Hausdorff volume in non equiregular sub-Riemannian manifolds

Roberta Ghezzi, Frédéric Jean

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Abstract

In this paper we study the Hausdorff volume in a non equiregular sub-Riemannian manifold and we compare it with a smooth volume. We first give the Lebesgue decomposition of the Hausdorff volume. Then we study the regular part, show that it is not commensurable with the smooth volume, and give conditions under which it is a Radon measure. We finally give a complete characterization of the singular part. We illustrate our results and techniques on numerous examples and cases (e.g. to generic sub-Riemannian structures).

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1 Introduction

The present work is motivated by the analysis of intrinsic volumes in sub-Riemannian geometry. Here a sub-Riemannian manifold is a triplet $(M, \mathcal{D}, g)$, where M is a smooth manifold, $\mathcal{D}$ a Lie-bracket generating distribution on M and g a Riemannian metric on $\mathcal{D}$ (note that our framework will permit us to consider rank-varying distributions as well). As in Riemannian geometry, one defines the length of absolutely continuous paths which are almost everywhere tangent to $\mathcal{D}$ by integrating the g-norm of their tangent vectors. Then, the sub-Riemannian distance d is defined as the infimum of length of paths between two given points. Since $\mathcal{D}$ is Lie-bracket generating, for every point $p \in M$ there exists $r(p) \in \mathbb{N}$ such that

$$\{0\} = \mathcal{D}_p^0 \subset \mathcal{D}_p^1 \subset \cdots \subset \mathcal{D}_p^{r(p)} = T_pM, \tag{1}$$

where $\mathcal{D}_p^i = \{X(p) \mid X \in \mathcal{D}^i\}$ and $\mathcal{D}^i \subset \mathrm{Vec}(M)$ is the submodule defined recursively by $\mathcal{D}^1 = \mathcal{D}$, $\mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}, \mathcal{D}^i]$. The sub-Riemannian manifold is equiregular if the dimensions $\dim \mathcal{D}_p^i$ do not depend on p.

Intrinsic measures on sub-Riemannian manifolds are those which are associated with the sub-Riemannian structure. There are essentially two ways to build such measures: either using the metric structure defined by the sub-Riemannian distance which provides Hausdorff and spherical Hausdorff measures, or by means of the algebraic structure associated with the distribution which allows to construct the so called Popp's measure (see [23]) on equiregular manifolds.

Intrinsic measures have been widely studied in the equiregular case, where the algebraic structure is well understood (in Carnot groups [4, 7, 10] and on equiregular manifolds [11]). The relevance of the study of intrinsic volumes, e.g., top-dimensional Hausdorff measures, is due to their use in PDE's analysis. For instance, to generalize the Laplace-Beltrami operator in sub-Riemannian geometry, one needs a (sufficiently smooth) intrinsic volume: this motivates the analysis of regularity of the Hausdorff volume in [1]. We also mention the recent work [8] where smoothness of intrinsic volumes is needed to apply some nice PDE's proof ideas. In the non equiregular case, no study of intrinsic volumes exists so far. We refer the reader to [16] for a survey of many facts and interesting questions. Hausdorff measures are also studied in [13, 19] along curves.

The main aim of this paper is the analysis of the Hausdorff volume in non equiregular sub-Riemannian manifolds.

For convenience, let us first recall the equiregular case. Let $(M, \mathcal{D}, g)$ be an equiregular sub-Riemannian manifold. The Hausdorff dimension of M can be algebraically computed in terms of the flag (1) of the distribution by $\dim_H M = Q$, where

$$Q = \sum_{i=1}^{r} i(\dim \mathcal{D}_p^i - \dim \mathcal{D}_p^{i-1}),$$

see [22]. The Hausdorff volume, denoted by vol_H, is by definition the top-dimensional spherical Hausdorff measure $\mathcal{S}^{\dim_H M}$. Assume M to be oriented. A natural way to understand the behavior of vol_H is to compare it with a smooth volume μ on M, i.e., a measure defined on open sets by $\mu(A) = \int_A \omega$, where $\omega \in \Lambda^n M$ is a positively oriented non degenerate n-form. The equiregular assumption implies that vol_H and any smooth volume μ are mutually absolutely continuous. Moreover the Radon-Nikodym derivative of vol_H with respect to μ at a point p, denoted by $\frac{d\mathrm{vol}_H}{d\mu}(p)$, can be computed explicitly by the formula (see [1])

$$\lim_{r \to 0} \frac{\mathrm{vol}_H(B(p,r))}{\mu(B(p,r))} = \frac{2^Q}{\hat{\mu}^p(\widehat{B}_p)}, \tag{2}$$

where $B(p,r)$ is the sub-Riemannian ball centered at p of radius r, $\widehat{B}_p$ is the unit ball in the nilpotent approximation at p and $\hat{\mu}^p$ is a measure obtained through a blow-up procedure of μ at p. As a consequence, $\frac{d\mathrm{vol}_H}{d\mu}$ is continuous on M and hence locally bounded and locally bounded away from zero on M. With the language of [22, 23], this implies that vol_H and μ are commensurable and, in particular, that vol_H is a Radon measure, i.e., $\mathrm{vol}_H(K) < \infty$ for every compact set K. Therefore, when the manifold is equiregular, vol_H is well understood in the sense that it behaves essentially as a smooth volume. Nevertheless, further regularity of $\frac{d\mathrm{vol}_H}{d\mu}$ is not granted see [1].

In this paper we study the Hausdorff volume in a non equiregular sub-Riemannian manifold. A point p is called regular if the growth vector $q \mapsto (\dim \mathcal{D}^1_q, \dots, \dim \mathcal{D}^{r(q)}_q)$ is constant in a neighborhood of p, otherwise p is called singular. The natural assumption under which we perform our work is that the manifold is stratified by equisingular submanifolds, where both the growth vector of the distribution and the growth vector of the distribution restricted to the submanifold are constant. These submanifolds were introduced in [16] and, thanks to their simple Lie algebraic structure, they constitute the fundamental block that allows us to carry through our investigation.

When the set S of singular points is not empty, the Hausdorff dimension of M is obviously the maximum between the Hausdorff dimension of S and the Hausdorff dimension of the set $\mathsf{R} = M \setminus \mathsf{S}$ of regular points.

The first question is whether vol_H is absolutely continuous with respect to a smooth volume μ. It turns out that this may not be the case and vol_H may have a singular part. More precisely, under the assumption that S is μ-negligible, then vol_H admits the following Lebesgue decomposition

$$\mathrm{vol}_H = \mathrm{vol}_{H \llcorner \mathsf{R}} + \mathrm{vol}_{H \llcorner \mathsf{S}},$$

that is, $\mathrm{vol}_{H \llcorner \mathsf{R}} \ll \mu$ and $\mathrm{vol}_{H \llcorner \mathsf{S}} \perp \mu$ (see Corollary 3.9). As a consequence, vol_H is absolutely continuous with respect to a smooth volume μ if and only if the Hausdorff dimension of the singular set is smaller than the Hausdorff dimension of the set of regular points.

The next problem is to analyze the behavior of the absolutely continuous part $\mathrm{vol}_{H \llcorner \mathsf{R}}$ (which is nontrivial if and only if $\dim_H \mathsf{R} \geq \dim_H \mathsf{S}$). Thanks to (2), this amounts to study the function $q \mapsto \hat{\mu}^q(\widehat{B}_q)$ near singular points. To this aim, we show that the asymptotics of this function is characterized by the one of the determinants of adapted bases at regular points (see Proposition 3.12). As a direct consequence, we deduce that $\frac{d\mathrm{vol}_{H \llcorner \mathsf{R}}}{d\mu}(q)$ blows up when q approaches the singular set (see Proposition 3.15). In particular, $\frac{d\mathrm{vol}_{H \llcorner \mathsf{R}}}{d\mu}$ is not essentially bounded near the singular set, that is, unlike the equiregular case, μ and $\mathrm{vol}_{H \llcorner \mathsf{R}}$ are no longer commensurable. Going further in the regularity analysis, we find out that $\mathrm{vol}_{H \llcorner \mathsf{R}}$ may even fail to be locally integrable with respect to μ and therefore $\mathrm{vol}_{H \llcorner \mathsf{R}}$ may fail to be a Radon measure. More precisely, we exhibit a sufficient condition involving the nonholonomic order of $\frac{d\mathrm{vol}_{H \llcorner \mathsf{R}}}{d\mu}$ and the algebraic structure of the distribution at a singular point for non-integrability of $\mathrm{vol}_{H \llcorner \mathsf{R}}$ (see Proposition 4.4), and another one involving the usual order of functions and the codimension of the singular set (see Proposition 4.10). For instance, these conditions are satisfied when the Hausdorff dimension of R is not greater than the Hausdorff dimension of S or when S topologically splits M (see Corollaries 4.6 and 4.11). We also exhibit a sufficient condition for the integrability of $\mathrm{vol}_{H \llcorner \mathsf{R}}$ (see Proposition 4.9), but there is a gap between integrability and non-integrability conditions. In Figure 1 we summarize the relations between the Hausdorff volume and μ in all cases.

For generic sub-Riemannian manifolds, we apply our technique and we characterize the integrability of the absolutely continuous part by comparing the dimension of the manifold to dimensions of free Lie algebras (see Proposition 6.1).

As for the singular part $\mathrm{vol}_{H \llcorner \mathsf{S}}$, the stratification assumption permits to focus on each equisingular submanifold $N \subset \mathsf{S}$. For an equisingular submanifold $N \subset \mathsf{S}$, we give an algebraic characterization of the Hausdorff dimension of N and we compare the Hausdorff volume with a smooth volume on the submanifold (see Theorem 5.3). We show actually that on the restricted metric space $(N, d|_N)$ the situation is very similar to the one in equiregular manifolds: the Hausdorff volume is absolutely continuous with respect to any smooth volume on N and we have an expression of the Radon-Nikodym derivative in terms of the nilpotent approximation. Results of this part of the paper have been announced in [14].

<table>
<tr><td>$\mathsf{S} = \emptyset$</td><td>$\mathsf{S} \neq \emptyset$
$Q_{\mathsf{S}} < Q_{\mathsf{R}}$</td><td>$\mathsf{S} \neq \emptyset$
$Q_{\mathsf{S}} = Q_{\mathsf{R}}$</td><td>$\mathsf{S} \neq \emptyset$
$Q_{\mathsf{S}} > Q_{\mathsf{R}}$</td></tr>
<tr><td>$\mathrm{vol}_H \ll \mu$</td><td>$\mathrm{vol}_H \ll \mu$</td><td>$\mathrm{vol}_H \not\ll \mu$</td><td>$\mathrm{vol}_H \perp \mu$</td></tr>
<tr><td>vol_H, μ
commensurable</td><td>vol_H, μ
not commensurable</td><td>$\mathrm{vol}_{H \llcorner \mathsf{R}}, \mu$
not commensurable</td><td rowspan="3"></td></tr>
<tr><td rowspan="2">vol_H Radon</td><td>vol_H Radon
Prop. 4.9</td><td rowspan="2">vol_H not Radon
Cor. 4.6</td></tr>
<tr><td>vol_H not Radon
Prop. 4.4 and 4.10</td></tr>
</table>

Figure 1: Summary of relations between vol_H and a smooth volume μ on an oriented stratified sub-Riemannian manifold ($Q_{\mathsf{S}}, Q_{\mathsf{R}}$ denote the Hausdorff dimensions of S and R, respectively)

The structure of the paper is the following. In Section 2 we briefly recall Hausdorff measures on metric spaces and the basic concepts in sub-Riemannian geometry. Section 3 is devoted to the analysis of the Hausdorff volume in the equiregular case and of the Lebesgue decomposition of the Hausdorff volume in the non equiregular case. In Section 4 we perform the study of regularity of the Radon-Nikodym derivative and we provide sufficient conditions for integrability and non-integrability. Then, Section 5 deals with equisingular submanifolds by providing first a complete study of the algebraic and metric structure of such submanifolds, and then an analysis of the properties of the Hausdorff volume. Finally, in Section 6 we apply the methods of Section 4 to the study of the generic smooth case and we list some examples. We end with an appendix containing the proof of a technical result, Proposition A.1 which is a uniform Ball-Box Theorem on equisingular submanifolds and which is a key ingredient to study the behavior of $\frac{d\mathrm{vol}_{H \llcorner \mathsf{R}}}{d\mu}$ at singular points.

