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Algorithms for finding optimal paths in network games with p players

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Abstract

We study the problem of finding optimal paths in network games with p players. Some polynomial-time algorithms for finding optimal paths and optimal by Nash strategies of the players in network games with p players are proposed.

1 Introduction

We study the problem of finding optimal paths in network games with p players, which generalizes the well-known combinatorial problem on the shortest paths in a weighted directed graph and the min-max paths problem in network games with two players [1–3]. This problem arose as an auxiliary one when studying cyclic games [2–4] and solving some of network transport problems [5]. We propose polynomial-time algorithms for finding optimal paths in network games and optimal by Nash strategies of players [6].

2 Problem formulation

Let $G = (V, E)$ be a directed graph with the vertex set V, $|V| = n$, and the edge set E, $|E| = m$, where p cost functions

$$c_1 : E \rightarrow R^1;\ c_2 : E \rightarrow R^1; \ldots\ c_p : E \rightarrow R^1$$

are defined on the edge set. Assume that a vertex $v_0 \in V$ is chosen so that for any vertex $v \in V$ there exists a directed path $P_G(v, v_0)$ from

v to v_0. Moreover, we divide the vertex set V into p disjoint subsets $V_1, V_2, \ldots, V_p$ ($V = \cup_{i=1}^{p} V_i,\ V_i \cap V_j = \emptyset, i \neq j$).

Let $s_1, s_2, \ldots, s_p$ be p maps defined on $V_1, V_2, \ldots, V_p$, respectively:

$$\begin{array}{ll} s_1 : v \to V_G(v) & \text{for } v \in V_1; \\ s_2 : v \to V_G(v) & \text{for } v \in V_2; \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ s_p : v \to V_G(v) & \text{for } v \in V_p, \end{array}$$

where $V_G(v)$ is the set of extremities of edges $e = (v, u)$, originating in v, i.e. $V_G(v) = \{u \in V |\ e = (v, u) \in E\}$. Denote by $T_s = (V, E_s)$ the subgraph generated by the edges $e = (v, s_i(v))$ for $v \in V \setminus \{v_0\}$ and $i = \overline{1, p}$. Obviously, for an arbitrary vertex $w \in V$ either a unique directed path $P_T(w, v_0)$ exists in T_s, or such a path does not exist in T_s. In the second case, if we pass through the edges from w, we get a unique directed cycle C_s.

For arbitrary $s_1, s_2, \ldots, s_p$ and $w \in V$ we define the quantities

$$H_w^1(s_1, s_2, \ldots, s_p),\ H_w^2(s_1, s_2, \ldots, s_p), \ldots,\ H_w^p(s_1, s_2, \ldots, s_p)$$

in the following way. If the path $P_T(w, v_0)$ exists in T_s, then put

$$H_w^i(s_1, s_2, \ldots, s_p) = \sum_{e \in P_T(w, v_0)} c_i(e), \quad i = \overline{1, p}.$$

If the directed path $P_T(w, v_0)$ from w to v_0 does not exist in T_s and $\sum_{e \in C_s} c_i(e) > 0$, then we put $H_w^i(s_1, s_2, \ldots, s_p) = \infty$; if $\sum_{e \in C_s} c_i(e) < 0$ we put $H_w^i(s_1, s_2, \ldots, s_p) = -\infty$. In the case when $\sum_{e \in C_s} c_i(e) = 0$ we consider that $H_w^i(s_1, s_2, \ldots, s_p) = \sum_{i \in P_s'} c_i(e)$, where P_s' is the directed path connecting w and the cycle C_s.

We consider the problem of finding the maps $s_1^*, s_2^*, \ldots, s_p^*$ for which

$$H_w^i(s_1^*, s_2^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_p^*) \leq$$

$$\leq H_w^i(s_1^*, s_2^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_p^*), \quad \forall s_i,\ i = \overline{1, p}.$$

So we study the problem of finding optimal by Nash solutions $s_1^*, s_2^*, \ldots, s_p^*$.

