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A note on the colorability of mixed hypergraph using k colors*

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Abstract

The colorability problem on mixed hypergraphs is discussed. A criterion of colorability of mixed hypergraph with k colors is given.

1 Introduction

A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is a set called the *vertex set* of $\mathcal{H}$, and each of $\mathcal{C}, \mathcal{D}$ is a list of subsets of X, the $\mathcal{C}$*-edges* and $\mathcal{D}$*-edges*, respectively. A *proper* k*-coloring* of a mixed hypergraph is a mapping from X to a set of k *colors* so that each $\mathcal{C}$*-edge* has two vertices with COMMON color and each $\mathcal{D}$-edge has two vertices with DIFFERENT colors. If $\mathcal{H}$ admits no proper coloring then it is called *uncolorable*. A *strict* k*-coloring* of a mixed hypergraph is a proper coloring using all k colors. The minimum number of colors in a strict coloring of $\mathcal{H}$ is its *lower chromatic number* $\chi(\mathcal{H})$; the maximum number of colors is its *upper chromatic number* $\bar{\chi}(\mathcal{H})$. Classical coloring theory of hypergraphs with edge set $\mathcal{E}$ is the special case where the set of $\mathcal{C}$*-edges* is empty and we color the mixed hypergraph $(X, \emptyset, \mathcal{E})$ [1]. Coloring of mixed hypergraphs was introduced in [3, 4].

Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be an arbitrary mixed hypergraph and $\mathcal{C} = \{C_1, C_2, \ldots, C_l\}$ and $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$ be two nonempty families of edges. We denote $I = \{1, 2, \ldots, l\}$ and $J = \{1, 2, \ldots, m\}$

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The colorability problem was first formulated in [4]: given a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, the problem to decide whether there exists at least one proper coloring of $\mathcal{H}$ is called the **colorability problem**.

The colorability problem represents a new type of problem in coloring theory since the number of colors is not specified. It contains, as a special case, the problem to decide whether a classic hypergraph admits a proper coloring with a given number of colors. Namely, any $\mathcal{D}$-hypergraph $\mathcal{H} = (X, \mathcal{D})$, is k-colorable if and only if the mixed hypergraph $\mathcal{H}' = (X, \binom{X}{k+1}, \mathcal{D})$ is colorable. The problem of characterizing uncolorable mixed hypergraphs was first formulated in [5, 6]. Some cases of the colorability problem appeared in [2, 3, 7].

As it is shown there, together with a general approach, quite different methods are required to determine the conditions for colorability in different classes of mixed hypergraphs. Nevertheless, one of the basic goals is to find the list of all minimal uncolorable mixed hypergraphs from a given class and describe the colorable structures in terms of forbidden subhypergraphs.

In this paper we give a criterion of colorability of mixed hypergraph using k colors. We show that the colorability problem with k colors on mixed hypergraph can be reduced to classical colorability problem using k colors on an auxiliary hypergraph.

2 The main result

For given hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ we construct an auxiliary hypergraph $\mathcal{H}' = (X, \mathcal{E})$. We show that the mixed hypergraph $\mathcal{H}$ is colorable with *k colors* if and only if hypergraph $\mathcal{H}$ is colorable with k color in classical sense.

Let us consider the coloring problem on mixed hypergraph $\mathcal{H}$ with *k colors*. We give a construction of the auxiliary hypergraph $\mathcal{H}' = (Y, \mathcal{E})$. Denote by $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ the mixed hypergraph, where $X = \{x_1, x_2, \ldots, x_n\}$, $\mathcal{C} = \{C_1, C_2, \ldots, C_l\}$, $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$. If $|C_i| \geq k+1$, $i = \overline{1, l}$ then $Y = X$ and $\mathcal{E} = \mathcal{D}$ in $\mathcal{H}'$. If in $\mathcal{H}$ there exist *$\mathcal{C}$-edges* C_i such that $|C_i| \leq k$ then for each of them we add a set of vertices $Y_i = \{y_i^i, y_2^i, \ldots, y_{t_i}^i\}$, where $t_i = k + 1 - |C_i|$. To $\mathcal{H}'$ we also add the edges of size two in the following way:

1) every pair of vertices from Y_i is an edge, i.e. $(y_k^i, y_l^i) \in \mathcal{H}'$;

2) every vertex y_l^i from Y_i is connected by edge of size two with every vertex of $\mathcal{C}$-*edge* C_i.

For given $\mathcal{C}$-*edge* C_i the set of edges of size two obtained according 1) 2) we denote by E_i.

So the set of vertices Y of the auxiliary hypergraph $\mathcal{H}'$ consists of set X and union of Y_i, i.e. $Y = \left(\bigcup_i Y_i\right) \bigcup X$. The set of edges $\mathcal{E}$ of the hypergraph $\mathcal{H}'$ consists of set of edges $\mathcal{D}$ and union of E_i, i.e. $\mathcal{E} = \left(\bigcup_i E_i\right) \bigcup \mathcal{D}$. Set $\mathcal{H}' = (Y, \mathcal{E})$.

The following theorem holds

Theorem 1 *The mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is colorable with k colors if and only if the auxiliary hypergraph $\mathcal{H}' = (Y, \mathcal{E})$ is colorable with k colors.*

Proof. Let the mixed hypergraph $\mathcal{H}$ be colorable with k colors and fix an arbitrary coloring φ. In $\mathcal{H}'$, the corresponding vertices $x \in X$ remain the same. In addition for every Y_i we use $k + 1 - |C_i|$ different colors which are missing in C_i.

Let us consider that the auxiliary hypergraph $\mathcal{H}' = (Y, \mathcal{E})$ is colored with k *colors*. It is easy to observe that if we delete the sets Y_i from $\mathcal{H}'$, then we obtain the set X and a coloring of vertices in $\mathcal{H}$ which is a proper coloring for the mixed hypergraph $H = (X, \mathcal{C}, \mathcal{D})$. Indeed if C_i is an arbitrary $\mathcal{C}$-*edge* then there exist at least two vertices with the same color in C_i what completes the proof.

$\square$

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