2 Definitions

2.1 Hausdorff and smooth volumes

Let us first recall some basic facts on Hausdorff measures. Let (M, d) be a metric space. We denote by $\operatorname{diam} S$ the diameter of a set $S \subset M$. Let $\alpha \geq 0$ be a real number. For every set $E \subset M$, the *α-dimensional Hausdorff measure* $\mathcal{H}^\alpha$ of E is defined as $\mathcal{H}^\alpha(E) = \lim_{\epsilon \to 0^+} \mathcal{H}^\alpha_\epsilon(E)$, where

$$\mathcal{H}^\alpha_\epsilon(E) = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} S_i)^\alpha \ : \ E \subset \bigcup_{i=1}^{\infty} S_i, \ S_i \text{ nonempty set}, \ \operatorname{diam} S_i \leq \epsilon \right\},$$

and the *α-dimensional spherical Hausdorff measure* is defined as $\mathcal{S}^\alpha(E) = \lim_{\epsilon \to 0^+} \mathcal{S}^\alpha_\epsilon(E)$, where

$$\mathcal{S}^\alpha_\epsilon(E) = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} S_i)^\alpha \ : \ E \subset \bigcup_{i=1}^{\infty} S_i, \ S_i \text{ is a ball}, \ \operatorname{diam} S_i \leq \epsilon \right\}.$$

For every set $E \subset M$, the non-negative number

$$D = \sup\{\alpha \geq 0 \mid \mathcal{H}^\alpha(E) = \infty\} = \inf\{\alpha \geq 0 \mid \mathcal{H}^\alpha(E) = 0\}$$

is called the *Hausdorff dimension of E*. Notice that $\mathcal{H}^D(E)$ may be 0, > 0, or ∞. By construction, for every subset $S \subset M$,

$$\mathcal{H}^\alpha(S) \leq \mathcal{S}^\alpha(S) \leq 2^\alpha \mathcal{H}^\alpha(S), \tag{3}$$

hence the Hausdorff dimension can be defined equivalently using spherical measures. In the sequel we will call *Hausdorff volume* the spherical Hausdorff measure $\mathcal{S}^{\dim_H M}$ and we will denote this measure by vol_H.

Given a subset $N \subset M$, we consider the metric space $(N, d|_N)$. Denoting by $\mathcal{H}^\alpha_N$ and $\mathcal{S}^\alpha_N$ the Hausdorff and spherical Hausdorff measures in this space, by definition we have

$$\begin{aligned} \mathcal{H}^\alpha{\llcorner_N}(E) &:= \mathcal{H}^\alpha(E \cap N) = \mathcal{H}^\alpha_N(E \cap N), \\ \mathcal{S}^\alpha{\llcorner_N}(E) &:= \mathcal{S}^\alpha(E \cap N) \leq \mathcal{S}^\alpha_N(E \cap N). \end{aligned} \tag{4}$$

Notice that the inequality (4) is strict in general, as coverings in the definition of $\mathcal{S}^\alpha_N$ are made with sets B which satisfy $B = B(p, \rho) \cap N$ with $p \in N$, whereas coverings in the definition of $\mathcal{S}^\alpha{\llcorner_N}$ include sets of the type $B(p, \rho) \cap N$ with $p \notin N$ ($\mathcal{S}^\alpha{\llcorner_N}$ is very similar to centered Hausdorff measures, see [9]). Moreover, thanks to (3), for every $S \subset N$, there holds $\mathcal{H}^\alpha_N(S) \leq \mathcal{S}^\alpha_N(S) \leq 2^\alpha \mathcal{H}^\alpha_N(S)$. Hence

$$\mathcal{H}^\alpha(S) \leq \mathcal{S}^\alpha_N(S) \leq 2^\alpha \mathcal{H}^\alpha(S),$$

and $\mathcal{S}^\alpha_N$ is absolutely continuous with respect to $\mathcal{H}^\alpha{\llcorner_N}$. We will denote by vol^N_H the Hausdorff volume $\mathcal{S}^{\dim_H N}_N$.

When M is an oriented manifold we can introduce another kind of volume. We say that a measure μ is a *smooth volume* on M if there exists a positively-oriented non degenerate n-form $\omega \in \Lambda^n M$ on M (i.e. ω is a volume form) such that, for every Borel set $E \subset M$, $\mu(E) = \int_E \omega$.

Finally we will say that two measures μ, ν on M are *commensurable* if μ, ν are mutually absolutely continuous, i.e., $\mu \ll \nu$ and $\nu \ll \mu$, and if both Radon-Nikodym derivatives $\frac{d\mu}{d\nu}$ and $\frac{d\nu}{d\mu}$ exist and are locally essentially bounded. When it is the case, for every compact set K there exists $C > 0$ such that

$$\frac{1}{C}\mu{\llcorner_K} \leq \nu{\llcorner_K} \leq C\mu{\llcorner_K}.$$

In particular, thanks to (3), $\mathcal{H}^\alpha$ and $\mathcal{S}^\alpha$ are mutually absolutely continuous and commensurable.

2.2 Sub-Riemannian manifolds

Usually, a sub-Riemannian manifold is a triplet (M, Δ, g^R), where M is a smooth (i.e., $\mathcal{C}^\infty$) manifold, Δ is a subbundle of TM of rank $m < \dim M$ and g^R is a Riemannian metric on Δ. Using g^R, the length of horizontal curves, i.e., absolutely continuous curves which are almost everywhere tangent to Δ, is well-defined. When Δ is Lie bracket generating, the map $d : M \times M \to \mathbb{R}$ defined as the infimum of length of horizontal curves between two given points is a continuous distance (Rashevsky-Chow Theorem), and it is called sub-Riemannian distance.

In this paper we study sub-Riemannian manifolds with singularities. Thus it is natural to work in a larger setting, where the map $q \mapsto \Delta_q$ itself may have singularities. This leads us to the following generalized definition [2, 6].

Definition 2.1. A *sub-Riemannian* structure on a manifold M is a triplet $(\mathbf{U}, \langle\cdot,\cdot\rangle, f)$ where $(\mathbf{U}, \langle\cdot,\cdot\rangle)$ is a Euclidean vector bundle over M (i.e., a vector bundle $\pi_\mathbf{U} : \mathbf{U} \to M$ equipped with a smoothly-varying scalar product $q \mapsto \langle\cdot,\cdot\rangle_q$ on the fibre $\mathbf{U}_q$) and f is a morphism of vector bundles $f : \mathbf{U} \to TM$, i.e. a smooth map linear on fibers and such that, for every $u \in \mathbf{U}$, $\pi(f(u)) = \pi_\mathbf{U}(u)$, where $\pi : TM \to M$ is the usual projection.

Let $(\mathbf{U}, \langle\cdot,\cdot\rangle, f)$ be a sub-Riemannian structure on M. We define the submodule $\mathcal{D} \subset \mathrm{Vec}(M)$ as

$$\mathcal{D} = \{f \circ \sigma \mid \sigma \text{ smooth section of } \mathbf{U}\}, \tag{5}$$

and for $q \in M$ we set $\mathcal{D}_q = \{X(q) \mid X \in \mathcal{D}\} \subset T_qM$. Clearly $\mathcal{D}_q = f(\mathbf{U}_q)$. The length of a tangent vector $v \in \mathcal{D}_q$ is defined as

$$g_q(v) := \inf\{\langle u, u\rangle_q \mid f(u) = v, u \in \mathbf{U}_q\}. \tag{6}$$

An absolutely continuous curve $\gamma : [a, b] \to M$ is *horizontal* if $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ for almost every t. If $\mathcal{D}$ is Lie bracket generating, that is

$$\forall\, q \in M \quad \mathrm{Lie}_q \mathcal{D} = T_qM, \tag{7}$$

then the map $d : M \times M \to \mathbb{R}$ defined as the infimum of length of horizontal curves between two given points is a continuous distance as in the classic case. In this paper, all sub-Riemannian manifolds are assumed to satisfy the Lie bracket generating condition (7).

Remark 2.2. Definition 2.1 includes the following cases.

- Classic sub-Riemannian structures: in this case $\mathbf{U}$ is a subbundle of TM and f is the inclusion. With the notations used at the beginning of this section, this amounts to take $\mathbf{U} = \Delta$, and $\langle \cdot, \cdot \rangle = g^R$. Then the module $\mathcal{D}$ coincides with the module of smooth sections of the subbundle Δ and $\mathcal{D}_q$ has constant dimension. Moreover, for every $q \in M$ and $v \in \mathcal{D}_q$, $g_q(v) = g_q^R(v, v)$.
- Sub-Riemannian structures associated with a family of vector fields $X_1, \ldots, X_m$: in this case $\mathbf{U} = M \times \mathbb{R}^m$ is the trivial bundle of rank m over M, $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product on $\mathbb{R}^m$, and $f : M \times \mathbb{R}^m \to M$ is defined as $f(q, u) = \sum_{i=1}^m u_i X_i(q)$. Here $\mathcal{D}$ is the module generated by $X_1, \ldots X_m$ and g is given by
$$g_q(v) = \inf \left\{ \sum_{i=1}^m u_i^2 \;\middle|\; v = \sum_{i=1}^m u_i X_i(q) \right\}.$$

Let $(\mathbf{U}, \langle \cdot, \cdot \rangle, f)$ be a sub-Riemannian structure on a manifold M, and $\mathcal{D}$, g the corresponding module and quadratic form as defined in (5) and (6). In analogy with the classic sub-Riemannian case and to simplify notations, in the sequel we will refer to the sub-Riemannian manifold as the triplet $(M, \mathcal{D}, g)$. This is justified since all the constructions and definitions below rely only on $\mathcal{D}$ and g.

Given $i \geq 1$, define recursively the submodule $\mathcal{D}^i \subset \mathrm{Vec}(M)$ by
$$\mathcal{D}^1 = \mathcal{D}, \quad \mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}, \mathcal{D}^i].$$
Fix $p \in M$ and set $\mathcal{D}_p^i = \{X(p) \mid X \in \mathcal{D}^i\}$. The Lie-bracket generating assumption implies that there exists an integer $r(p)$ such that
$$\{0\} = \mathcal{D}_p^0 \subset \mathcal{D}_p^1 \subset \cdots \subset \mathcal{D}_p^{r(p)} = T_p M. \tag{8}$$
The sequence of subspaces (8) is called the *flag of* $\mathcal{D}$ *at* p. Set $n_i(p) = \dim \mathcal{D}_p^i$ and
$$Q(p) = \sum_{i=1}^{r(p)} i(n_i(p) - n_{i-1}(p)). \tag{9}$$
This integer will play a crucial role for determining the Hausdorff dimension of (M, d). To write $Q(p)$ in a different way, we define the *weights* of the flag (8) at p as the integers $w_1(p), \ldots, w_n(p)$ such that $w_i(p) = s$ if $\dim \mathcal{D}_p^{s-1} < i \leq \dim \mathcal{D}_p^s$. Then $Q(p) = \sum_{i=1}^n w_i(p)$.

We say that a point p is *regular* if, for every i, $n_i(q)$ is constant as q varies in a neighborhood of p. Otherwise, the point is said to be *singular*. The sub-Riemannian manifold is called *equiregular* if every point is regular.

When the dimensions $n_i(q)$ are constant on M, the module $\mathcal{D}^i$ coincides with the module of vector fields
$$\{X \in \mathrm{Vec}(M) \mid X(q) \in \mathcal{D}_q^i \;\forall\, q \in M\},$$
i.e., vector fields that are tangent to the distribution $q \mapsto \mathcal{D}_q^i$. Yet the identification between the module $\mathcal{D}^i$ and the map $q \mapsto \mathcal{D}_q^i$ is no more meaningful when the dimension of $\mathcal{D}_q^i$ varies as a function of q (see the discussion in [6, page 48]). Indeed, in the rank-varying case, a vector field tangent to $\mathcal{D}_q^i$ at every $q \in M$ may fail[1] to be in the module $\mathcal{D}^i$. Definition 2.1 allows to take account of structures where the dimensions $n_i(q)$ (and in particular $n_1(q)$) may vary.

[1]For instance, on $M = \mathbb{R}$, take the module $\mathcal{D} \subset \mathrm{Vec}\,\mathbb{R}$ generated by $X(x) = x^2 \partial_x$. Then the vector field $Y(x) = x\partial_x$ is clearly tangent to the distribution $x \mapsto \mathcal{D}_x$ but does not belong to $\mathcal{D}$.

