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Optimal Correction of Infeasible Systems in the Second Order Conic Linear Setting*

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Abstract

In this paper we consider correcting infeasibility in a second order conic linear inequality by minimal changes in the problem data. Under certain conditions, it is proved that the minimal correction can be done by solving a lower dimensional convex problem. Finally, several examples are presented to show the efficiency of the new approach.

Keywords: Second Order Cone Program, Infeasibility, Interior Point Methods.

1 Introduction

Correcting infeasibility by minimal changes in problem data is a well studied problem and various approaches have been developed to do this task [2, 5, 6]. The aim of this paper is to consider the optimal correction of infeasible linear inequalities in the second order conic setting. Thus let us first introduce second order cone program that has been widely used in modeling many real world problems [1, 4].

Definition 1. *A second order cone in R^n is defined as*

$$Q_n = \{x \in R^n | \ \ ||\bar{x}|| \leq x_1\}, \text{ where } \ \bar{x} = (x_2, \cdots, x_n)^T.$$

It has the following fundamental properties that enables one to extend interior point algorithms from linear program (LP) to second order cone program (SOCP) [1, 4]:

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- It is convex and closed.
- It is self-dual.
- It is pointed and has nonempty interior.

It is worth to note that $x \in Q_n$ is usually denoted by $x \succeq_{Q_n} 0$. An SOCP in the standard primal form similar to primal LP is given by

$$\begin{array}{ll} \min & c_1^T x_1 + \cdots + c_r^T x_r \\ & A_1 x_1 + \cdots + A_r x_r = b, \\ & x_i \succeq_{Q_{n_i}} 0, \quad i = 1, \cdots, r, \end{array}$$

where $A_i \in R^{m \times n_i}, b \in R^m,\ c \in R^{n_i}$ and its dual is given by

$$\begin{array}{ll} \max & b^T y \\ & A_i^T y + s_i = c_i, \quad i = 1, \cdots, r, \\ & s_i \succeq_{Q_{n_i}} 0,\ i = 1, \cdots, r. \end{array}$$

This dual pair in the compact form is written as

$$\begin{array}{ll} \min & c^T x \\ & Ax = b, \\ & x \succeq_Q 0, \end{array} \tag{1}$$

where $A = [A_1, \cdots, A_r]$, $x = (x_1^T, \cdots, x_r^T)^T$ and $Q = Q_{n_1} \times \cdots \times Q_{n_r}$ and

$$\begin{array}{ll} \max & b^T y \\ & A^T y + s = c, \\ & s \succeq_Q 0, \end{array} \tag{2}$$

where $s = (s_1^T, \cdots, s_r^T)^T$ and $c = (c_1^T, \cdots, c_r^T)^T$. Moreover, dual without slack variable can be written as

$$\begin{array}{ll} \max & b^T y \\ & c - A^T y \succeq_Q 0. \end{array} \tag{3}$$

It is worth to note that weak duality theorem holds for this dual pair analogous to the LP case, but the strong duality theorem requires stronger assumptions as follows:

- **Assumption 1:** A_i's $i = 1, \cdots, r$ are linearly independent.
- **Assumption 2:** Both primal and dual problems are strictly feasible i.e., there exist a primal feasible vector $x_1, \cdots, x_r$ such that $x_i \succ_{Q_{n_i}} 0$ for $i = 1, \cdots, r$ and there exist a dual feasible vector y and $s_1, \cdots, s_r$ such that $s_i \succ_{Q_{n_i}} 0$ for $i = 1, \cdots, r$.

Now under these two assumptions the strong duality theorem holds for SOCP [4].

