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On stability of Nash equilibrium situations and Pareto optimal situations in finite games *

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Abstract

A non-cooperative finite game of several persons is considered in the case, where payoff functions are linear. Extreme levels of independent perturbations of payoff functions parameters, which remain Nash and Pareto optimality of a situation, are specified. Necessary and sufficient conditions of such stability are stated.

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1 Introduction

There are some approaches of formulating of choice functions in the theory of non-cooperative games. One of them is the widely known principle of situation stability to deviations of strategy of each player, operating alone, called Nash equilibrium [1]. But there are situations, which are more profit for all the players, than Nash equilibrium ones. Pareto optimal situations can be such situations [2]. These situations are characterized by the following property: no one situation is more preferable for all players. In [3-5], stability of Pareto optimal solutions to perturbations of input data is investigated for vector combinatorial optimization problems. By analogy with those works, extreme levels of independent perturbations of linear payoff functions parameters, which remain Nash and Pareto optimality of a situation, are specified here.

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2 Nash equilibrium situation

Consider a non-cooperative finite game in the normal form [1,6-9].

Let $X_i \subset \boldsymbol{R}$ be a finite set of (pure) strategies of the player $i \in N_n := \{1, 2, \dots, n\}$, $n \geq 2, |X_i| > 1$. Let $f_i(x) = C_i x$ be a payoff function of the player i, defined on the set of game situations $X = \prod_{i=1}^{n} X_i$. Here C_i is the i-th row of the matrix $C = (c_{ij})_{n\times n} \in \boldsymbol{R}^{nn}$; $x = (x_1, x_2, \dots, x_n)^T$, $x_i \in X_i$, $i \in N_n$. The game consists in following: the players choose their strategies x_i from the sets X_i, $i \in N_n$ simultaneously and independently to each other. So the situation x is formed. Then each player i receives the profit $f_i(x)$ and the game finishes. Any such game is called a game with the matrix C.

According to Nash [1], a situation $x^0 \in X$ is called *the equilibrium situation* of the game with matrix C, if the following equality holds for any index $i \in N_n$:

$$\max\{C_i x : x \in W(x^0, i)\} = C_i x^0, \tag{1}$$

where

$$W(x^0, i) = \prod_{j=1}^{n} W_j(x^0, i),$$

$$W_j(x^0, i) = \begin{cases} X_j & \text{if } j = i, \\ x_j^0 & \text{if } j \neq i. \end{cases}$$

It can be easily seen, that a situation $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ of the game with matrix C is equilibrium situation, if and only if the strategy of each player $i \in N_n$ is expressed by

$$x_i^* = \begin{cases} \max\{z : z \in X_i\} & \text{if } c_{ii} > 0, \\ \min\{z : x \in X_i\} & \text{if } c_{ii} < 0, \\ z \in X_i & \text{if } c_{ii} = 0. \end{cases}$$

We denote by $NE^n(C)$ the set of all such situations in the game with matrix C.

An equilibrium situation x^* is called *strict*, if maximum in equality (1) is achieved at a single point for any index $i \in N_n$, i.e. if $|NE^n(C)| = 1$.

Thus, the next statements follows.

Proposition 1 *$NE^n(C) \neq \emptyset$ for any matrix $C \in \boldsymbol{R}^{nn}$.*

Proposition 2 *An equilibrium situation of the game with matrix C is strict if and only if any element of the main diagonal of matrix C is nonzero.*

By analogy with [3-5], under stability of a fixed equilibrium situation of the game with matrix C, we understand its following property: the situation remains equilibrium under "small" independent perturbations of elements of the matrix C. Mentioned perturbations are represented by addition of the matrix C with matrices from the set

$$\mathcal{B}(\epsilon) = \{B \in \boldsymbol{R}^{nn} : ||B|| < \epsilon\},\ \epsilon > 0. \tag{2}$$

It can be easily seen, that if an element of the matrix C is located outside the main diagonal, then its perturbations do not influence on equilibrium of any situation. Therefore we define the norm of the matrix $B = (b_{ij})_{n\times n} \in \boldsymbol{R}^{nn}$ in definition (2) by the following:

$$||B|| = \max\{|b_{ii}| : i \in N_n\}.$$

So an equilibrium situation $x \in NE^n(C)$ is called *stable* (to perturbations of payoff function parameters), if the following formula is valid:

$$\exists \epsilon > 0\ \forall B \in \mathcal{B}(\epsilon)\ (x \in NE^n(C + B)).$$

By analogy with [4-6], under *stability radius of an equilibrium situation $x \in NE^n(C)$* we understand the number

$$\rho_1^n(x, C) = \begin{cases} \sup \Omega_1, & \text{if } \Omega_1 \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Omega_1 = \{\epsilon > 0 : \forall B \in \mathcal{B}(\epsilon)\ (x \in NE^n(C + B))\}$.

Proposition 3 *Stability radius of a strict equilibrium situation x of the game with matrix $C \in \boldsymbol{R}^{nn}$ $(c_{ii} \neq 0,\ i \in N_n)$ is expressed by the formula*

$$\rho_1^n(x, C) = \min\{|c_{ii}| : i \in N_n\}.$$

Hence, any strict equilibrium situation is stable.

