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On quasistability radius of a vector trajectorial problem with a principle of optimality generalizing Pareto and lexicographic principles*

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Abstract

A multicriterion linear combinatorial problem with a parametric principle of optimality is considered. This principle is defined by a partitioning of partial criteria onto Pareto preference relation groups within each group and the lexicographic preference relation between them. Quasistability of the problem is investigated. This type of stability is a discrete analog of Hausdorff lower semi-continuity of the multiple-valued mapping that defines the choice function. A formula of quasistability radius is derived for the case of the metric l_∞. Some known results are stated as corollaries.

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1 Introduction

Traditionally stability of an optimization problem is understood as continuous dependence of solutions on parameters of the problem. The most general approaches to stability analysis of optimization problems are based on properties of multiple-valued mappings that define optimality principles [1–4].

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Mathematical analysis does not present methods sufficient to investigate stability of a discrete optimization problem. It is greatly due to complexity of discrete models, which can behave unpredictably under small variations of initial data [4, 5]. At the same time, if terminology of general topology is not used, then the formulation of a stability problem can be significantly simplified in the case of a space of acnodes. There are different types of stability of discrete optimization problems (e. g. [4–9]). Stability of a discrete problem in the broad sense means that there exists a neighborhood of the initial data in the space of problem parameters such that any problem with parameters from this neighborhood possesses some invariance with respect to the initial problem. In particular, upper (lower) semicontinuity of an optimal mapping is equivalent to nonappearance of new (preserving of initial) optimal solutions under "small" perturbations of the mapping parameters. So concepts of stability [4–8] and quasistability [6–8, 10, 11] of discrete optimization problems arise.

In this article we consider an n-criterion trajectorial linear problem with partitioning of criteria into groups according to given Pareto preference relation within each group and the lexicographic preference relation between them. Two special cases of such partitioning correspond to Pareto and lexicographic optimality principles. A formula for quasistability radius of this problem is derived for the case of independent perturbations of initial data in the metric l_∞. Some known results are stated as corollaries.

Note that similar formulas were derived earlier in [12–16] for stability and quasistability radii of vector trajectorial and game-theoretic problems with other parametric principles of optimality ("from Condorset to Pareto", "from Pareto to Slater", "from Pareto to Nash").

2 Basic definitions and notations

Let a vector criterion

$$f(t, A) = (f_1(t, A_1), f_2(t, A_2), \ldots, f_n(t, A_n)) \to \min_{t \in T}$$

with partial criterion

$$f_i(t, A_i) = \sum_{j \in N(t)} a_{ij}, \quad i \in N_n = \{1, 2, ..., n\},\ n \geq 1,$$

be defined on a system of subsets (trajectories) $T \subseteq 2^E$, $|T| \geq 2$, $E = \{e_1, e_2, \ldots, e_m\}$, $m \geq 2$. Here $N(t) = \{j \in N_m : e_j \in t\}$, A_i is the i-th row of a matrix $A = [a_{ij}] \in \mathbf{R}^{n \times m}$. Put $f_i(\emptyset, A_i) = 0$.

Let $s \in N_n$, $\mathcal{I} = \{I_1, I_2, \ldots, I_s\}$ be a partitioning of the set N_n into s nonintersecting nonempty sets, i. e.

$$N_n = \bigcup_{r \in N_s} I_r,$$

where $I_r \neq \emptyset$, $r \in N_s$; $p \neq q \Rightarrow I_p \cap I_q = \emptyset$. To any such partitioning we put in correspondence the binary relation $\Omega^n_{\mathcal{I}}$ of strict preference in the space $\mathbf{R}^n$ between different vectors $y = (y_1, y_2, \ldots, y_n)$ and $y' = (y'_1, y'_2, \ldots, y'_n)$ as follows

$$y\ \Omega^n_{\mathcal{I}}\ y' \quad \Leftrightarrow \quad y_{I_k} \succ y'_{I_k},$$

where $k = \min\{i \in N_s :\ \ y_{I_i} \neq y'_{I_i}\}$; y_{I_k} and y'_{I_k} are the projections of the vectors y and y' correspondingly onto the coordinate axes of the space $\mathbf{R}^n$ with numbers from the group I_k; $\succ$ is a relation, which induces Pareto optimality principle in the space $\mathbf{R}^{|I_k|}$:

$$y_{I_k} \succ y'_{I_k} \quad \Leftrightarrow \quad y_{I_k} \neq y'_{I_k}\ \ \&\ \ y_{I_k} \geq y'_{I_k}.$$