Example 2.3 (Grushin plane). Let $M = \mathbb{R}^2$, $\mathbf{U} = \mathbb{R}^2 \times \mathbb{R}^2$ endowed with the canonical Euclidean structure, and f be the morphism defined as follows. If σ_1, σ_2 is a global orthonormal basis on $\mathbf{U}$, we set $f(\sigma_1(x,y)) = \partial_x$ and $f(\sigma_2(x,y)) = x\partial_y$. Then $\mathcal{D}_{(x,y)}$ is two dimensional for every $x \neq 0$, whereas $\dim \mathcal{D}_{(0,y)} = 1$.

Remark 2.4. At a regular point p, the equality $n_i(p) = n_{i+1}(p)$ implies that the local distribution $q \mapsto \mathcal{D}^i_q$ is involutive, and so that $n_i(p) = n_j(p)$ for any $j \geq i$. From the Lie bracket generating assumption (7) we deduce $n_i(p) < n_{i+1}(p)$ for $i < r(p)$, which in turn implies $Q(p) \leq n^2$.

Given any sub-Riemannian manifold $(M, \mathcal{D}, g)$ there always exist a (possibly very big) integer m and vector fields $X_1, \ldots, X_m$ such that $\mathcal{D}$ is globally generated by $X_1, \ldots, X_m$ and

$$g_q(v) = \inf \left\{ \sum_{i=1}^m u_i^2 \;\middle|\; v = \sum_{i=1}^m u_i X_i(q) \right\} \qquad \text{for every } q \in M,\ v \in T_qM.$$

We call such a family $X_1, \ldots, X_m$ a *(global) generating family* for the sub-Riemannian structure $(\mathcal{D}, g)$. The existence of a generating family is a consequence of [2, Corollary 3.16]. For an alternative proof see also [27].

Consider a generating family $X_1, \ldots, X_m$ for $(\mathcal{D}, g)$. A multi-index I of length $|I| = j \geq 1$ is an element of $\{1, \ldots, m\}^j$. With any multi-index $I = (i_1, \ldots, i_j)$ is associated an iterated Lie bracket $X_I = [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{j-1}}, X_{i_j}] \ldots]$ (we set $X_I = X_{i_1}$ if $j = 1$). The set of vector fields $\{X_I \mid |I| \leq j\}$ generates the module $\mathcal{D}^j$. As a consequence, if the values of $X_{I_1}, \ldots, X_{I_n}$ at $p \in M$ are linearly independent, then $\sum_i |I_i| \geq Q(p)$.

Let Y be a vector field. We define the *length of* Y by

$$\ell(Y) = \min\{i \in \mathbb{N} \mid Y \in \mathcal{D}^i\}.$$

In particular, $\ell(X_I) \leq |I|$. By an *adapted basis* to the flag (8) at p, we mean n vector fields $Y_1, \ldots, Y_n$ such that their values at p satisfy

$$\mathcal{D}^i_p = \operatorname{span}\{Y_j(p) \mid \ell(Y_j) \leq i\}, \quad \forall\, i = 1, \ldots, r(p).$$

In particular, $\sum_{i=1}^n \ell(Y_i) = Q(p)$. As a consequence, a family of Lie brackets $X_{I_1}, \ldots, X_{I_n}$ such that $X_{I_1}(p), \ldots, X_{I_n}(p)$ are linearly independent is an adapted basis to the flag (8) at p if and only if $\sum_i |I_i| = Q(p)$.

Let $h : U \to \mathbb{R}$ be a continuous function on a neighborhood of p. The *nonholonomic order of* h *at* p is

$$\operatorname{ord}_p(h) := \inf\{s \in (0, +\infty) \mid h(q) = O(d(p,q)^s)\}.$$

If h is a smooth function, then $\operatorname{ord}_p(h)$ admits an algebraic characterization, namely,

$$\operatorname{ord}_p(h) = \min\{s \in \mathbb{N} \mid \exists\, i_1, \ldots, i_s \in \{1 \ldots, m\} \text{ such that } (X_{i_1} X_{i_2} \cdots X_{i_s} h)(p) \neq 0\}.$$

A smooth function h is called *privileged* at p if

$$\operatorname{ord}_p(h) = \max\{s \in \mathbb{N} \mid dh(\mathcal{D}^s_p / \mathcal{D}^{s-1}_p) \neq 0\},$$

where $\mathcal{D}^s_p / \mathcal{D}^{s-1}_p$ is a vector subspace such that $\mathcal{D}^s_p = \mathcal{D}^{s-1}_p \oplus (\mathcal{D}^s_p / \mathcal{D}^{s-1}_p)$. We say that coordinates $\varphi = (x_1, \ldots, x_n) : U \to \mathbb{R}^n$ are *privileged at* p if they are centered at p and linearly adapted (that is the coordinate vector fields are an adapted basis to the flag at p), and if every coordinate function x_i is privileged at p.

Note that in this definition of privileged coordinates, the coordinate functions are not ordered by increasing nonholonomic order. However the set $\{\operatorname{ord}_p(x_i), i = 1, \ldots, n\}$ coincides with $\{w_1(p), \ldots, w_n(p)\}$, so we can relabel the weights in such a way that $\operatorname{ord}_p(x_i) = w_i(p)$. We then say that the weights are *labeled according to the coordinates* φ.

For every point $p \in M$ the metric tangent cone[2] to (M, d) at p exists and is isometric to $(T_pM, \widehat{d}_p)$ where $\widehat{d}_p$ is the sub-Riemannian distance associated with the nilpotent approximation of the structure at p (see

[2] in Gromov's sense, see [15]

[6, Definition 5.15] for the definition of nilpotent approximation). This was shown in [22] for equiregular sub-Riemannian manifolds and in [6] for the general case. Isometries between the metric tangent cone and $(T_pM, \widehat{d}_p)$ are given by privileged coordinates at p.

By construction, the metric tangent cone is endowed with a family of dilations δ_λ with respect to which the distance $\widehat{d}_p$ is homogeneous. If $\varphi : U \to \mathbb{R}^n$ is a system of privileged coordinates at p then dilations are given by

$$\delta_\lambda(x_1, \dots, x_n) = (\lambda^{w_1(p)} x_1, \dots, \lambda^{w_n(p)} x_n),$$

where $w_i(p) = \mathrm{ord}_p(x_i)$.

We refer the reader to [2, 6, 23] for a primer in sub-Riemannian geometry.

3 Lebesgue decomposition of the Hausdorff volume

In this section we investigate the relation between the Hausdorff volume and a smooth volume on a sub-Riemannian manifold. We first recall the equiregular case, where the situation is well understood, then we consider the case where singular points are present. We write the Lebesgue decomposition of the Hausdorff volume with respect to the smooth volume and we start the analysis of both the absolutely continuous part and the singular part.

3.1 Equiregular case

Assume $(M, \mathcal{D}, g)$ is an equiregular connected sub-Riemannian manifold. Then $q \mapsto Q(q)$ is constant on M (see (9)) and we denote by Q its constant value. Moreover at every point $p \in M$ the metric tangent cone to (M, d), which is isometric to $(T_pM, \widehat{d}_p)$, has a structure of a Carnot group. Assume M is also oriented and let μ be a smooth volume on M. The associated volume form ω induces canonically a left-invariant volume form $\hat{\omega}_p$ on T_pM. We denote by $\hat{\mu}^p$ the smooth volume on T_pM defined by $\hat{\omega}_p$. To clarify this construction, we refer the reader to Proposition 5.1 in Section 5.

Theorem 3.1. *Let $(M, \mathcal{D}, g)$ be an equiregular connected oriented sub-Riemannian manifold and let μ be a smooth volume on M. Then*

(i) $\dim_H M = Q$ *and* $\mathrm{vol}_H = \mathcal{S}^Q$*;*

(ii) vol_H *is a Radon measure on M;*

(iii) $\mathrm{vol}_H \ll \mu$ *and* $\mu \ll \mathrm{vol}_H$*;*

(iv) the Radon-Nikodym derivative $\frac{d\mathrm{vol}_H}{d\mu}(p)$ *coincides with the density* $\lim_{\epsilon\to 0} \frac{\mathrm{vol}_H(B(p,\epsilon))}{\mu(B(p,\epsilon))}$*, whose value is*

$$\lim_{\epsilon\to 0} \frac{\mathrm{vol}_H(B(p,\epsilon))}{\mu(B(p,\epsilon))} = \frac{2^Q}{\hat{\mu}^p(\widehat{B}_p)}, \quad \forall\, p \in M, \tag{10}$$

where $\widehat{B}_p$ denotes the ball centered at 0 and of radius 1 in T_pM with respect to $\widehat{d}_p$.

Remark 3.2. As we will see in Theorem 5.3 below, one can interpret the constant 2 in (10) as the diameter of the ball $\widehat{B}_p$ with respect to the distance $\widehat{d}_p$. Since the nilpotent approximation is a Carnot group here, we then have $2^Q = \mathrm{diam}_{\widehat{d}_p}(\widehat{B}_p)^Q = \mathcal{S}^Q_{\widehat{d}_p}(\widehat{B}_p)$ (here $\mathcal{S}^Q_{\widehat{d}_p}$ denotes the spherical Hausdorff measure in T_pM with respect to the distance $\widehat{d}_p$), which gives a clear interpretation to (10).

Theorem 3.1 is a special case of Theorem 5.3 proved in Section 5. Note that all the statements in Theorem 3.1 are well-known, but they have been proved only fragmentarily in several references: properties (i) and (iii) are stated in [22] (for a rigorous proof see [23]), property (iv) in [1]. Up to the authors' knowledge, property (ii) has never been stated as is and it is a consequence of basic covering arguments in geometric measure theory. In particular, this property is needed to apply the differentiation theorem for Radon measures [26, Theorem 4.7 p.24] and ensures that $\frac{d\mathrm{vol}_H}{d\mu}(p)$ coincides with $\lim_{\epsilon\to 0} \frac{\mathrm{vol}_H(B(p,\epsilon))}{\mu(B(p,\epsilon))}$.

Remark 3.3. Properties (i), (ii), (iii) hold true if we replace vol_H by $\mathcal{H}^{\dim_H M}$. In particular, applying the differentiation theorem for Radon measures [26, Theorem 4.7 p.24] we get that (i) the limit $\lim_{\epsilon\to 0}\frac{\mathcal{H}^Q(B(p,\epsilon))}{\mu(B(p,\epsilon))}$ exists μ-almost everywhere and it coincides with the Radon-Nikodym derivative of $\mathcal{H}^Q$ with respect to μ. Nevertheless, we do not have an explicit representation of such limit as we have for the spherical Hausdorff case in (10).

A first consequence of Theorem 3.1 is that the Radon-Nikodym derivative of vol_H with respect to μ is continuous on M, see [1, Corollary 2 and Section 4.1]. This is due to the fact that both the nilpotent approximation at q and the tangent measure $\hat{\mu}^q$ depend smoothly on q. Studying higher regularity of $\frac{d\mathrm{vol}_H}{d\mu}$ is the main subject in [1], to which we refer the interested reader.

A further consequence is that μ and vol_H are commensurable (see the last remark in [22]). This follows directly by the fact that $q\mapsto\hat{\mu}^q(\widehat{B}_q)$ is positive and continuous on M, so that $\frac{d\mathrm{vol}_H}{d\mu}(q)$ and its inverse $\frac{d\mu}{d\mathrm{vol}_H}(q)$ are locally bounded.

In conclusion, when the manifold is equiregular, the Hausdorff volume essentially behaves as a smooth measure. This fails when singular points are present, as we see in the next section.

3.2 Non equiregular case

Assume $(M,\mathcal{D},g)$ is an oriented sub-Riemannian manifold and μ is a smooth volume on M. The manifold is split into the disjoint union of two sets

$$M=\mathsf{R}\cup\mathsf{S},$$

where R,S denote respectively the set of regular and singular points. Since the functions $q\mapsto\dim\mathcal{D}^i_q$ are lower semi-continuous, R is an open and dense subset of M and conversely S is a closed subset of empty interior.

We will assume in this paper that $\mu(\mathsf{S})=0$. This assumption is satisfied for a very wide class of sub-Riemannian manifolds: for instance for analytic sub-Riemannian manifolds, for generic sub-Riemannian structures on a given manifold (see Section 6.1), or when $q\mapsto\mathcal{D}_q$ is a distribution of corank 1 [25, Proposition 3.3]. Nevertheless one can build up sub-Riemannian manifolds where $\mu(\mathsf{S})\neq 0$ as in the next example.