This problem can be interpreted as a dynamical game of p players with integral-time cost function, where $w = v(0)$ is the starting position of the game at the moment $t = 0$, and $v(1), v(2), \ldots \in V$ are the corresponding positions of players at the moments $t = 1, 2, \ldots$ If $w \in V_i$ then the move is done by the player i. The moves of players mean the passage from the position w to the position $v(1) = v_1$, so that $(w, v_1) = e \in E$. In the general case, at the moment t the move is done by the player i if $v(t) \in V_i$. The game can be finite or infinite. If the position v_0 was reached at the finite moment t, i.e. $v(t) = v_0$, then the game is finite and the cost of the position w for the player i is $p_i(w) = \sum_{\tau=1}^{t} c_i(v(\tau-1), v(\tau))$. If the position v_0 cannot be reached, then the cost of the position w for the player i is $p_i(w) = \lim_{t\to\infty} \sum_{\tau=1}^{t} c_i(v(\tau-1), v(\tau))$. Each player has the aim to minimize the cost of the position w.

So the functions

$$H_w^1(s_1, s_2, \ldots, s_p), H_w^2(s_1, s_2, \ldots, s_p), \ldots H_w^p(s_1, s_2, \ldots, s_p)$$

define a game in the normal form with p players. We name this game a c-game of p players on the network game $(G, V_1, V_2, \ldots, V_p, c_1, c_2, \ldots, c_p, w)$. If the c-game is given by the network game $(G, V_1, V_2, \ldots, V_p, c_1, c_2, \ldots, c_p, w)$, then we have a game in the positional form.

Note that if $V = V_1$ then we have the shortest path problem [9]. If $V = V_1 \cup V_2$ and $c_2 = -c_1$, then we have the min-max path problem on the network [4,8].

3 The main results

The maps $s_1, s_2, \ldots, s_p$ are named the strategies of players $1, 2, \ldots, p$, respectively. Let us show that if in the network game $(G, V_1, V_2, \ldots, V_p, c_1, c_2, \ldots, c_p, w)$ with the starting position w the functions $c_1, c_2, \ldots, c_p$

are positive then there exist the optimal strategies $s_1^*, s_2^*, \dots, s_p^*$ of players $1, 2, \dots, p$.

Theorem 1 *Let $(G, V_1, V_2, \dots, V_p, c_1, c_2, \dots, c_p, w)$ be a network game for which the vertex v_0 is attainable from any vertex $w \in V$ and the functions $c_1, c_2, \dots, c_p$ are positive. Then for the players $1, 2, \dots, p$ there exist optimal by Nash strategies $s_1^*, s_2^*, \dots, s_p^*$, and the graph $G_{s^*} = (V, E_{s^*})$ corresponding to these strategies has the structure of a directed tree with the root vertex v_0.*

Proof. We prove this theorem by using the induction on the number p of players in the c-game. The problem of finding the optimal by Nash strategies in the c-game in the case when $p = 1$ becomes the problem of finding the minimum path tree from the vertices $w \in V$ to the vertex v_0 in G with positive edge lengths $c_1(e)$, $e \in E$. For this problem, as it is well known, there exists the optimal solution, hence the theorem holds for $p = 1$.

Let us assume that the theorem holds for any $p \leq k$, $k \geq 1$, and let us show that it is true for $p = k + 1$.

Let us have the network game with $p = k+1$ players. We shall consider the problem of finding the optimal by Nash strategies of players $2, 3, \dots, p$, fixing the possible admissible strategies $s_1^1, s_1^2, \dots, s_1^q$ of the first player.

Let us note, that if the first player fixes his first possible strategy, i.e. $s_1 = s_1^1$, and if we consider the problem of finding the optimal by Nash strategies for the rest of the players, then in the positional form the obtained game will represent a c-game for $p - 1$ players, since the positions of the first player can be considered as the positions of any other player (we consider them as the positions of the second player).