The introduced binary relation $\Omega^n_{\mathcal{I}}$ determines ordering of the formed groups of criteria such that any previous group is significantly more important that any consequent group. This relation generates the set of $\mathcal{I}$-optimal trajectories

$$T^n(A, \mathcal{I}) = \{t \in T : \forall t' \in T \quad (f(t, A)\ \overline{\Omega^n_{\mathcal{I}}}\ f(t', A))\},$$

where $\overline{\Omega^n_{\mathcal{I}}}$ is the negation of the relation $\Omega^n_{\mathcal{I}}$.

It is evident that $T^n(A, \mathcal{I}_P)$, $\mathcal{I}_P = \{N_n\}$ ($s = 1$), is Pareto set, i. e. the set of efficient trajectories

$$P^n(A) = \{t \in T :\ \forall t' \in T \quad (f(t, A) \overline{\succ} f(t', A))\},$$

and $T^n(A, \mathcal{I}_L)$, $\mathcal{I}_L = \{\{1\}, \{2\}, \ldots, \{n\}\}$ $(s = n)$, is the set of lexicographically optimal trajectories

$$L^n(A) = \{t \in T : \ \forall t' \in T \quad (f(t, A) \overline{\vdash} f(t', A))\},$$

where $\vdash$ is the lexicographic order in the space $\mathbf{R}^n$. This order is defined as follows

$$y \vdash y' \quad \Leftrightarrow \quad y_k > y'_k,$$
$$k = \min\{i \in N_n : \ y_i \neq y'_i\}.$$

So under the parametrization of optimality principle we understand assigning the characteristic of binary relation that in special cases induces well-known Pareto and lexicographic optimality principles.

It is easy to show that the binary relation $\Omega^n_{\mathcal{I}}$ is antireflexive, asymmetric, transitive, and hence it is acyclic. And since the set T is finite, the set $T^n(A, \mathcal{I})$ is non-empty for any matrix A and any partitioning $\mathcal{I}$ of the set N_n.

Hereinafter by $Z^n(A, \mathcal{I})$ we denote the problem of finding the set $T^n(A, \mathcal{I})$.

Clearly, $T^1(A, \{1\})$ is the set of optimal trajectories of the scalar linear trajectorial problem $Z^1(A, \{1\})$, where $A \in \mathbf{R}^m$. Many extreme combinatoric problems on graphs, boolean programming and scheduling problems and others are reduced to $Z^1(A, \{1\})$ [7, 9, 10, 17]).

The following properties follow directly from the above definitions.

Property 1. *$T^n(A, \mathcal{I}) \subseteq P_1(A) \subseteq T$, where*

$$P_1(A) = \{t \in T : \ \forall t' \in T \quad (f_{I_1}(t, A) \overline{\succ} f_{I_1}(t', A))\}.$$

Property 2. *If $f_{I_1}(t, A) \succ f_{I_1}(t', A)$, then $f(t, A) \ \Omega^n_{\mathcal{I}} \ f(t', A)$.*

Property 3. *If $f(t, A) \ \Omega^n_{\mathcal{I}} \ f(t', A)$, then $f_{I_1}(t, A_i) \geq f_{I_1}(t', A_i)$.*

Property 4. *A trajectory $t \notin T^n(A, \mathcal{I})$ if and only if there exists a trajectory t' such that $f(t, A) \ \Omega^n_{\mathcal{I}} \ f(t', A)$.*

Property 5. *A trajectory $t \in T^n(A, \mathcal{I})$ if and only if for any trajectory t' the relation $f(t, A) \ \overline{\Omega^n_{\mathcal{I}}} \ f(t', A)$ holds.*

Denote

$$S_1(A) = \{t \in P_1(A) : \ \forall t' \in T \setminus \{t\} \quad \big(f_{I_1}(t, A) \neq f_{I_1}(t', A)\big)\}.$$

Property 6. $S_1(A) \subseteq T^n(A, \mathcal{I})$.