Example 3.4. Let $S\subset\mathbb{R}^4$ be any closed set having empty interior and positive Lebesgue measure. By Whitney's Extension Theorem (see for instance [21, Proposition A.8]), there exists a function $f\in\mathcal{C}^\infty(\mathbb{R}^4)$ such that $S=f^{-1}(0)$. Consider the sub-Riemannian structure on $\mathbb{R}^4$ for which the following vector fields are a generating family,

$$X_1=\partial_1,\quad X_2=\partial_2+x_1\partial_3+\frac{x_1^2}{2}\partial_4,\quad X_3=f(x)\partial_4.$$

One easily checks that the singular set coincides with S and thus it is of positive measure.

We are interested in properties of $\mathrm{vol}_H=\mathcal{S}^{\dim_H M}$, hence we first compute $\dim_H M$. Clearly, the Hausdorff dimension of M is $\max\{\dim_H\mathsf{R},\dim_H\mathsf{S}\}$. Then, using property (i) in Theorem 3.1 the Hausdorff dimension of the regular set can be computed algebraically as

$$\dim_H\mathsf{R}=\sup\{\dim_H O,\ O\text{ connected component of }\mathsf{R}\}=\max_{q\in\mathsf{R}}Q(q).$$

This follows from the fact that R is a countable union of open connected components, and by the inequality $Q(q)\leq n^2$ at every $q\in\mathsf{R}$ (see Remark 2.4). To remind its algebraic characterization, we denote by Q_{R} the number $\max_{q\in\mathsf{R}}Q(q)$ (which equals $\dim_H\mathsf{R}$). We present in Section 5 an analogous method to compute the Hausdorff dimension of S under some stratification assumptions.

Applying Theorem 3.1 to every connected component of R we obtain the following result.

Proposition 3.5. *Let $(M,\mathcal{D},g)$ be an oriented sub-Riemannian manifold and let μ be a smooth volume on M. Assume $\mu(\mathsf{S})=0$. Then $\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}\ll\mu$, the Radon-Nikodym derivative of $\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}$ with respect to μ exists μ-almost everywhere, and it coincides with the density $\lim_{\epsilon\to 0}\frac{\mathcal{S}^{Q_{\mathsf{R}}}(B(q,\epsilon))}{\mu(B(q,\epsilon))}$ for every $q\in\mathsf{R}$.*

Moreover, if O is a connected component of R such that $\dim_H O = Q_{\mathsf{R}}$, then

$$\frac{d\mathcal{S}^{Q_{\mathsf{R}}}\llcorner_{\mathsf{R}}}{d\mu}(q) = \frac{2^{Q_{\mathsf{R}}}}{\hat{\mu}^q(\widehat{B}_q)} \qquad \text{for every } q \in O. \tag{11}$$

Remark 3.6. The measure $\mathcal{S}^{Q_{\mathsf{R}}}\llcorner_{\mathsf{R}}$ may not be a Radon measure on M (see Section 4 below). Thus the existence of a Radon-Nikodym derivative and the fact that it coincides with the density are not consequences of $\mathcal{S}^{Q_{\mathsf{R}}}\llcorner_{\mathsf{R}} \ll \mu$.

Remark 3.7. In the statement above we can replace $\mathcal{S}^{Q_{\mathsf{R}}}$ by $\mathrm{vol}_{H\llcorner\mathsf{R}}$. Indeed, if $\dim_H \mathsf{S} > Q_{\mathsf{R}}$, then $\mathrm{vol}_H = \mathcal{S}^{\dim_H \mathsf{S}}$. As a consequence, $\mathrm{vol}_{H\llcorner\mathsf{R}} \equiv 0$ and $\mathrm{vol}_{H\llcorner\mathsf{R}}(B(q,\epsilon)) = 0$, for every $q \in \mathsf{R}$, so that the density becomes trivial. On the other hand, if $\dim_H \mathsf{S} \le Q_{\mathsf{R}}$ then $\mathrm{vol}_{H\llcorner\mathsf{R}} = \mathcal{S}^{Q_{\mathsf{R}}}\llcorner_{\mathsf{R}}$, that is, the two measures coincides.

Remark 3.8. When there is a connected component $O \subset \mathsf{R}$ with $\dim_H O < Q_{\mathsf{R}}$, the density $\lim_{\epsilon\downarrow 0} \frac{\mathcal{S}^{Q_{\mathsf{R}}}(B(q,\epsilon))}{\mu(B(q,\epsilon))}$ vanishes at every point $q \in O$ and so $\frac{d\mathcal{S}^{Q_{\mathsf{R}}}\llcorner_{\mathsf{R}}}{d\mu} \equiv$ on O.

Proof of Proposition 3.5. Apply Theorem 3.1 to each connected component of R. Then, by property (ii), $\mathcal{S}^{Q_{\mathsf{R}}}$, as a measure on the metric space $(\mathsf{R}, d|_{\mathsf{R}})$, is Radon, that is, $\mathcal{S}^{Q_{\mathsf{R}}}$ is finite on compact subsets contained in R. Moreover, by property (iii), $\mathcal{S}^{Q_{\mathsf{R}}}$, as a measure on R, is absolutely continuous with respect to μ. Obviously, μ is also a Radon measure on M (and thus on R) which is also locally doubling (see [24]). Thus we can apply the differentiation theorem for Radon measures (see for instance [26, Theorem 4.7 p.24] with $X = \mathsf{R}$, $\mu_2 = \mathrm{vol}_H$ and $\mu_1 = \mu$) and deduce that, on R, the Radon-Nikodym derivative of $\mathcal{S}^{Q_{\mathsf{R}}}$ with respect to μ coincides with the density $\lim_{\epsilon\to 0} \frac{\mathcal{S}^{Q_{\mathsf{R}}}(B(q,\epsilon))}{\mu(B(q,\epsilon))}$. In other words, for every Borel set $E \subset \mathsf{R}$,

$$\mathcal{S}^{Q_{\mathsf{R}}}(E) = \int_E \lim_{\epsilon\to 0} \frac{\mathcal{S}^{Q_{\mathsf{R}}}(B(q,\epsilon))}{\mu(B(q,\epsilon))} d\mu(q).$$

Since R is open and $\mu(M \setminus \mathsf{R}) = 0$, we deduce that for every Borel set $E \subset M$,

$$\mathcal{S}^{Q_{\mathsf{R}}}\llcorner_{\mathsf{R}}(E) = \int_{E\cap\mathsf{R}} \lim_{\epsilon\to 0} \frac{\mathcal{S}^{Q_{\mathsf{R}}}(B(q,\epsilon))}{\mu(B(q,\epsilon))} d\mu(q),$$

which gives the conclusion. □

Since R is open and Hausdorff measures are Borel regular, R and S are μ- and vol_H-measurable. Hence for every set $E \subset M$

$$\mathrm{vol}_H(E) = \mathrm{vol}_H(E \cap \mathsf{R}) + \mathrm{vol}_H(E \cap \mathsf{S}),$$

or equivalently, $\mathrm{vol}_H = \mathrm{vol}_{H\llcorner\mathsf{R}} + \mathrm{vol}_{H\llcorner\mathsf{S}}$. Moreover, since $\mu(\mathsf{S}) = 0$, $\mathrm{vol}_{H\llcorner\mathsf{S}}$ is concentrated on S whereas μ is concentrated on R. Therefore μ and $\mathrm{vol}_{H\llcorner\mathsf{S}}$ are mutually singular. We thus get directly the following fact.

Corollary 3.9. *Let $(M, \mathcal{D}, g)$ be an oriented sub-Riemannian manifold and let μ be a smooth volume on M. Assume $\mu(\mathsf{S}) = 0$. Then μ and $\mathrm{vol}_{H\llcorner\mathsf{S}}$ are mutually singular and the Lebesgue decomposition of vol_H with respect to μ is*

$$\mathrm{vol}_H = \mathrm{vol}_{H\llcorner\mathsf{R}} + \mathrm{vol}_{H\llcorner\mathsf{S}}.$$

As a consequence, (i) if $\dim_H \mathsf{R} < \dim_H \mathsf{S}$ then vol_H and μ are mutually singular; (ii) if $\dim_H \mathsf{R} > \dim_H \mathsf{S}$ then $\mathrm{vol}_H \ll \mu$.

Corollary 3.9 provides the Lebesgue decomposition of vol_H with respect to a smooth measure μ. In the sequel we are interested in studying the absolutely continuous part and the singular part of vol_H. Note that the only case where $\mathrm{vol}_{H\llcorner\mathsf{R}} \not\equiv 0$ is when $\dim_H \mathsf{R} \ge \dim_H \mathsf{S}$. In this case $\mathrm{vol}_{H\llcorner\mathsf{R}} = \mathcal{S}^{Q_{\mathsf{R}}}\llcorner_{\mathsf{R}}$. Thus the latter is the measure we study in Section 4.

3.3 Stratification assumption

To go further in the characterization of vol_H, the first question is how to compute the Hausdorff dimension $\dim_H M$ and how to relate $\dim_H M$ to algebraic properties of the sub-Riemannian structure.

Recall that to compute $\dim_H \mathsf{R}$ one simply considers R as the disjoint union of open sets where the growth vector $(n_1(q), \dots, n_{r(q)}(q))$ is constant and then compute the Hausdorff dimension of each component $O \subset \mathsf{R}$ by the algebraic formula

$$\dim_H O = \sum_{i=1}^{r(q)} i(n_i(q) - n_{i-1}(q)).$$

This idea can be carried out on the whole manifold, provided that M can be stratified by suitable submanifolds. This motivates the following definition.

Let $N \subset M$ be a smooth connected submanifold and $q \in N$. The *flag at q of $\mathcal{D}$ restricted to N* is the sequence of subspaces

$$\{0\} \subset (\mathcal{D}_q^1 \cap T_q N) \subset \cdots \subset (\mathcal{D}_q^{r(q)} \cap T_q N) = T_q N. \tag{12}$$

Set

$$n_i^N(q) = \dim(\mathcal{D}_q^i \cap T_q N) \qquad \text{and} \qquad Q_N(q) = \textstyle\sum_{i=1}^{r(q)} i(n_i^N(q) - n_{i-1}^N(q)). \tag{13}$$

We say that N is *equisingular*[3] if

(i) for every i, the dimension $n_i(q)$ is constant as q varies in N;

(ii) for every i, the dimension $n_i^N(q)$ is constant as q varies in N.

In this case, we denote by Q_N the constant value of $Q_N(q)$, $q \in N$. We will see in Theorem 5.3 that Q_N is actually the Hausdorff dimension of N.

Remark 3.10. If N is an open connected submanifold of M then N is equisingular if and only if it is equiregular, that is, condition (i) is equivalent to condition (ii). For submanifolds N of dimension smaller than $\dim M$ being equisingular implies that N is contained in the singular set (see Example 3.14 below for a sub-Riemannian structure where the singular set is itself an equisingular submanifold).

Definition 3.11. We say that the sub-Riemannian manifold $(M, \mathcal{D}, g)$ is *stratified by equisingular submanifolds* if there exists a locally finite stratification $\mathsf{S} = \cup_i \mathsf{S}_i$ where every S_i is an equisingular submanifold.

In the rest of the paper we will make the assumption that $(M, \mathcal{D}, g)$ is stratified by equisingular submanifolds. The interest of such an assumption has already been raised by Gromov [16, 1.3.A]. It holds true in particular for generic smooth sub-Riemannian manifolds and for analytic ones. It obviously implies $\mu(\mathsf{S}) = 0$ for any smooth measure μ, hence Corollary 3.9 and all the analysis of the previous section are valid. To shorten the notations we gather all the hypotheses we need on the sub-Riemannian manifold in the following assumption:

(A) The sub-Riemannian manifold $(M, \mathcal{D}, g)$ is stratified by equisingular submanifolds, M is oriented, and $Q(q) = Q_{\mathsf{R}}$ at every point $q \in \mathsf{R}$.

As a first consequence of this assumption, we obtain $\dim_H \mathsf{S} = \sup_i \dim_H \mathsf{S}_i$ and $\mathrm{vol}_{H \llcorner \mathsf{S}} = \sum_i \mathrm{vol}_{H \llcorner \mathsf{S}_i}$. Thus, to characterize $\mathrm{vol}_{H \llcorner S}$, it suffices to understand what happens on each stratum. This will be done Section 5 where we develop the analysis of Hausdorff volumes on equisingular submanifolds started in [14].