So for $s_1 = s_1^1$ we obtain a new c-game with $p - 1$ players on the network game $(G^1, V_2^1, V_3, \dots, V_p, c_2^1, c_3^1, \dots, c_p^1, w)$ where $V_2^1 = V_1 \cup V_2$ and $G^1 = (V, E^1)$ is the digraph, obtained from G by deleting the edges $e = (u, v) \in E$ for which $u \in V_1$ and $v \neq s_1^1(u)$; $c_i^1 : E^1 \to R^1$ are the functions obtained respectively from the functions c_i as a result of the contraction of the set E to the set E^1, i.e. $c_i^1(e) = c_i(e)$, $\forall e \in E^1$, $i = \overline{2, p}$. If we consider this game in the normal form, then it is a game with $p-1$ players, determined by the cost functions $H_w^2(s_1^1, s_2, s_3, \dots, s_p)$, $H_w^3(s_1^1, s_2, s_3, \dots, s_p), \dots,$ $H_w^p(s_1^1, s_2, s_3, \dots, s_p)$, $s_2 \in S_2, s_3 \in S_3, \dots, s_p \in S_p$, where $S_2, S_3, \dots, S_p$ are the sets of admissible strategies of players $2, 3, \dots, p$, respectively. According to the induction assumption, for this game with $p-1 = k$ players there exist optimal by Nash strategies $s_2^{1^*}, s_3^{1^*}, \dots, s_p^{1^*}$ and the digraph $G_{s_1^*} = (V, E_{s_1^*})$ which corresponds to the strategies $s_1^1, s_2^{1^*}, s_3^{1^*}, \dots, s_p^{1^*}$ has the structure of a directed tree with the root vertex v_0.

In an anologous way we consider the case when the first player fixes his second possible strategy s_1^2, i.e. $s_1 = s_1^2$. Then, according to the induction assumption, we find the optimal by Nash strategies $s_2^{2^*}, s_3^{2^*}, \dots, s_p^{2^*}$ of players $2, 3, \dots, p$ in the c-game given in the normal form, which is determined by the cost functions $H_w^2(s_1^2, s_2, s_3, \dots, s_p)$, $H_w^3(s_1^2, s_2, s_3, \dots, s_p), \dots,$ $H_w^p(s_1^2, s_2, s_3, \dots, s_p)$ and the digraph $G_{s_2^*} = (V, E_{s_2^*})$, corresponding to the strategies $s_1^2, s_2^{2^*}, s_3^{2^*}, \dots, s_p^{2^*}$, has the structure of a directed tree with the root vertex v_0.

Further we consider the case when the first player fixes his third possible strategy and we find the optimal strategies $s_2^{3^*}, s_3^{3^*}, \dots, s_p^{3^*}$ and the directed tree $G_{s_3^*} = (V, E_{s_3^*})$, which corresponds to the strategies $s_1^3, s_2^{3^*}, s_3^{3^*}, \dots, s_p^{3^*}$.

Continuing this process we find the following sets of strategies of players $1, 2, \dots, p$

$$\begin{array}{c} (s_1^1, s_2^{1^*}, s_3^{1^*}, \dots, s_p^{1^*}), \\ (s_1^2, s_2^{2^*}, s_3^{2^*}, \dots, s_p^{2^*}), \\ \dots\dots\dots\dots\dots\dots\dots \\ (s_1^q, s_2^{q^*}, s_3^{q^*}, \dots, s_p^{q^*}) \end{array}$$

and the corresponding directed trees $G_{s_1^*}, G_{s_2^*}, \dots, G_{s_q^*}$ with the root vertex v_0.

Among all these sets of players' strategies in the c-game we choose the set $(s_1^{j^*}, s_2^{j^*}, s_3^{j^*}, \dots, s_p^{j^*})$ for which

$$H_w^1(s_1^{j^*}, s_2^{j^*}, \dots, s_p^{j^*}) = \min_{1 \le i \le q} H_w^1(s_1^i, s_2^{i^*}, \dots, s_p^{i^*}). \tag{1}$$

Let us show that the strategies $s_1^{j^*}, s_2^{j^*}, \dots, s_p^{j^*}$ are optimal by Nash for players $1, 2, \dots, p$ in the initial c-game.