Proof. Assume the converse, i. e. $t \in S_1(A)$ and $t \notin T^n(A, \mathcal{I})$. Then according to property 4 there exists a trajectory $t' \neq t$ such that

$$f(t, A) \ \Omega^n_{\mathcal{I}} \ f(t', A).$$

Hence due to property 3 we have

$$f_{I_1}(t, A) \geq f_{I_1}(t', A).$$

Taking into account the inclusion $t \in P_1(A)$ we obtain

$$f_{I_1}(t, A) = f_{I_1}(t', A),$$

i. e. $t \notin S_1(A)$, which contradicts the assumption.

Property 7. $\forall \ t \in S_1(A) \quad \forall \ t' \in T \setminus \{t\} \quad \exists \ i \in I_1 \quad \big(f_i(t', A_i) > f_i(t, A_i)\big)$.

For any number $\varepsilon > 0$, define the set of perturbation matrixes

$$\mathcal{B}(\varepsilon) = \{B \in \mathbf{R}^{n \times m} : \ ||B|| < \varepsilon\},$$

where $||B|| = \max\{|b_{ij}| \ : \ (i, j) \in N_n \times N_m\}, \ B = [b_{ij}]$.

As in [8, 10, 14, 17], under the quasistability radius of the problem $Z^n(A, \mathcal{I})$ we understand the number

$$\rho^n(A, \mathcal{I}) = \begin{cases} \sup K^n(A, \mathcal{I}) & \text{if } K^n(A, \mathcal{I}) \neq \emptyset, \\ 0 & \text{if } K^n(A, \mathcal{I}) = \emptyset, \end{cases}$$

where

$$K^n(A, \mathcal{I}) = \{\varepsilon > 0 : \forall \ B \in \mathcal{B}(\varepsilon) \ (T^n(A, \mathcal{I}) \subseteq T^n(A + B, \mathcal{I}))\}.$$

3 Lemmas

For any trajectories t and t' we define the numbers

$$\Delta(t,t') = |(t \cup t') \setminus (t \cap t')|,$$

$$d^n(t,t',A) = \max_{i \in I_1} \frac{f_i(t',A_i) - f_i(t,A_i)}{\Delta(t,t')}.$$

Lemma 1. *If $d^n(t,t',A) \geq \varphi > 0$, then the following relation holds for any perturbation matrix $B \in \mathcal{B}(\varphi)$:*

$$f(t,A+B)\ \overline{\Omega^n_{\mathcal{I}}}\ f(t',A+B).$$

Proof. Directly from the definition of the number $d^n(t,t',A)$ we have

$$\exists k \in I_1 \quad \big(f_k(t',A_k) - f_k(t,A_k) \geq \varphi\Delta(t,t')\big)\,. \tag{1}$$

Further suppose that the assertion of the lemma is false, i. e. there exists matrix $B^* = [b^*_{ij}] \in \mathcal{B}(\varphi)$ such that $f(t,A+B^*)\ \Omega^n_{\mathcal{I}}\ f(t',A+B^*)$. Then by virtue of property 3 and linearity of the functions $f_i(t,A),\ i \in N_n$, we derive

$$0 \geq f_i(t',A_i+B^*_i) - f_i(t,A_i+B^*_i) =$$

$$= f_i(t',A_i) - f_i(t,A_i) + f_i(t',B^*_i) - f_i(t,B^*_i) \geq$$

$$\geq f_i(t',A_i) - f_i(t,A_i) - ||B_i||\Delta(t,t') >$$

$$> f_i(t',A_i) - f_i(t,A_i) - \varphi\Delta(t,t'), \quad i \in I_1,$$

i. e.

$$\forall i \in I_1 \quad (f_i(t',A_i) - f_i(t,A_i) < \varphi\Delta(t,t')),$$

which contradicts (1).