As concerns the regular part of the Hausdorff volume, note that all connected components O of R have the same Hausdorff dimension Q_{R}, which implies that (11) holds for every $q \in \mathsf{R}$. Also, it makes sense to study $\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}$ rather than $\mathrm{vol}_{H \llcorner \mathsf{R}}$. Indeed, if $\dim_H \mathsf{S} \leq Q_{\mathsf{R}}$ then $\mathrm{vol}_H = \mathcal{S}^{Q_{\mathsf{R}}}$. Conversely, if $\dim_H \mathsf{S} > Q_{\mathsf{R}}$ then the absolutely continuous part $\mathrm{vol}_{H \llcorner \mathsf{R}}$ vanishes identically, but $\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}$ still satisfies Proposition 3.5.

[3] Note that in our previous paper [14, Definition 1], these submanifolds are called *strongly equiregular*.

We will focus our analysis on the regularity properties of the derivative $\frac{d\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}}{d\mu}$ as they translate directly into properties of the measure $\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}$. In particular, $\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}$ is commensurable to μ if and only if $\frac{d\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}}{d\mu}$ is locally essentially bounded and bounded away for zero and $\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}$ is a Radon measure if and only if $\frac{d\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}}{d\mu}$ is locally integrable with respect to μ. To carry out the analysis of this Radon-Nikodym derivative the main tool is the proposition below.

For simplicity, we introduce the following notations. For two real numbers t and t' we write $t \asymp_C t'$ if $t'/C \leq t \leq Ct'$, where $C > 0$ is a constant. Moreover, given a family of vector fields $X_1, \dots, X_m$ and a n-tuple $\mathcal{I} = (I_1, \dots, I_n)$ of multi-indices we denote by $\mathcal{X}_{\mathcal{I}}$ the n-tuple of vector fields $(X_{I_1}, \dots, X_{I_n})$ (where X_{I_j} is the Lie bracket corresponding to the multi-index I_j).

Proposition 3.12. *Assume (A) and consider a generating family $X_1, \dots, X_m$ for the sub-Riemannian structure. For every compact subset $K \subset M$, there exists a constant $C > 0$ such that*

$$\frac{d\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}}{d\mu}(q) \asymp_C \frac{1}{\nu(q)} \quad \forall\, q \in K \cap \mathsf{R},$$

where

$$\nu(q) = \sqrt{\sum_{\mathcal{I}\in\mathcal{F}} \omega(\mathcal{X}_{\mathcal{I}})(q)^2} \qquad \text{and} \qquad \mathcal{F} = \left\{ \mathcal{I} = (I_1, \dots, I_n) \;\middle|\; \sum_i |I_i| = Q_{\mathsf{R}} \right\}.$$

The proof is postponed to Section 5.3 . Actually, since $\frac{d\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}}{d\mu}(q) = \frac{2^{Q_{\mathsf{R}}}}{\hat{\mu}^q(\widehat{B}_q)}$ for $q \in \mathsf{R}$, Proposition 3.12 follows from a particular instance of Proposition 5.7, see also Remark 5.8.

Remark 3.13. Since the family $\mathcal{F}$ is finite, one easily computes the nonholonomic order of the function $q \mapsto \nu(q)$ at $p \in \mathsf{S}$ as

$$\mathrm{ord}_p \nu = \min_{\mathcal{I}\in\mathcal{F}} \mathrm{ord}_p(\omega(\mathcal{X}_{\mathcal{I}})).$$

Moreover, in the statement of Proposition 3.12, ν can be replaced by

$$\bar{\nu}(q) = \max_{\mathcal{I}\in\mathcal{F}} |\omega(\mathcal{X}_{\mathcal{I}})(q)|.$$

Example 3.14 (the Martinet space). Let us compute ν on a specific example. Consider the sub-Riemannian structure on $\mathbb{R}^3$ given by the generating family

$$X_1 = \partial_1, \quad X_2 = \partial_2 + \frac{x_1^2}{2}\partial_3.$$

We choose $\omega = dx_1 \wedge dx_2 \wedge dx_3$, that is, the canonical volume form on $\mathbb{R}^3$. The growth vector is equal to $(2, 2, 3)$ on the plane $\{x_1 = 0\}$, and it is $(2, 3)$ elsewhere. As a consequence, S coincides with $\{x_1 = 0\}$ and is equisingular, since $n_1^{\mathsf{S}}(0, x_2, x_3) = n_2^{\mathsf{S}}(0, x_2, x_3) = 1, n_3^{\mathsf{S}}(0, x_2, x_3) = 2$. At regular points we have $Q_{\mathsf{R}} = 4$ and the only adapted basis at regular points is $(X_1, X_2, [X_1, X_2])$. Then

$$\nu(x_1, x_2, x_3) \asymp_C \bar{\nu}(x_1, x_2, x_3) = |x_1|.$$

By Proposition 3.12, on R the Radon-Nikodym derivative $\frac{d\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}}{d\mu}$ is locally equivalent to $1/\nu$. Henceforth it is sufficient to study essential boundedness and integrability of $1/\nu$. It turns out that this function is not locally essentially bounded around the singular set.

Proposition 3.15. *Under assumption (A), for every $p \in \mathsf{S}$ and every neighborhood $U \subset M$ of p,*

$$\operatorname{essup} \frac{d\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}}{d\mu}\Big|_U = +\infty.$$

As a consequence, $\mathcal{S}^{Q_{\mathsf{R}}}\llcorner\mathsf{R}$ and μ are not commensurable when $\mathsf{S} \neq \emptyset$.

Proof. Let $\mathcal{I} = (I_1, \dots, I_n) \in \mathcal{F}$. Recall that if the values of $X_{I_1}, \dots, X_{I_n}$ at a point $q \in M$ are linearly independent, then $\sum_i |I_i| = Q_{\mathsf{R}} \geq Q(q)$ (see Section 2.2). Let $p \in \mathsf{S}$. Since $Q_{\mathsf{R}} < Q(p)$, the vectors $X_{I_1}(p), \dots, X_{I_n}(p)$ are linearly dependent. Thus $\nu(q) \to 0$ as $q \to p$ and the conclusion follows from Proposition 3.12. □

Thus $\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}$ and μ are never commensurable in non equiregular sub-Riemannian manifolds. It makes sense now to ask whether $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$ is locally integrable with respect to μ. The answer to this question is more involved and is the object of Section 4.

Remark 3.16. Let us mention that, on R, there is a smooth volume which is intrinsically associated with the sub-Riemannian structure (i.e., to the pair $(\mathcal{D}, g)$), namely the Popp measure $\mathcal{P}$. An explicit formula for this measure is given in [5]. More precisely, thanks to [5, Theorem 1], the Radon-Nikodym derivative $\frac{d\mathcal{P}}{d\mu}$ satisfies

$$\frac{d\mathcal{P}}{d\mu}(q) \asymp_C \frac{1}{\nu(q)}, \qquad \text{for every } q \in K \cap \mathsf{R},$$

where K is any compact set in M. As a consequence, $\mathcal{P}$ and $\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}$ are commensurable on M ($\mathcal{P}$ is extended to a measure on M by setting $\mathcal{P} \llcorner \mathsf{S} = 0$). In particular, $\mathcal{P}$ and μ are not commensurable when $\mathsf{S} \neq \emptyset$, and $\mathcal{P}$ is Radon if and only if $\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}$ is.

Remark 3.17. In Proposition 3.12 and Proposition 3.15, the only conditions of assumption (A) needed are $Q(q) = Q_{\mathsf{R}}$ for every $q \in \mathsf{R}$ and M oriented. The stratification assumption is redundant.

4 Local integrability of the Radon-Nikodym derivative

Thanks to Theorem 3.1, $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$ is continuous on R and thus locally μ-integrable on R. Nevertheless, by Proposition 3.15 $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$ explodes when approaching S. Therefore, the question remains whether $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$ is locally μ-integrable around singular points, or equivalently whether $\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}$ is a Radon measure. The aim of this section is to provide sufficient conditions for $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$ to be locally μ-integrable or not around S.

In Section 4.1 we express in some suitable coordinates an integral whose finiteness is equivalent to the local integrability of $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$. We then use this expression in Section 4.2 to deduce explicit algebraic conditions for both integrability and non-integrability of $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$.

4.1 Computation in local coordinates

Consider an equisingular submanifold N and a point $p \in N$. Thanks to Proposition 3.12, under assumption (A) the local μ-integrability of $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$ near p is equivalent to the one of $\frac{1}{\nu}$. The latter can be conveniently characterized in well-chosen systems of coordinates.

Lemma 4.1. *Assume (A). Let N be an equisingular k-dimensional submanifold, $p \in N$, and $\varphi = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ be local coordinates centered at p such that, near p,*

$$\varphi(N) \subset \{z = 0\}.$$

Then $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$ is locally μ-integrable near p if and only if, for $R > 0$ small enough,

$$\int_0^R \lambda^{n-k-1} \left(\int_{|y| \leq R} \left(\int_{\{|z|=1\}} \frac{d\sigma(z)}{\nu(\Phi(y, \lambda z))} \right) dy \right) d\lambda < +\infty, \tag{14}$$

where $\Phi = \varphi^{-1}$, $|y| = \max_{i \in \{1,\dots,k\}} |y_i|$, $|z| = \max_{i \in \{k+1,\dots,n\}} |z_i|$, $\lambda z = (\lambda z_{k+1}, \dots, \lambda z_n)$, and $d\sigma(z)$ denotes the $(n-k-1)$-Lebesgue measure on $\{|z| = 1\}$.

If moreover φ is a system of privileged coordinates at p, then $\frac{d\mathcal{S}^{Q_R}\llcorner R}{d\mu}$ is locally μ-integrable near p if and only if, for $R>0$ small enough,

$$\int_0^R \lambda^{Q(p)-Q_N-1}\left(\int_{\|y\|\le R}\left(\int_{\{\|z\|=1\}}\frac{d\tilde{\sigma}(z)}{\nu(\Phi(y,\delta_\lambda z))}\right)dy\right)d\lambda \quad < \quad +\infty, \tag{15}$$

where

$$\|y\|=\max_{i\in\{1,\dots,k\}}|y_i|^{w_i(p)},\quad \|z\|=\max_{i\in\{k+1,\dots,n\}}|z_i|^{w_i(p)},\qquad \delta_\lambda z=(\lambda^{w_{k+1}(p)}z_{k+1},\dots,\lambda^{w_n(p)}z_n),$$

$w_i(p)$ are the weights at p labeled according to the coordinates φ, and $d\tilde{\sigma}(z)$ denotes the $(n-k-1)$-Lebesgue measure on $\{\|z\|=1\}$.

Remark 4.2. Since N is equisingular, the functions $q\mapsto w_i(q)$ are constants on N and the integers $Q(q)$ and $Q_N=Q_N(q)$ (see (13)) may be computed as

$$Q(q)=\sum_{i=1}^n w_i(q)\qquad\text{and}\qquad Q_N=\sum_{i=1}^k w_i(q).$$

Proof. According to the notation in the statement above, we set $\varphi=(y_1,\dots,y_k,z_{k+1},\dots,z_n)$. This allows to distinguish which coordinates parameterize N, namely the first group y, and which ones are transversal to N, namely the second group z.

Let U be a small neighborhood of p. Thanks to the expression of $\frac{d\mathcal{S}^{Q_R}\llcorner R}{d\mu}$ provided by Proposition 3.12, the finiteness of $\mathcal{S}^{Q_R}\llcorner R(U)$ reduces to the finiteness of the integral $\int_U\frac{1}{\nu(q)}d\mu(q)$. Applying the change of variables φ in the integral and taking upper and lower bounds for $|\det J\Phi(y,z)|$ near $(0,0)$, we obtain

$$\mathcal{S}^{Q_R}\llcorner R(U)\quad\asymp_C\quad\int_{\varphi(U)}\frac{1}{\nu(\Phi(y,z))}\,dydz=\int_{y\in\Omega_1}\left(\int_{z\in\Omega_2}\frac{1}{\nu(\Phi(y,z))}dz\right)dy,$$

where we have written $\varphi(U)$ as $\Omega_1\times\Omega_2$, with $\Omega_1\subset\mathbb{R}^k$ and $\Omega_2\subset\mathbb{R}^{n-k}$ open subsets.