Indeed,

$$H_w^i(s_1^{j^*}, s_2^{j^*}, \ldots, s_{i-1}^{j^*}, s_i^{j^*}, s_{i+1}^{j^*}, \ldots, s_p^{j^*}) \le$$

$$\le H_w^i(s_1^{j^*}, s_2^{j^*}, \ldots, s_{i-1}^{j^*}, s_i, s_{i+1}^{j^*}, \ldots, s_p^{j^*}),$$

$$\forall s_i \in S_i,\ 2 \le i \le p,$$

since $s_2^{j^*}, s_3^{j^*}, \ldots, s_p^{j^*}$ are the optimal by Nash strategies in the c-game for $s_1 = s_1^j$ and moreover, according to (1)

$$H_w^1(s_1^{j^*}, s_2^{j^*}, \ldots, s_p^{j^*}) \le H_w^1(s_1, s_2^{j^*}, \ldots, s_p^{j^*}),\ \forall s_1 \in S_1.$$

The digraph $G_{s_j^*} = (V, E_{s_j^*})$ corresponding to the strategies $s_1^{j^*}, s_2^{j^*}, \ldots, s_p^{j^*}$ has the structure of a directed tree with the root vertex v_0. The theorem is proved.

Theorem 2 *Let $(G, V_1, V_2, \ldots, V_p, c_1, c_2, \ldots, c_p, w)$ be a network game for which the vertex v_0 is attainable from any vertex $w \in V$ and the functions $c_1, c_2, \ldots, c_p$ are positive. Then on the vertex set V of the network game there exist p real functions*

$$\varepsilon^1 : V \to R^1,\ \varepsilon^2 : V \to R^1, \ldots, \varepsilon^p : V \to R^1,$$

which satisfy the conditions:

a) $\varepsilon^k(v) - \varepsilon^k(u) + c_k(u,v) \ge 0,\ \forall (u,v) \in E_k,\ k = \overline{1,p}$, where $E_k = \{e = (u,v) \in E \mid u \in V_k, v \in V\}$;

b) $\min\limits_{v \in V_G(u)} \{\varepsilon^k(v) - \varepsilon^k(u) + c_k(u,v)\} = 0,\ \forall u \in V_k,\ k = \overline{1,p}$;

c) the subgraph $G^0 = (V, E^0)$ generated by the edge set $E^0 = E_1^0 \cup E_2^0 \cup \ldots \cup E_p^0$, $E_k^0 = \{e = (u,v) \in E_k \mid \varepsilon^k(v) - \varepsilon^k(u) + c_k(u,v) = 0\}$, $k = \overline{1,p}$ has the property that the vertex v_0 is attainable from any vertex $w \in V$.

The optimal by Nash strategies of players in the c-game with the network $(G, V_1, V_2, \ldots, V_p, c_1, c_2, \ldots, c_p, w)$ can be found as follows: in G^0 an arbitrary directed tree $T = (V, E^*)$ is chosen and in T the maps

$$\begin{array}{ll} s_1^* : v \to V_T(v) & \textit{for } v \in V_1, \\ s_2^* : v \to V_T(v) & \textit{for } v \in V_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ s_p^* : v \to V_T(v) & \textit{for } v \in V_p \end{array}$$

are fixed.

Proof. By Theorem 1, in the c-game there exist optimal by Nash strategies $s_1^*, s_2^*, \ldots, s_p^*$ of players $1, 2, \ldots, p$, and in G these strategies generate a directed tree $T_{s^*} = (V, E_{s^*})$ with the root vertex v_0. In this tree we find the functions $\varepsilon^1 : V \to R^1$, $\varepsilon^2 : V \to R^1, \ldots, \varepsilon^p : V \to R^1$, where $\varepsilon^i(v)$ equals the sum of costs $c_i(e)$ of edges e, which belong to the only directed path in T_{s^*}, connecting the vertices w and v_0. It is easy to verify that these numbers satisfy conditions a) and b).

Note, that the directed tree T_{s^*} is a subgraph of the digraph $G^0 = (V, E^0)$, therefore condition c) holds too. Moreover, if in G^0 a different from T_{s^*} directed tree $T_s = (V, E_s)$ with the root vertex v_0 is chosen, then this tree generates another optimal by Nash strategy of players $1, 2, \ldots, p$. The theorem is proved.