2 Optimal Correction of an Infeasible Conic Linear Inequality

Suppose we have the following infeasible conic linear inequality

$$Ax - b \succeq_{Q_m} 0, \quad x \in R^n. \tag{4}$$

To correct this infeasible system to a feasible one by minimal changes in the vector b, it is sufficient to solve

$$\begin{aligned} \min_{x,r} \quad & ||r|| \\ & Ax - b - r \succeq_{Q_m} 0. \end{aligned} \tag{5}$$

This is obviously equivalent to

$$\begin{aligned} \max_{x,r,t} \quad & -t \\ & Ax - b - r \succeq_{Q_m} 0, \\ & ||r|| \le t, \end{aligned}$$

which further can be written in the following dual SOCP form (3):

$$\begin{aligned} & \max \quad -t \\ & \begin{pmatrix} -b \\ 0 \\ 0_{m\times 1} \end{pmatrix} - \begin{pmatrix} 0_{m\times 1} & I_m & -A \\ -1 & 0^T_{m\times 1} & 0^T_{n\times 1} \\ 0_{m\times 1} & -I_m & 0_{m\times n} \end{pmatrix} \begin{pmatrix} t \\ r \\ x \end{pmatrix} \succeq_{Q_m \times Q_{m+1}} 0, \end{aligned} \tag{6}$$

which can be solved efficiently using any interior point based software packages for SOCP, like Mosek or SeDuMi [3, 7].

Now let us see whether it would be possible to have the optimal r value by solving a lower dimensional convex problem as in the nonnegative orthant case. In the following theorem we discuss this question.

Theorem 1. *The optimal r value in (5) is either given by*

$$r = \begin{pmatrix} \frac{(A(1,:)x^* - b(1)) - ||\bar{A}x^* - \bar{b}||}{2} \\ \frac{\bar{A}x^* - \bar{b}}{2} - \frac{(A(1,:)x^* - b(1))(\bar{A}x^* - \bar{b})}{2||\bar{A}x^* - \bar{b}||} \end{pmatrix},$$

where x^* is the optimal solution of

$$\begin{aligned} \min \quad & \frac{1}{\sqrt{2}}(||\bar{A}x - \bar{b}|| - (A(1,:)x - b(1))), \\ & |A(1,:)x - b(1)| \leq ||\bar{A}x - \bar{b}|| \end{aligned} \tag{7}$$

with $A(1,:)$ and $b(1)$ denoting the first row of A and the first element of b respectively and $\bar{A} = A(2:m,:)$ and $\bar{b} = b(2:m)$, or

$$r = Ax^* - b, \tag{8}$$

where x^* is an optimal solution of

$$\begin{aligned} \min \quad & ||Ax - b|| \\ & -A(1,:)x + b(1) \geq ||\bar{A}x - \bar{b}||. \end{aligned} \tag{9}$$

Proof. Problem (5) can be written as

$$\begin{aligned} \min_x \quad & \min_r ||r|| \\ & Ax - b - r \succeq_{Q_m} 0. \end{aligned}$$

Now let us first consider the inner minimization problem. It is equivalent to

$$\begin{aligned} \min_{t,r} \quad & t \\ & Ax - b - r \succeq_{Q_m} 0 \\ & ||r|| \leq t \end{aligned}$$

or the following dual SOCP, since x is a constant vector for the inner minimization problem:

$$\begin{array}{ll} \max & -t \\ \multicolumn{2}{l}{\begin{pmatrix} Ax-b \\ 0 \\ 0_{m\times 1} \end{pmatrix} - \begin{pmatrix} 0_{m\times 1} & I_m \\ -1 & 0_{1\times m} \\ 0_{m\times 1} & -I_m \end{pmatrix} \begin{pmatrix} t \\ r \end{pmatrix} \succeq_{Q_m \times Q_{m+1}} 0} \end{array} \quad (10)$$

and its corresponding primal problem is given by

$$\begin{array}{ll} \min & (Ax-b)^T y_1 \\ & \begin{pmatrix} 0^T_{m\times 1} & -1 & 0^T_{m\times 1} \\ I_m & 0^T_{1\times m} & -I_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0_{m\times 1} \end{pmatrix}, \\ & y_1 \in Q_m, \quad (y_2, y_3^T)^T \in Q_{m+1}. \end{array} \quad (11)$$