Proposition 4 *An equilibrium situation is stable if and only if it is strict.*

Proposition 5 *If at least one diagonal element of the matrix $C \in \boldsymbol{R}^{nn}$ is zero, then there is no stable equilibrium situation.*

3 Pareto optimal situation

There is no single meaning of choice function in games theory. Evident examples exist (see [6-9] and example 1 below), when there are more advantageous situations for participants of the game, than equilibrium ones. Such situations can be Pareto optimal [2]. A situation x^0 in the game with matrix C is called *Pareto optimal*, and also efficient, if it belongs to the set

$$P^n(C) = \{x \in X : \pi(x, C) = \emptyset\},$$

where

$$\pi(x, C) = \{x' \in X : Cx \leq Cx',\ Cx \neq Cx'\}.$$

It is evident, that $P^n(C) \neq \emptyset$ for any finite game with matrix $C \in \boldsymbol{R}^{nn}$.

By analogy with [3-5], under *stability radius of an efficient situation* $x \in P^n(C)$, we understand the number

$$\rho_2^n(x, C) = \begin{cases} \sup \Omega_2 & \text{if } \Omega_2 \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Omega_2 = \{\epsilon > 0 : \forall B \in \mathcal{B}(\epsilon)\ (x \in P^n(C + B))\}.$$

The set $\mathcal{B}(\epsilon)$ is defined here by formula (2), but under the norm of a matrix $B = (b_{ij})_{n\times n}$ we understand Chebyshev norm:

$$||B||_\infty = \max\{|b_{ij}| : (i,j) \in N_n \times N_n\}.$$

A minimization problem was considered in [3]. We discuss here a maximization one. Therefore, applying the result of [3], we obtain

Proposition 6 *Stability radius of any efficient situation $x \in P^n(C)$ in the game with matrix $C \in \boldsymbol{R}^n$ is expressed by the formula*

$$\rho_2^n(x, C) = \min_{x' \in X\backslash\{x\}} \max_{i \in N_n} \frac{C_i(x - x')}{||x - x'||_1},$$

where $||z||_1 = \sum\limits_{i=1}^{n} |z_i|,\ z = (z_1, z_2, \ldots, z_n)$.

A situation $x \in P^n(C)$ of the game with matrix C is called *stable*, if $\rho_2^n(x, C) > 0$; it is called *Smale optimal* (or *strongly efficient*) [10], if there are no situations $x' \in X\backslash\{x\}$, such that

$$Cx = Cx'.$$

From proposition 5, we obtain

Corollary. *Pareto optimal situation is stable if and only if it is Smale optimal.*

Consider some examples of the two-person game called bimatrix. Let $X_i = \{0, 1\},\ i \in N_2,\ x^{(1)} = (0, 0)^T,\ x^{(2)} = (0, 1)^T,\ x^{(3)} = (1, 0)^T,\ x^{(4)} = (1, 1)^T$. We write payoff functions in the form of matrix

$$\begin{bmatrix} (C_1x^{(1)},\ C_2x^{(1)}) & (C_1x^{(2)},\ C_2x^{(2)}) \\ (C_1x^{(3)},\ C_2x^{(3)}) & (C_1x^{(4)},\ C_2x^{(4)}) \end{bmatrix}.$$

Example 1. Let $C = \begin{pmatrix} 2 & -6 \\ -2 & 1 \end{pmatrix}$. Then we obtain bimatrix game with payoff functions

$$\begin{bmatrix} (0, 0) & (-6, 1) \\ (2, -2) & (-4, -1) \end{bmatrix}.$$

Therefore $NE^2(C) = x^{(4)},\ P^2(C) = \{x^{(1)}, x^{(2)}, x^{(3)}\}$. It is evident, that efficient strategy $x^{(1)}$ is more advantageous for players, than equilibrium situation $x^{(4)}$, although it is not an equilibrium one. Such a game is known in the literature (see, for example, [7-9]) as "dilemma of prisoner". By virtue of propositions 3 and 6, we have

$$\rho_1^2(x^{(4)}, C) = 1,\ \rho_2^2(x^{(1)}, C) = \rho_2^2(x^{(3)}, C) = 2,\ \rho_2^2(x^{(2)}, C) = 1.$$

Example 2. Let $C = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$. Then payoff functions matrix is

$$\begin{bmatrix} (0,0) & (-1,0) \\ (2,-1) & (1,-1) \end{bmatrix}.$$

From here, $NE^2(C) = \{x^{(3)}, x^{(4)}\},\ P^2(C) = \{x^{(1)}, x^{(3)}\}$. On the basis of propositions 4 and 6, we obtain

$$\rho_1^2(x^{(3)}, C) = \rho_1^2(x^{(4)}, C) = 0,\ \rho_2^2(x^{(1)}, C) = \frac{1}{2},\ \rho_2^2(x^{(3)}, C) = 1.$$

Example 3. Let $C = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}$. Then payoff functions matrix is

$$\begin{bmatrix} (0,0) & (3,1) \\ (2,5) & (5,6) \end{bmatrix}.$$

Therefore $NE^2(C) = P^2(C) = \{x^{(4)}\}$. According to propositions 3 and 6, we derive

$$\rho_1^2(x^{(4)}, C) = 1,\ \rho_2^2(x^{(4)}, C) = 3.$$

Example 4. Let $C = \begin{pmatrix} -2 & -1 \\ 1 & -3 \end{pmatrix}$. Then payoff functions matrix is:

$$\begin{bmatrix} (0,0) & (-1,-3) \\ (-2,1) & (-3,-2) \end{bmatrix}.$$

Therefore $NE^2(C) = \{x^{(1)}\},\ P^2(C) = \{x^{(1)}, x^{(3)}\}$. Referring to propositions 3 and 6 once again, we obtain $\rho_1^2(x^{(1)}, C) = 2,\ \rho_2^2(x^{(1)}, C) = \frac{3}{2},\ \rho_2^2(x^{(3)}, C) = 1$.

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