4 The algorithm for finding optimal strategies in the case of a network without directed cycles

Let G be a digraph without directed cycles. In this case the vertices of G may be numbered from 1 to n, so that for any two vertices i and j, $i < j$ in G there no directed path from i to j exists. The following algorithm is used to number the vertices of G.

Step 1. Number the vertex v_0 by 1. Set $k = 2$.

Step k $(k \geq 2)$**.** Find a vertex $v \in V$ for which the set $V_G(v)$ contains only numbered vertices. Number the vertex v by k. Set $k = k + 1$. If $k > n$, STOP. Otherwise go to step k.

In the network with the vertices numbered by this algorithm the directed tree $T_s = (V, E_s)$ of optimal strategies is easily constructed by the following algorithm.

Algorithm 1

Step 1. Assign to each vertex i, $i = \overline{1, n}$, a set of labels $l_1(i), l_2(i), \ldots, l_p(i)$ as follows:

$$l_j(1) = 0, \ j = \overline{1, p},$$

$$l_j(i) = +\infty, \quad \forall i = \overline{2, n}, \ \forall j = \overline{1, p}.$$

Set $k = 1$, $V_s = \emptyset$, $E_s = \emptyset$.

Step 2. Modify the labels of vertices $v \in V_G^-(k)$ $(V_G^-(k) = \{u \in V \mid (u, k) \in E\})$ by using the formula

$$l_j(v) = \min\{l_j(v), c_j(v, k) + l_j(k)\}, \ \text{ if } v \in V_j, \ j = \overline{1, p}.$$

Step 3. Set $V_s = V_s \cup \{k\}$. If $k > 1$, then add to E_s the edge (k, v) for which

$$l_j(k) = l_j(v) + c_j(k, v), \ \text{ if } k \in V_j, \ j = \overline{1, p}$$

and change the labels $l_i(k), \ i = \overline{1, p}, \ i \neq j$ using the formula

$$l_i(k) = l_i(v) + c_i(k, v).$$

Set $k = k + 1$. If $k = n$ then STOP. Otherwise go to step 2.

Let us prove that this algorithm gives the optimal by Nash strategies of players $1, 2, \ldots, p$. The labels $l_j(v), \ j = \overline{1, p}, \ v \in V$ satisfay conditions a) and b) of Theorem 2. Indeed, for any vertex $v \in V_j \setminus \{v_0\}$ we have

$$l_j(u) - l_j(v) + c_j(v, u) \geq 0, \quad \forall u \in V_G(v),$$

and at least for a vertex $u \in V_G(v)$

$$l_j(u) - l_j(v) + c_j(v,u) = 0.$$

The digraph $T_s = (V, E_s)$ generated by the edges $(v,u) \in E$, for which the last equality holds, is connected. Hence, in Theorem 2 we can put $\varepsilon^j(v) = l_j(v),\ j = \overline{1,p},\ \forall v \in V$.

Algorithm 1 gives an optimal solution of the problem. This algorithm has the computational complexity $O(pnm)$.

5 The algorithms for finding optimal strategies in networks with an arbitrary structure

Let us have a c-game with p players and let the digraph G have an arbitrary structure, i.e. G may contain directed cycles. In this case the problem can be reduced to the problem of finding optimal strategies in a network game without directed cycles.

We construct an auxiliary network $\bar{G} = (W, F)$ without directed cycles, where W and F are defined as follows:

$$W = \cup_{i=1}^{n+1} W^i, \quad W^i \cap W^j = \emptyset \ \text{ for } \ i \neq j;$$

$$W^i = \{w_1^i, w_2^i, \dots, w_n^i\}, \quad i = \overline{1, n+1};$$

$$F = \{(w_k^i, w_l^j) \mid (v_k, v_l) \in E,\ i,j = \overline{1,n+1},\ i > j\}.$$

The construction of $\bar{G}$ can be interpreted in the following manner: the vertex set W contains the vertex set V, doubled $n+1$ times; in $\bar{G}$ the vertices w_k^i and w_l^j are joined by the edge (w_k^i, w_k^j) if and only if $i > j$ and in the initial graph G the edge (v_k, v_l) is present.