Now if for the given vector $x \in R^n$, we have $|A(1,:)x - b(1)| < ||\bar{A}x - \bar{b}||$, then the optimal solutions of (10) and (11) are given by

$$\begin{aligned} r &= \begin{pmatrix} \frac{(A(1,:)x-b(1)) - ||\bar{A}x-\bar{b}||}{2} \\ \frac{\bar{A}x-\bar{b}}{2} - \frac{(A(1,:)x-b(1))(\bar{A}x-\bar{b})}{2||\bar{A}x-\bar{b}||} \end{pmatrix}, \\ t &= ||r|| \end{aligned}$$

and

$$y_2 = 1, \quad y_1 = y_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\frac{\bar{A}x-\bar{b}}{||\bar{A}x-\bar{b}||} \end{pmatrix},$$

since they are both feasible and having equal objective values. It is easy to check that

$$||r|| = \frac{1}{\sqrt{2}}(||\bar{A}x - \bar{b}|| - (A(1,:)x - b(1))).$$

Thus in this case to have the optimal r value in (5), it is sufficient to solve (7). However, if $-A(1,:)x + b(1)| \geq ||\bar{A}x - \bar{b}||$, then the optimal solution for (10) and (11) are given by

$$r = Ax - b, \qquad t = ||r||$$

and $y_2 = 1, \quad y_1 = y_3 = -\frac{Ax-b}{||Ax-b||}$, since they are both feasible and having equal objective values. Thus in this case to find optimal r value in (5), it is sufficient to solve (9). □

As we see, for the optimal correction of (4), unlike linear inequalities in the nonnegative orthant, we can not necessarily find the optimal r value by solving a lower dimensional convex problem. However under certain conditions it would be possible. These conditions are given in the next corollary.

Corollary 1. *If for all $x \in R^n, A(1,:)x - b(1) > 0$ or $|A(1,:)x - b(1)| \leq ||\bar{A}x - \bar{b}||$, then the optimal r value in (5) is given by*

$$r = \begin{pmatrix} \frac{(A(1,:)x^* - b(1)) - ||\bar{A}x^* - \bar{b}||}{2} \\ \frac{\bar{A}x^* - \bar{b}}{2} - \frac{(A(1,:)x^* - b(1))(\bar{A}x^* - \bar{b})}{2||\bar{A}x^* - \bar{b}||} \end{pmatrix},$$

where x^* is the optimal solution of

$$\min \quad \frac{1}{\sqrt{2}}(||\bar{A}x - \bar{b}|| - (A(1,:)x - b(1))). \tag{12}$$

One can see that (12) is equivalent to the following dual SOCP:

$$\begin{aligned} &\max \quad -\frac{1}{\sqrt{2}}z \\ &-b - \begin{pmatrix} -A(1,:) & -1 \\ -\bar{A} & 0_{(m-1)\times 1} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \succeq_{Q_m} 0. \end{aligned} \tag{13}$$

Remark 1. *Obviously the dimension of problem (13) is lower than the dimension of (6). Therefore doing the minimal correction by solving (13) should be much faster than (6), as it is verified by our numerical experiments.*

3 Illustrative Examples

In Table 1 we have listed the results of several randomly generated test problems with different dimensions using MATLAB version 7.2. For all problems, matrices are generated randomly and we set their first row equal to zero. Then we consider the vector b with all coordinates equal to one of appropriate dimension. Obviously $Ax - b \succeq_{Q_m} 0$ is infeasible since its first element is negative. To solve (6) and (13), which are exactly in dual form SOCP, we have used SeDuMi version 1.05 [7], which is an interior point methods based software package. SeDuMi's input format can be either (1) or (3), which in our case is (3). For all test problems we report the norm of r and the time taken to find it. As we see, by increasing the dimension of the problems, finding r by using the lower dimensional model (13) is extremely faster than (6).

Table 1. Comparison of problems (6) and (13)

m, n	Problem (6) (time(sec), $\|\|r\|\|$)	Problem (13) (time(sec), $\|\|r\|\|$)
50,30	$(0.2, 4.0697)$	$(0.1, 4.0697)$
100,80	$(0.3, 3.6762)$	$(0.1, 3.6762)$
300,200	$(1.9, 7.0938)$	$(0.5, 7.0938)$
500,300	$(5.9, 10.9243)$	$(1.2, 10.9243)$
1000,700	$(132.1, 12.9153)$	$(10.1, 12.9153)$

4 Conclusions

In this paper, we have considered correcting infeasible systems in second order conic linear setting by minimal changes in the vector b. It is proved that under certain conditions, the minimal correction can be done by solving a lower dimensional convex problem. Numerical examples show that the new approach is extremely faster than the original model, especially on large scale problems.

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