Since $\varphi(U)$ contains and is contained in sets of the form $\{|y|<R\}\times\{|z|<R\}$, it is sufficient to consider the case where $\Omega_1=\{|y|<R\}$ and $\Omega_2=\{|z|<R\}$ for some small $R>0$. Criterion (14) follows directly by a change of coordinates $z=\lambda\tilde{z}$, $|\tilde{z}|=1$, in the integral on Ω_2.

Assume now that the coordinates φ are privileged at p. As above, it is sufficient to consider the case where $\Omega_1=\{\|y\|<R\}$ and $\Omega_2=\{\|z\|<R\}$ for some small $R>0$. Recall that in privileged coordinates, dilations take the form

$$\delta_\lambda(y,z)=(\lambda^{w_1(p)}y_1,\dots,\lambda^{w_k(p)}y_k,\lambda^{w_{k+1}(p)}z_{k+1},\dots,\lambda^{w_n(p)}z_n).$$

By construction, the pseudo-norm $\|\cdot\|$ is homogeneous with respect to δ_λ, and we have $\Omega_2=\{\delta_\lambda z\mid\|z\|=1,\lambda\in[0,R)\}$. Thus, changing variables in the integral, we obtain

$$\int_{y\in\Omega_1}\left(\int_{z\in\Omega_2}\frac{1}{\nu(\Phi(y,z))}dz\right)dy=\int_{\|y\|\le R}\left(\int_0^R\lambda^{\sum_{j=k+1}^n w_j-1}\left(\int_{\{\|z\|=1\}}\frac{d\sigma(z)}{\nu(\Phi(y,\delta_\lambda z))}\right)d\lambda\right)dy.$$

By Remark 4.2, $\sum_{j=k+1}^n w_j=Q(p)-Q_N$ and we obtain criterion (15). □

As pointed out in Remark 3.17, in Lemma 4.1 we do not need the stratification assumption, but only the other two conditions in assumption (A). Nevertheless, whenever M is stratified by equisingular submanifolds and $p\in M$, there exists a unique equisingular submanifold N containing p.

We end this section with the construction of a system of coordinate around p satisfying the assumptions of the previous lemma. This construction will also prove to be useful in Section 5.

Lemma 4.3. *Let N be an equisingular submanifold. Then there exist a neighborhood U of p and a smooth map*

$$\begin{aligned} \varphi : U \cap N &\to \mathcal{C}^\infty(U, \mathbb{R}^n) \\ q &\mapsto \varphi_q : U \to \mathbb{R}^n \end{aligned}$$

such that, for every $q \in U \cap N$,

(I) *φ_q is a system of privileged coordinates centered at q;*

(II) *$\varphi_q(U \cap N) \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_{k+1} = \cdots = x_n = 0\}$, where $k = \dim N$.*

Moreover, for every $q \in U \cap N$ and for every $q' \in U$,

$$(\varphi_q(q'))_i = (\varphi_p(q'))_i, \quad i = k+1, \dots, n. \tag{16}$$

Proof. For every $q \in N$, let us consider the flag of $\mathcal{D}$ restricted to N

$$(\mathcal{D}^1_q \cap T_qN) \subset \cdots \subset (\mathcal{D}^r_q \cap T_qN) = T_qN.$$

Recall that the integers $n^N_i = \dim(\mathcal{D}^i_q \cap T_qN)$ do not depend on q in N. Thus, in a neighborhood U of p, there exist a family of vector fields $Z_1, \dots, Z_k$ on U such that, for every $q \in N$ and for every $i = 1, \dots, r$,

$$\mathcal{D}^i_q \cap T_qN = \operatorname{span}\{Z_j(q) \mid \ell(Z_j) \le i\}.$$

In particular, we have $T_qN = \operatorname{span}\{Z_1(q), \dots, Z_k(q)\}$, and

$$Q_N = \sum_{i=1}^{r} i(n^N_i - n^N_{i-1}) = \sum_{i=1}^{k} \ell(Z_i).$$

Possibly reducing U, since $q \mapsto \dim \mathcal{D}^i_q$ is constant on N, there exist vector fields $Z_{k+1}, \dots, Z_n$ such that the family $Z_1, \dots, Z_n$ is adapted to the flag of $\mathcal{D}$ at every $q \in N \cap U$ (see (8)).

Using these bases, we define for $q \in N \cap U$, a local diffeomorphism $\Phi_q : \mathbb{R}^n \to M$ by

$$\Phi_q(x) = \exp(x_nZ_n) \circ \cdots \circ \exp(x_1Z_1)\,(q). \tag{17}$$

The inverse $\varphi_q = \Phi_q^{-1}$ of Φ_q provides a system of privileged coordinates at q (see [17]) and in these coordinates N coincides locally with the set $\{x_{k+1} = \cdots = x_n = 0\}$. Moreover, the map $(q, x) \mapsto \Phi_q(x)$ is smooth on $N \cap U$, which completes the proof of points (I) and (II).

Finally, let us prove (16). Let $q \in U \cap N$ and $q' \in U$. We denote respectively by x and y the coordinates of q' centered at q and p, that is, $q' = \Phi_q(x) = \Phi_p(y)$. By construction, the point $\exp(x_kZ_k) \circ \cdots \circ \exp(x_1Z_1)\,(q)$ belongs to $U \cap N$, and hence may be written as $\Phi_p(z)$ with $z_{k+1} = \cdots = z_n = 0$. Therefore,

$$q' = \Phi_q(x) = \exp(x_nZ_n) \circ \cdots \circ \exp(x_{k+1}Z_{k+1}) \circ \exp(z_kZ_k) \circ \cdots \circ \exp(z_1Z_1)\,(p),$$

which implies $y_i = x_i$ for $i = k+1, \dots, n$, i.e. (16). □

4.2 Sufficient conditions

In the sequel, we provide sufficient conditions to obtain either local integrability or non integrability of $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$ around a fixed singular point. These conditions will be of different nature (either differential or metric), depending on whether they are obtained with (14) or (15).

Note that by assumption (A), S admits a stratification by equisingular submanifolds, hence every singular point belongs to an equisingular submanifold which is determined in a unique way. Recall also from Proposition 3.12 that

$$\nu(q) = \sqrt{\sum_{\mathcal{I} \in \mathcal{F}} \omega(\mathcal{X}_{\mathcal{I}})(q)^2} \qquad \text{and} \qquad \mathcal{F} = \left\{ \mathcal{I} = (I_1, \dots, I_n) \;\middle|\; \sum_i |I_i| = Q_{\mathsf{R}} \right\}.$$

4.2.1 Conditions on nonholonomic orders

Consider a singular point $p \in \mathsf{S}$ and an equisingular submanifold $N \subset \mathsf{S}$ containing p. Define

$$\varrho_{\min}(p, N) = \lim_{\epsilon \to 0} \min\{\mathrm{ord}_q \nu \mid q \in N \cap B(p, \epsilon)\},$$

where $\mathrm{ord}_q(h)$ is the nonholonomic order of the function h at q. Note that by Remark 3.13, $\mathrm{ord}_p \nu = \min_{\mathcal{I} \in \mathcal{F}} \mathrm{ord}_p(\omega(\mathcal{X}_\mathcal{I}))$, so that

$$\varrho_{\min}(p, N) = \lim_{\epsilon \to 0} \min\{\mathrm{ord}_q(\omega(\mathcal{X}_\mathcal{I})) \mid q \in N \cap B(p, \epsilon),\ \mathcal{I} \in \mathcal{F}\}.$$

Next result gives an algebraic sufficient condition for a neighborhood of p to have infinite $\mathcal{S}^{Q_\mathsf{R}}$ measure. This condition involves the order of ν at singular points around p ($\varrho_{\min}(p, N)$), the flag of the distribution at p ($Q(p)$) and the flag of the distribution restricted to the singular set at p (Q_N).

Proposition 4.4. *Assume (A). Let $p \in \mathsf{S}$ and let N be an equisingular submanifold containing p. If $\varrho_{\min}(p, N) \geq Q(p) - Q_N$, then $\frac{d\mathcal{S}^{Q_\mathsf{R}} \llcorner \mathsf{R}}{d\mu}$ is not μ-locally integrable around p.*

Proof. We will apply condition (15) with the coordinate system $\varphi_p = (y, z)$ built in Lemma 4.3. Fix R small enough. By definition,

$$\nu(\Phi_p(y, \delta_\lambda z))^2 = \sum_{\mathcal{I} \in \mathcal{F}} \omega(\mathcal{X}_\mathcal{I})(\Phi_p(y, \delta_\lambda z))^2,$$

where $\mathcal{F}$ is the family of n-tuples of multi-indices $\mathcal{I} = (I_1, \ldots, I_n)$ such that $\sum_j |I_j| = Q_\mathsf{R}$. Since p belongs to the singular set, for any family $\mathcal{I} \in \mathcal{F}$, the vectors $X_{I_1}(p), \ldots, X_{I_n}(p)$ are linearly dependent. Similarly, for every y near 0, the vectors $X_{I_1}(\Phi_p(y, 0)), \ldots, X_{I_n}(\Phi_p(y, 0))$ are linearly dependent. As a consequence, $\nu(\Phi_p(y, 0)) \equiv 0$.

Given $y \in \{\|y\| \leq R\}$, denote by $\varrho(y)$ the greatest number such that there exists a constant $C_y > 0$ satisfying

$$\nu(\Phi_p(y, \delta_\lambda z)) \leq C_y \lambda^{\varrho(y)},$$

for every $\lambda \leq 1$ and $\|z\| = 1$. Thanks to (16),

$$\Phi_p(y, z) = \Phi_q(0, z),$$

where $q = \Phi_p(y, 0)$. Hence $\Phi_p(y, \delta_\lambda z) = \Phi_q(0, \delta_\lambda z) = \Phi_q(\delta_\lambda(0, z))$, and $\varrho(y)$ is simply the nonholonomic order of ν at $\Phi_p(y, 0)$ and, in particular, $\varrho(y) \geq \varrho_{\min}$ (we omit the dependance w.r.t. (p, N)). Assume $\varrho_{\min} < \infty$. Since $q \mapsto \nu^2(q)$ is smooth, there exists $C > 0$ such that

$$\max_{\|y\| \leq R, \|z\|_p = 1} \nu(\Phi_p(y, \delta_\lambda z)) \leq C \lambda^{\varrho_{\min}}.$$

Thus

$$\int_0^R \lambda^{Q(p) - Q_N - 1} \left(\int_{\|y\| \leq R} \left(\int_{\{\|z\| = 1\}} \frac{d\tilde{\sigma}(z)}{\nu(\Phi(y, \delta_\lambda z))} \right) dy \right) d\lambda \geq C \int_0^1 \lambda^{Q(p) - Q_N - 1 - \varrho_{\min}} d\lambda.$$

Therefore, applying (15) in Lemma 4.1, condition $\varrho_{\min} \geq Q(p) - Q_N$ gives the non integrability. If $\varrho_{\min} = +\infty$, then we directly infer the conclusion, as the inequality

$$\int_0^R \lambda^{Q(p) - Q_N - 1} \left(\int_{\|y\| \leq R} \left(\int_{\{\|z\| = 1\}} \frac{d\tilde{\sigma}(z)}{\nu(\Phi(y, \delta_\lambda z))} \right) dy \right) d\lambda \geq C \int_0^1 \lambda^{Q(p) - Q_N - 1 - \alpha} d\lambda$$

holds for every positive integer α. □

Remark 4.5. For every $q = \Phi_p(y,0) \in N$ and every $\mathcal{I} \in \mathcal{F}$, $\mathrm{ord}_q(\omega(\mathcal{X}_\mathcal{I})) \geq Q(p) - Q_{\mathsf{R}}$ and thus

$$\varrho_{\min}(p,N) \geq Q(p) - Q_{\mathsf{R}}.$$

To see this, let $\widehat{X}_1, \dots, \widehat{X}_m$ be a nilpotent approximation at $q \in N$ of the generating family $X_1, \dots, X_m$. Denote by $\widehat{\mathcal{X}}_\mathcal{I}$ the family $\widehat{X}_{I_1}, \dots, \widehat{X}_{I_n}$, where $\mathcal{I} = (I_1, \dots, I_n)$ and $\widehat{X}_{I_j}$ is the Lie bracket between $\widehat{X}_1, \dots, \widehat{X}_m$. Then

$$\omega(\mathcal{X}_\mathcal{I})(\Phi_p(y, \delta_\lambda z)) = \omega(\mathcal{X}_\mathcal{I})(\Phi_q(0, \delta_\lambda z)) = \lambda^{Q(q)-Q_{\mathsf{R}}} \left(\omega(\widehat{\mathcal{X}}_\mathcal{I})(\Phi_q(0, \delta_\lambda z)) + \lambda b_{\mathcal{I},q}(\lambda, z) \right),$$

where $b_{\mathcal{I},q}$ is a smooth function. Finally, the equisingularity of N ensures that $Q(q) = Q(p)$.