Delete from $\bar{G}$ those vertices $w \in W$, for which the oriented path $P_{\bar{G}}(w, w_0^{n+1})$ from w to w_0^{n+1} does not exist. Divide the vertex set W into p subsets $W_1, W_2, \dots, W_p$ as follows:

$$\begin{array}{l} W_1 = \{w_k^i \in W \mid v_k \in V_1,\ i = \overline{1,n+1}\}; \\ W_2 = \{w_k^i \in W \mid v_k \in V_2,\ i = \overline{1,n+1}\}; \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ W_p = \{w_k^i \in W \mid v_k \in V_p,\ i = \overline{1,n+1}\}. \end{array}$$

Obviously, $\cup_{i=1}^{n+1} W_i = W$ and $W_i \cap W_j = \emptyset,\ i, j = \overline{1, n+1},\ i \neq j$.

Define on the edge set F the cost functions:

$$\begin{array}{l} c_1(w_k^i, w_l^j) = c_1(v_k, v_l), \quad \forall (w_k^i, w_l^j) \in F; \\ c_2(w_k^i, w_l^j) = c_2(v_k, v_l), \quad \forall (w_k^i, w_l^j) \in F; \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ c_p(w_k^i, w_l^j) = c_p(v_k, v_l), \quad \forall (w_k^i, w_l^j) \in F. \end{array}$$

In the obtained network the problem of finding the optimal paths from all the vertices $w \in W$ to the vertex w_0^{n+1} can be solved by using Algoritm 1.

Let $l_j(w)$ be the length of an optimal path from $w \in W_j$ to w_0^{n+1}.

Since the cost functions both in the initial and the auxiliary networks are positive, then for all vertices $w_k^i \in W_j,\ i = \overline{1, n+1}$, the lengths of optimal paths are constant and equal the length of the optimal path which connects the vertex v_k with v_0 in the initial graph G, i.e.

$$l_j(w_k^i) = l_j(v_k).$$

This algorithm is inconvenient because of the great number of vertices in the auxiliary network.

Further we present a simpler algorithm for finding the optimal strategies of players.

Algorithm 2

Preliminary step. Assign to every vertex $v \in V$ a set of labels $\varepsilon^1(v), \varepsilon^2(v), \dots, \varepsilon^p(v)$ as follows:

$$\varepsilon^i(v_0) = 0, \quad \forall i = \overline{1, p},$$

$$\varepsilon^i(v) = \infty, \quad \forall v \in V \setminus \{v_0\},\ i = \overline{1, p}.$$

General step. For every vertex $v \in V \setminus \{v_0\}$ change the labels $\varepsilon^i(v),\ i = \overline{1, p}$, in the following way. If $v \in V_k$ then find the vertex $\bar{v}$ for which

$$\varepsilon^k(\bar{v}) + c_k(v, \bar{v}) = \min_{u \in V_G(v)} \{\varepsilon^k(u) + c_k(v, u)\}.$$

If $\varepsilon^k(v) > \varepsilon^k(\bar{v}) + c_k(v, \bar{v})$, then replace $\varepsilon^k(v)$ by $\varepsilon^k(\bar{v}) + c_k(v, \bar{v})$ and $\varepsilon^i(v)$ by $\varepsilon^i(v) + c_i(v, \bar{v})$, $i = \overline{1,p}$, $i \neq k$. If $\varepsilon^k(v) \leq \varepsilon^k(\bar{v}) + c_k(v, \bar{v})$, then the labels are not changing.

Repeat the general step n times. Then the labels $\varepsilon^i(v)$, $i = \overline{1,p}$, $v \in V$, become constant. Let us note that these labels satisfy the conditions of Theorem 2. Hence, using the labels $\varepsilon^i(v)$, $i = \overline{1,p}$, $v \in V$, and Theorem 2 we construct optimal by Nash strategies of players $1, 2, \ldots, p$. Algorithm 2 has the computational complexity $O(pn^2m)$.

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