Lemma 2. *Let $t \in T^n(A,\mathcal{I})$, $t' \in T \setminus \{t\}$. For any number $\alpha > d^n(t,t',A)$ there exists a matrix $B^* \in \mathbf{R}^{n\times m}$ with norm $||B^*|| = \alpha$ such that*

$$f(t,A+B^*)\ \Omega^n_{\mathcal{I}}\ f(t',A+B^*). \tag{2}$$

Proof. We construct the perturbation matrix $B^* = [b^*_{ij}] \in \mathbf{R}^{n\times m}$ by the formula

$$b^*_{ij} = \begin{cases} -\alpha & \text{if } i \in I_1,\ e_j \in t' \setminus t, \\ \alpha & \text{if } i \in I_1,\ e_j \in t \setminus t', \\ 0 & \text{otherwise.} \end{cases}$$

Then $||B^*|| = \alpha$ and

$$f_i(t', B^*_i) - f_i(t, B^*_i) = -\alpha\Delta(t, t'), \quad i \in I_1.$$

From here we get

$$\frac{1}{\Delta(t,t')}(f_i(t', A_i + B^*_i) - f_i(t, A_i + B^*_i)) = \frac{f_i(t', A_i) - f_i(t, A_i)}{\Delta(t,t')} - \alpha \leq$$

$$\leq d^n(t, t', A) - \alpha < 0, \quad i \in I_1,$$

i. e. $f_{I_1}(t, A + B^*) \succ f_{I_1}(t', A + B^*)$. This implies (2) by virtue of property 2.

4 Theorem

Theorem. *For any partitioning $\mathcal{I}$ of the set N_n, $n \geq 1$, into s groups, $s \in N_n$, the quasistability radius $\rho^n(A, \mathcal{I})$ of a problem $Z^n(A, \mathcal{I})$ is expressed by the formula*

$$\rho^n(A, \mathcal{I}) = \min_{t\in T^n(A,\mathcal{I})} \ \min_{t'\in T\setminus\{t\}} d^n(t, t', A). \tag{3}$$

Proof. Denote the right hand side of (3) by φ for short. Before proving the theorem we note that since the sets $T^n(A, \mathcal{I})$ and $T \setminus \{t\}$ are non-empty, the number φ is correctly defined and nonnegative.

First we prove the inequality

$$\rho^n(A, \mathcal{I}) \geq \varphi. \tag{4}$$

Without loss of generality assume that $\varphi > 0$ (otherwise inequality (4) is obvious). From the definition of the number φ, it follows that for any trajectories $t \in T^n(A, \mathcal{I})$ and $t' \neq t$ the inequalities

$$d^n(t, t', A) \geq \varphi > 0$$

hold. Applying lemma 1 we get

$$\forall\ B \in \mathcal{B}(\varphi) \quad \forall\ t \in T^n(A, \mathcal{I}) \quad \forall\ t' \in T \quad (f(t, A+B)\ \overline{\Omega^n_{\mathcal{I}}}\ f(t', A+B)).$$

Therefore $t \in T^n(A+B, \mathcal{I})$ by virtue of property 5. Thus we conclude

$$\forall\ B \in \mathcal{B}(\varphi) \quad (T^n(A, \mathcal{I}) \subseteq T^n(A+B, \mathcal{I})).$$

This formula proves (4).

It remains to show that

$$\rho^n(A, \mathcal{I}) \leq \varphi. \tag{5}$$

Let $\varepsilon > \alpha > \varphi$ and trajectories $t \in T^n(A, \mathcal{I})$, $t' \neq t$ be such that $d^n(t, t', A) = \varphi$. Then according to lemma 2 there exists a matrix B^* with norm $||B^*|| = \alpha$ such that (2) holds, i. e. $t \notin T^n(A+B^*, \mathcal{I})$. Hence we have

$$\forall \varepsilon > \varphi \quad \exists B^* \in \mathcal{B}(\varepsilon) \quad (T^n(A, \mathcal{I}) \not\subseteq T^n(A+B^*, \mathcal{I}))\,.$$

This proves inequality (5). Summarizing (4) and (5) we obtain (3).

