_{e_2}(t), \dots, c_{e_m}(t))$ has positive and bounded components for $t \in [0, N-1]$. If $T = T^*$ then $k = T^*$ and $t(x_s) = 0$. Let $H^* = (X, E^*)$ be the tree obtained by **Algorithm 1**, then an arbitrary path $P^*(x, x_f)$ in the H^* represents the optimal path from x to x_f for optimal control problem on G.*

Proof. We prove the theorem by using the induction principle on the number of steps k of the algorithm. In the case when $k = 0$ the theorem is evident.

Let us consider the theorem holds for any $k \leq r$ and let us show it is true for $k = r + 1$. If $H^r = (X^r, E^r)$ is the tree obtained after r steps and $H^{r+1} = (X^{r+1}, E^{r+1})$ is the tree obtained after $r + 1$ steps of the algorithm, then $X^0 = X^{r+1} \backslash X^r$ and $E^0 = E^{r+1} \backslash E^r$ represents the vertex set and the edge set obtained by the algorithm at the step $r+1$. Let us show that if $y^{'}$ is an arbitrary vertex of X^0, then in H^{r+1} the unique directed path $P^*(y^{'}, x_f)$ from $y^{'}$ to x_f is optimal. Indeed, if this is not the case, then there exists an optimal path $Q(y^{'}, x_f)$ from $y^{'}$ to x_f, which does not contain the edge $e = (y^{'}, z^{'}) \in E^0$. The path $Q(y^{'}, x_f)$ can be represented as $Q(y^{'}, x_f) = Q^1(y^{'}, z^{'}) \cup \{(z^{'}, x^{'})\} \cup Q^2(x^{'}, x_f)$, where $x^{'}$ is the first vertex of the path $Q(y^{'}, x_f)$ belonging to X^r when we pass from $y^{'}$ to x_f. Let us show that

$$cost(Q(y^{'}, x_f)) > cost(P^*(y^{'}, x_f))$$

where

$$cost(Q(y^{'}, x_f)) = \sum_{t \, = \, T-m_Q-1}^{T-1} c_{e_t}(t),$$

$e_{T-m_Q-1}, e_{T-m_Q}, \ldots, e_{T-1}$ are the corresponding edges of the directed path $Q(y^{'}, x_f)$ when we pass from $y^{'}$ *to* x_f.

$$cost(P^*(y^{'}, x_f)) = \sum_{t \, = \, T-m_P-1}^{T-1} c_{e^{'}_t}(t),$$

$e^{'}_{T-m_P-1}, e^{'}_{T-m_P}, \ldots, e^{'}_{T-1}$ are the corresponding edges of the directed path $P^*(y^{'}, x_f)$ when we pass from $y^{'}$ *to* x_f.

Note by $R(x^{'}) = cost(Q^2(x^{'}, x_f))$

According to the algorithm, we can state

$$R(x^{'}) + c_{(y^{'}, \, x^{'})}(t(x^{'}) - 1) > R(z^{''}) + c_{(y^{'}, \, z^{''})}(t(z^{''}) - 1) = R(y^{'})$$

where $e^{'} = (y^{'}, z^{''})$ is the first edge of the path $P^*(y^{'}, x_f)$. Then

$$cost(Q^2(x^{'}, x_f) \cup \{(z^{'}, x^{'})\}) > cost(P^*(y^{'}, x_f))$$

$$R(x^{'}) + c_{(y^{'},\ x^{'})}(t(x^{'}) - 1) = cost(Q^2(x^{'}, x_f) \cup \{(y^{'}, x^{'})\})$$
$$R(x^{'}) = cost(P^*(x^{'}, x_f))$$

The cost functions $c_e(t), \forall e \in E$, are positive, therefore

$$cost(Q(y^{'}, x_f)) = cost(Q^1(y^{'}, z^{'}) \cup \{(z^{'}, x^{'})\} \cup Q^2(x^{'}, x_f)) \geq$$
$$cost(Q^2(x^{'}, x_f) \cup \{(z^{'}, x^{'})\}) > cost(P^*(y^{'}, x_f))$$

i.e. $Q(y^{'}, x_f)$ is not an optimal path from $y^{'}$ to x_f. That means that the tree $H^{r+1} = (X^{r+1}, E^{r+1})$ contains an optimal path from every $y^{'} \in X^{r+1}$ to x_f.
If $k = T^*$ then the fact that $t(x_s) = 0$ is evident (conform algorithm).

3.2 An algorithm for solving the problem when the number of stages is unknown

Algorithm 2

We study the problem when T is not fixed. In this case when T is unknown we shall assume $T \in [1, N-1]$. The problem in this case can be reduced to $N-1$ problems with $T = 1, T = 2, \ldots, T = N-1$ respectively. For every T^i fixed from [1,N-1] we calculate $t^i(x_s)$ and $R^i(x_s), \quad i \in [1, N-1]$.

Find
$$R^*(x_s) = \min_{i\ \in\ [1,N-1]} \{R^i(x_s) \mid t^i(x_s) = 0\},$$

which represented the minimal integral-time cost of system's passages from x_s to x_f and

$$T^* = \{T^i \in [1, N-1] \mid t^i(x_s) = 0 \ and \ R^i(x_s) = R^*(x_s)\}$$

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