Proposition 4.4, jointly with Remark 4.5 allows to deduce the following criterion for $\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}$ not to be Radon.

Corollary 4.6. *Assume (A). If* $\dim_H \mathsf{S} \geq \dim_H \mathsf{R}$*, then* $\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}$ *is not a Radon measure.*

Remark 4.7. Denoting by $\mathsf{S} = \cup \mathsf{S}_j$ the stratification of S by equisingular submanifolds, condition $\dim_H \mathsf{S} \geq \dim_H \mathsf{R}$ is equivalent to the existence of a stratum S_j such that $Q_{\mathsf{S}_j} \geq Q_{\mathsf{R}}$. Indeed, thanks to Theorem 5.3,

$$\dim_H \mathsf{S} = \max_j \dim_H \mathsf{S}_j = \max_j Q_{\mathsf{S}_j}.$$

Therefore, such condition encodes a metric information, in the sense that the Hausdorff dimension is defined only through the metric structure of the manifold, and it bears an algebraic interpretation (and hence a direct way to be tested) in terms of Q_{R} and the Q_{S_j}.

Proof. Under the assumptions, there exists an equisingular submanifold $N \subset \mathsf{S}$ such that $\dim_H N = \dim_H \mathsf{S}$. On the other hand, $Q_N = \dim_H N$ by Theorem 5.3. Using Remark 4.5, for every $p \in N$ we have

$$\varrho_{\min}(p,N) \geq Q(p) - Q_{\mathsf{R}} \geq Q(p) - Q_N,$$

hence the assumption of Proposition 4.4 is satisfied and $\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}$ is not locally integrable around p. □

As concerns sufficient conditions for local integrability, we use the same approach as in Proposition 4.4 and we look for uniform lower bounds on ν.

For a point $p \in \mathsf{S}$ and an equisingular submanifold N containing p, we define

$$\varrho_{\max}(p,N) = \inf \left\{ s \geq 0 \,:\, \inf_{y \in \Omega_1, \|z\|_p = 1} \left\{ \liminf_{\lambda \to 0} \frac{\nu(\Phi_p(y, \delta_\lambda z))}{\lambda^s} \right\} > 0 \right\},$$

where Φ_p and Ω_1 are constructed in Lemma 4.3. By construction, $\varrho_{\min}(p,N) \leq \varrho_{\max}(p,N)$.

Remark 4.8. The number $\varrho_{\max} = \varrho_{\max}(p,N)$ is defined precisely to get a uniform lower bound on ν. More precisely, if $\varrho_{\max} < \infty$, there exists $C > 0$ such that, for every $(y,z) \in \Omega_1 \times \{\|z\|_p = 1\}$ and $\lambda \in [0,1]$, there holds

$$\nu(\Phi_p(y, \delta_\lambda z)) \geq C \lambda^{\varrho_{\max}}. \tag{18}$$

Unlike $\varrho_{\min}$, in general $\varrho_{\max}$ does not bear an algebraic interpretation as a nonholonomic order. Nevertheless, it can be characterized in a more intrinsic way as

$$\varrho_{\max} = \inf \left\{ s \geq 0 \,:\, \inf_{\bar{q} \in B(p,\epsilon) \cap N} \left\{ \liminf_{q \to \bar{q}, q \notin \mathsf{S}} \frac{\nu(q)}{d(q,N)^s} \right\} > 0 \right\}.$$

As a direct consequence of (18) and Lemma 4.1, we deduce the following fact.

Proposition 4.9. *Assume (A). Let* $p \in \mathsf{S}$ *and let* N *be an equisingular submanifold containing* p*. If* $\varrho_{\max}(p,N) < Q(p) - Q_N$ *then* $\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}$ *is locally* μ*-integrable around* p*.*

Notice that the hypothesis $\varrho_{\max}(p,N) < Q(p) - Q_N$ implies in particular that $\varrho_{\max} < \infty$, and thus that in a neighborhood of p the set S coincides with N.

In general, condition $\varrho_{\max} < Q(p) - Q_N$ is not sharp, in the sense that there may be cases in which the condition fails and $\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}$ is locally integrable (see Example 6.6). Actually, the study of the generic case (see the proof of Proposition 6.1) shows that the integrability does not necessarily require a bound on $\varrho_{\max}$. Nevertheless, when ν is homogeneous, we have $\varrho_{\max} = \varrho_{\min}$ and thus Proposition 4.4 together with Proposition 4.9 provides a characterization of integrability. Namely, if ν is homogeneous, then $\mathcal{S}^{Q_{\mathsf{R}}}_{\mathsf{R}}$ is locally integrable at $p \in N$ if and only if $\varrho_{\min} < Q(p) - Q_N$.

4.2.2 Conditions on usual orders

Consider a singular point $p \in \mathsf{S}$ and an equisingular submanifold $N \subset \mathsf{S}$ containing p. Define

$$e_{\min}(p,N) = \lim_{\epsilon \to 0} \min\{\mathrm{ord}_q^{\mathrm{diff}}(\omega(\mathcal{X}_{\mathcal{I}})) \mid q \in N \cap B(p,\epsilon),\ \mathcal{I} \in \mathcal{F}\},$$

where $\mathrm{ord}_q^{\mathrm{diff}}$ is the usual notion of order of the differential calculus. In coordinates x, the order $\mathrm{ord}_q^{\mathrm{diff}} h$ of a smooth function h is the biggest integer k such that all partial derivatives of h of order smaller than k are zero. Equivalently, it is the biggest integer such that locally $h(x) = O(|x|^k)$. Obviously,

$$e_{\min}(p,N) \leq \varrho_{\min}(p,N).$$

Proposition 4.10. *Assume (A). Let $p \in \mathsf{S}$ and let N be an equisingular submanifold of codimension $n-k \geq 1$. If $e_{\min}(p,N) \geq n-k$, then $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$ is not locally μ-integrable around p.*

Proof. The proof is similar to the one of Proposition 4.4, replacing ord_q by $\mathrm{ord}_q^{\mathrm{diff}}$ and (15) by (14). □

Corollary 4.11. *Assume (A). If S contains a submanifold of codimension 1, then $\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}$ is not a Radon measure.*

Proof. By assumption (A), S is stratified by equisingular submanifolds. Therefore there exists an equisingular submanifold $N \subset \mathsf{S}$ of codimension one. Let $p \in N$. Take a coordinate system $(y_1, \ldots, y_{n-1}, z)$ that allows to identify N with $\{z = 0\}$. For every $\mathcal{I} \in \mathcal{F}$, since $\omega(\mathcal{X}_{\mathcal{I}})(\Phi_p(y,z)) \equiv 0$, it results from Malgrange Preparation Theorem that there exists $C > 0$ such that

$$|\omega(\mathcal{X}_{\mathcal{I}})(\Phi_p(y,z))| \leq C|z|,$$

for every y in a neighborhood of 0, and thus $e_{\min}(p,N) \geq 1 = n - (n-1)$. Proposition 4.10 applies. □

A sufficient condition of integrability similar to Proposition 4.9 can also be derived.

5 Hausdorff measures on submanifolds

Thanks to the analysis in Section 3.3, under assumption (A) the study of the singular part $\mathrm{vol}_H \llcorner \mathsf{S}$ reduces to the study of the restriction of this measure to every equisingular stratum $\mathsf{S}_i \subset \mathsf{S}$. Since $\mathrm{vol}_H \llcorner \mathsf{S}_i = 0$ or $+\infty$ if $\dim_H \mathsf{S}_i \neq \dim_H M$, we are actually led to study $\mathcal{S}^{\dim_H N} \llcorner N$ for an equisingular submanifold N. This is the object of this section.

We first describe the algebraic structure of an equisingular manifold N, and then give in Theorem 5.3 a complete description of $\mathcal{S}^{\dim_H N} \llcorner N$. Finally we give an estimate of the Radon-Nikodym derivative of $\mathcal{S}^{\dim_H N} \llcorner N$ with respect to a smooth volume on N.

5.1 Algebraic structure associated with equisingular submanifolds

Recall that the metric tangent cone to (M, d) at any point p exists and it is isometric to $(T_pM, \widehat{d}_p)$, where $\widehat{d}_p$ denotes the sub-Riemannian distance associated with a nilpotent approximation at p (see [6]).

The following proposition shows the relevance of equisingular submanifolds as particular subsets of (M, d) for which a metric tangent cone exists. Such metric space is isometrically embedded in a metric tangent cone to the whole M at the point.

Proposition 5.1. *Let $N \subset M$ be an equisingular submanifold of M of dimension k and $K \subset N$ be a compact set. The following properties hold.*

(i) *There exists a metric tangent cone to $(N, d|_N)$ at every $p \in N$ and it is isometric to $(T_pN, \widehat{d}_p|_{T_pN})$.*

(ii) *For every $p \in N$, the graded vector space*

$$\mathfrak{gr}_p^N(\mathcal{D}) := \bigoplus_{i=1}^{r(p)} (\mathcal{D}_p^i \cap T_pN)/(\mathcal{D}_p^{i-1} \cap T_pN)$$

is a nilpotent Lie algebra whose associated Lie group $\mathrm{Gr}_p^N(\mathcal{D})$ is diffeomorphic to T_pN.

(iii) *Denote by $\widehat{B}_p$ the unit ball for $\widehat{d}_p$. The function $p \mapsto \operatorname{diam}_{\widehat{d}_p}(T_pN \cap \widehat{B}_p)$ is continuous on N and*

$$\sup_{p\in K} \left| \frac{1}{\epsilon} \operatorname{diam}(N \cap B(p,\epsilon)) - \operatorname{diam}_{\widehat{d}_p}(T_pN \cap \widehat{B}_p) \right| \xrightarrow[\epsilon\to 0^+]{} 0; \tag{19}$$

(iv) *Assume N is oriented and let μ be a smooth volume on N associated with a k-form $\varpi \in \bigwedge^k N$. Then ϖ induces canonically a left-invariant k-form $\hat{\varpi}^p$ on $\mathrm{Gr}_p^N(\mathcal{D})$ and a corresponding smooth volume $\hat{\mu}^p$. Moreover, the function $p \mapsto \hat{\mu}^p(T_pN \cap \widehat{B}_p)$ is continuous on N, and*

$$\sup_{p\in K} \left| \frac{1}{\epsilon^{Q_N}} \mu(N \cap B(p,\epsilon)) - \hat{\mu}^p(T_pN \cap \widehat{B}_p) \right| \xrightarrow[\epsilon\to 0^+]{} 0. \tag{20}$$

Remark 5.2. When N is an open submanifold of M, assuming N equisingular is equivalent to saying that N contains only regular points. In that case, most of Proposition 5.1 is well-known: point (i) follows by the fact that the nilpotent approximation is a metric tangent cone; point (ii) says that the tangent cone shares a group structure; point (iii) is trivial since $\operatorname{diam}_{\widehat{d}_p}(\widehat{B}_p) = 2$ and $\operatorname{diam}(B(p,\epsilon)) = 2\epsilon$ for small ϵ; and the continuity of $p \mapsto \hat{\mu}^p(\widehat{B}_p)$ in (iv) has been remarked in [1]. Only the uniform convergence (20) is new in this case.

Proof. Note first that since the result is of local nature, it is sufficient that we prove it on a small neighborhood $U \cap N$ of a point $p_0 \in N$. It results from Lemma 4.3 that, for every p in a such a neighborhood, there exists a coordinate system $\varphi_p : U_p \to \mathbb{R}^n$ on a neighborhood $U_p \subset M$ of p, such that φ_p are privileged coordinates at p, $p \mapsto \varphi_p$ is smooth on $U \cap N$, and N is rectified in coordinates φ_p, that is $\varphi_p(N \cap U_p) \subset \{x \in \mathbb{R}^n \mid x_{k+1} = \cdots = x_n = 0\}$.