5 Corollaries

Corollary 1 [10]. *The quasistability radius of the problem $Z^n(A, \mathcal{I}_P)$, $n \geq 1$, of finding Pareto set $P^n(A)$ is expressed by the formula*

$$\rho^n(A, \mathcal{I}_P) = \min_{t \in P^n(A)} \ \min_{t' \in T \backslash \{t\}} \ \max_{i \in N_n} \frac{f_i(t', A_i) - f_i(t, A_i)}{\Delta(t, t')}.$$

Corollary 2 [18]. *The quasistability radius of the problem $Z^n(A, \mathcal{I}_L)$, $n \geq 1$, of finding the set of lexicographically optimal trajectories $L^n(A)$ is expressd by the formula*

$$\rho^n(A, \mathcal{I}_L) = \min_{t \in L^n(A)} \ \min_{t' \in T \backslash \{t\}} \frac{f_1(t', A_1) - f_1(t, A_1)}{\Delta(t, t')}.$$

A problem $Z^n(A,\mathcal{I})$ is called quasistable if $\rho^n(A,\mathcal{I}) > 0$. Thus quasistability of a problem $Z^n(A,\mathcal{I})$ is the property of preserving optimality by all $\mathcal{I}$-efficient trajectories under small variation of matrix A. In other words, quasistability is a discrete analog of Hausdorff lower semi-continuity of the multiple-valued mapping that assigns the set of $\mathcal{I}$-efficient trajectories to each set of the problem parameters.

Corollary 3. *For any partitioning $\mathcal{I}$ of the set N_n, $n \geq 1$, into s groups, $s \in N_n$, the following statements are equivalent for a problem $Z^n(A,\mathcal{I})$, $n \geq 1$:*

(i) *the problem $Z^n(A,\mathcal{I})$ is quasistable,*

(ii) $\forall\, t \in T^n(A,\mathcal{I}) \quad \forall\, t' \in T \setminus \{t\} \quad \exists\, i \in I_1 \quad \big(f_i(t', A_i) > f_i(t, A_i)\big),$

(iii) $T^n(A,\mathcal{I}) = S_1(A)$.

Proof. Equivalence of statements (i) and (ii) follows directly from the theorem.

The implication (ii) $\Rightarrow$ (iii) is proved by contradiction. Suppose that (ii) holds but (iii) does not.

From properties 1 and 6 we get

$$S_1(A) \subseteq T^n(A,\mathcal{I}) \subseteq P_1(A).$$

Then (since $T^n(A,\mathcal{I}) \neq S_1(A)$ is assumed) there exists a trajectory $t \in T^n(A,\mathcal{I}) \subseteq P_1(A)$ such that $t \notin S_1(A)$. It follows that there exists trajectory $t' \in P_1(A)$ such that

$$t' \neq t, \quad f_{I_1}(t, A) = f_{I_1}(t', A).$$

This contradicts statement (ii).

The implication (iii) $\Rightarrow$ (i) is obvious by virtue of property 7.

From corollary 3, we easily get the following known result (e. g. see [10]).

Corollary 4. *The problem $Z^n(A,\mathcal{I}_P)$, $n \geq 1$, of finding Pareto set $P^n(A)$ is quasistable if and only if $P^n(A) = S^n(A)$.*

Here $S^n(A)$ is Smale set [19], i. e. the set of strictly efficient trajectories:

$$S^n(A) = \{t \in P^n(A) : \ \forall t' \in T \setminus \{t\} \quad \big(f(t, A) \neq f(t', A)\big)\}.$$

Corollary 3 also implies

Corollary 5 [18]. *The problem $Z^n(A, \mathcal{I}_L)$, $n \geq 1$, of finding the set $L^n(A)$ of lexicographically optimal trajectories is quasistable if and only if*

$$|L^n(A)| = \left| \text{Arg} \min_{t \in T} \ f_1(t, A_1) \right| = 1.$$

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