Proof of (i) and (ii). Coordinates φ_p allow to identify $(T_pM, \widehat{d}_p)$ with $(\mathbb{R}^n, \widehat{d}_p)$. From [6, Theorem 7.32] we have the following estimate, which is instrumental in the proof below. There exists two positive constants ϵ_p, C_p and an integer r such that, if $x, y \in \mathbb{R}^n$ satisfy $M_p(x,y) := \max(\widehat{d}_p(0,x), \widehat{d}_p(0,y)) < \epsilon_p$, then

$$-C_pM_p(x,y)d(\varphi_p^{-1}(x), \varphi_p^{-1}(y))^{1/r} \le d(\varphi_p^{-1}(x), \varphi_p^{-1}(y)) - \widehat{d}_p(x,y) \le C_pM_p(x,y)\widehat{d}_p(x,y)^{1/r}. \tag{21}$$

Moreover, as noticed in [20, Sect. 2.2.2], since N is equisingular the functions $p \mapsto \epsilon_p$ and $p \mapsto C_p$ are continuous on N and r does not depend on $p \in N$ (r is the maximum of the weights $w_1(p), \ldots, w_n(p)$).

Now, using φ_p we identify locally M with $T_pM \simeq \mathbb{R}^n$ and N with $T_pN \simeq \mathbb{R}^k \times \{0\}$. Therefore, whenever $q_1, q_2 \in U_p \cap N$ we have

$$\widehat{d}_p(\varphi_p^{-1}(q_1), \varphi_p^{-1}(q_2)) = \widehat{d}_p|_{T_pN}(\varphi_p^{-1}(q_1), \varphi_p^{-1}(q_2)),$$

and obviously $d(q_1, q_2) = d|_N(q_1, q_2)$. Hence (21) holds when we restrict d to N and $\widehat{d}$ to T_pN. This allows to conclude [23, Proposition 8.12] that a metric tangent cone to $(N, d|_N)$ at p exists and it is isometric to $(T_pN, \widehat{d}_p|_{T_pN})$. The algebraic structure of $\mathfrak{gr}_p^N(\mathcal{D})$ and the fact that $\mathrm{Gr}_p^N(\mathcal{D})$ is diffeomorphic to $\mathbb{R}^k$ are obtained in exactly the same way as at regular point, see [23, Sect. 4.4]. Thus (i) and (ii) are proved.

Proof of (iii). Let $w_i(p)$ be the weights at p labeled according to the coordinates φ_p. Note that the dilations $\delta_s : x \mapsto (s^{w_1(p)}x_1, \ldots, s^{w_n(p)}x_n)$, $s > 0$, do not depend on $p \in N$ as N is equisingular. Thus the homogeneity of $\widehat{d}_p$ with respect to δ_s and estimate (21) imply that the function $(p, x, y) \mapsto \widehat{d}_p(x, y)$ is continuous on $N \times \mathbb{R}^n \times \mathbb{R}^n$. As a consequence,

$$\operatorname{diam}_{\widehat{d}_p}(T_pN \cap \widehat{B}_p) = \sup\left\{\widehat{d}_p(x, y) \mid x, y \in \mathbb{R}^k,\ \widehat{d}_p(0, x) \text{ and } \widehat{d}_p(0, y) \leq 1\right\}$$

is an upper semi-continuous function of p. Let us prove that is also lower semi-continuous. Fix $p \in N$ and let $(x, y) \in \mathbb{R}^k \times \mathbb{R}^k$ such that $\operatorname{diam}_{\widehat{d}_p}(T_pN \cap \widehat{B}_p) = \widehat{d}_p(x, y)$. For $q \in N$ we set

$$\alpha(q) = \min\left(\frac{\widehat{d}_p(0, x)}{\widehat{d}_q(0, x)}, \frac{\widehat{d}_p(0, y)}{\widehat{d}_q(0, y)}\right).$$

By construction we have $\widehat{d}_q(0, \delta_{\alpha(q)}x)$ and $\widehat{d}_q(0, \delta_{\alpha(q)}y) \leq 1$, hence

$$\operatorname{diam}_{\widehat{d}_q}(T_qN \cap \widehat{B}_q) \geq \widehat{d}_q(\delta_{\alpha(q)}x, \delta_{\alpha(q)}y) = \alpha(q)\widehat{d}_q(x, y).$$

When $q \to p$, the continuity of $q \mapsto \widehat{d}_q(x, y)$ implies that $\alpha(q) \to 1$ and $\widehat{d}_q(x, y) \to \widehat{d}_p(x, y)$. As a result,

$$\liminf_{q \to p} \operatorname{diam}_{\widehat{d}_q}(T_qN \cap \widehat{B}_q) \geq \widehat{d}_p(x, y) = \operatorname{diam}_{\widehat{d}_p}(T_pN \cap \widehat{B}_p),$$

that is, the function $p \mapsto \operatorname{diam}_{\widehat{d}_p}(T_pN \cap \widehat{B}_p)$ is lower semi-continuous and then continuous.

To prove (19), we write

$$\operatorname{diam}(N \cap B(p, \epsilon)) = \sup\left\{d(\varphi_p^{-1}(x), \varphi_p^{-1}(y)) \ \mid \ x, y \in \varphi_p\left(B(p, \epsilon)\right) \cap \mathbb{R}^k\right\}.$$

From the continuity of the constants in (21) we deduce the existence of a constant $C > 0$ and a function $\rho : \mathbb{R} \to \mathbb{R}$, $\rho(\epsilon) \to 0$ as $\epsilon \to 0$, which satisfy the following property: for every $p \in K$,

$$B_{\widehat{d}_p}(0, \epsilon(1 - \rho(\epsilon))) \subset \varphi_p\left(B(p, \epsilon)\right) \subset B_{\widehat{d}_p}(0, \epsilon(1 + \rho(\epsilon))), \tag{22}$$

and, if x, y belong to $B_{\widehat{d}_p}(0, \epsilon(1 + \rho(\epsilon)))$, then

$$|d(\varphi_p^{-1}(x), \varphi_p^{-1}(y)) - \widehat{d}_p(x, y)| \leq C(\epsilon(1 + \rho(\epsilon)))^{1+1/r}.$$

As a consequence, the diameter of $N \cap B(p, \epsilon)$ satisfies

$$\begin{aligned}\operatorname{diam}_{\widehat{d}_p}\left(T_pN \cap B_{\widehat{d}_p}(0, \epsilon(1 - \rho(\epsilon)))\right) - C(\epsilon(1 + \rho(\epsilon)))^{1+1/r} \ &\leq \ \operatorname{diam}(N \cap B(p, \epsilon)) \\ &\leq \ \operatorname{diam}_{\widehat{d}_p}\left(T_pN \cap B_{\widehat{d}_p}(0, \epsilon(1 + \rho(\epsilon)))\right) + C(\epsilon(1 + \rho(\epsilon)))^{1+1/r},\end{aligned}$$

for every $p \in K$. From the homogeneity of $\widehat{d}_p$,

$$\operatorname{diam}_{\widehat{d}_p}\left(T_pN \cap B_{\widehat{d}_p}(0, \epsilon(1 \pm \rho(\epsilon)))\right) = \epsilon(1 \pm \rho(\epsilon)) \operatorname{diam}_{\widehat{d}_p}\left(T_pN \cap \widehat{B}_p\right),$$

and we obtain (19) from the inequalities above. Thus point (iii) is proved.

Proof of (iv). By (ii) there exists a canonical isomorphism σ between $\bigwedge^k(T_p^*N)$ and $\bigwedge^k(\mathfrak{gr}_p^N(\mathcal{D})^*)$ (see the construction in [23, Section 10.5]). Let $\sigma(\varpi_p)$ be the image under such isomorphism of the value ϖ_p of ϖ at p. Then $\hat{\varpi}^p$ is defined as the left-invariant k-form on $\mathrm{Gr}_p^N(\mathcal{D})$ which coincides with $\sigma(\varpi_p)$ at the origin.

The next step is to show that the function $p \mapsto \hat{\mu}^p(T_pN \cap \widehat{B}_p)$ is continuous on N. In coordinates φ_p, we identify T_pN and $\mathrm{Gr}_p^N(\mathcal{D})$ with $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. Through this identification, $\sigma(\varpi_p)$ coincides with ϖ_p and may be written as $\varpi_p(\partial_{x_1}, \dots, \partial_{x_k})(dx_1 \wedge \cdots \wedge dx_k)_p$. As a consequence

$$\hat{\varpi}^p = \varpi_p(\partial_{x_1}, \dots, \partial_{x_k})\, dx_1 \wedge \cdots \wedge dx_k \quad \text{and} \quad \hat{\mu}^p = \varpi_p(\partial_{x_1}, \dots, \partial_{x_k})\mathscr{L}^k,$$

where $\mathscr{L}^k$ is the Lebesgue measure on $\mathbb{R}^k$.

Continuity of $p \mapsto \hat{\mu}^p(T_pN \cap \widehat{B}_p)$ on N reduces to continuity of $p \mapsto \mathscr{L}^k(T_pN \cap \widehat{B}_p)$ on N. As for the proof of the continuity of $p \mapsto \mathrm{diam}_{\widehat{d}_p}(T_pN \cap \widehat{B}_p)$, the crucial point is the homogeneity of $\widehat{d}_p$. Fix $p \in N$ and define the functions $\alpha, \beta : N \mapsto \mathbb{R}$ as follows. For $q \in N$,

$$\alpha(q) = \max\{\widehat{d}_q(0,x) \;\mid\; x \in \mathbb{R}^k,\ \widehat{d}_p(0,x) = 1\}, \qquad \beta(q) = \min\{\widehat{d}_q(0,x) \;\mid\; x \in \mathbb{R}^k,\ \widehat{d}_p(0,x) = 1\}.$$

The continuity of $q \mapsto \widehat{d}_q$ implies that α and β are continuous. Moreover, by homogeneity of the distances $\widehat{d}_q$, we have $\delta_{\beta(q)}\widehat{B}_q \cap \mathbb{R}^k \subset \widehat{B}_p \cap \mathbb{R}^k \subset \delta_{\alpha(q)}\widehat{B}_q \cap \mathbb{R}^k$. Recall that, for $x \in \mathbb{R}^k$ $\delta_s x = (s^{w_1(p)}x_1, \dots, s^{w_k(p)}x_k)$ and that $Q_N = \sum_{i=1}^k w_i(q)$ for every $q \in N$ (see remark 4.2). Hence

$$\frac{1}{\alpha(q)^{Q_N}}\mathscr{L}^k(\widehat{B}_p \cap \mathbb{R}^k) \le \mathscr{L}^k(\widehat{B}_q \cap \mathbb{R}^k) \le \frac{1}{\beta(q)^{Q_N}}\mathscr{L}^k(\widehat{B}_p \cap \mathbb{R}^k).$$

Since α and β are continuous and converge to 1 as $q \to p$, $\mathscr{L}^k(\widehat{B}_q \cap \mathbb{R}^k)$ converges to $\mathscr{L}^k(\widehat{B}_p \cap \mathbb{R}^k)$ as $q \to p$. This proves the continuity of $p \mapsto \mathscr{L}^k(T_pN \cap \widehat{B}_p)$ and so the one of $p \mapsto \hat{\mu}^p(T_pN \cap \widehat{B}_p)$.

It remains to prove (20). Let us write the measure μ in coordinates φ_p,

$$\mu(N \cap B(p,\epsilon)) = \int_{\varphi_p(N \cap B(p,\epsilon))} \varphi_{p*}\varpi(\partial_{x_1}, \dots, \partial_{x_k})(x)\, dx_1 \wedge \cdots \wedge dx_k.$$

Using (22) and the smoothness of the function $(p,x) \mapsto \varphi_{p*}\varpi(\partial_{x_1}, \dots, \partial_{x_k})(x)$, we obtain the following inequalities,

$$\begin{aligned}\int_{T_pN \cap B_{\widehat{d}_p}(0,\epsilon(1-\rho(\epsilon)))} \Big(\varpi_p(\partial_{x_1}, \dots, \partial_{x_k}) - \rho(\epsilon)\Big)\, dx_1 \wedge \cdots \wedge dx_k \;&\le\; \mu(N \cap B(p,\epsilon)) \\ &\le \int_{T_pN \cap B_{\widehat{d}_p}(0,\epsilon(1+\rho(\epsilon)))} \Big(\varpi_p(\partial_{x_1}, \dots, \partial_{x_k}) + \rho(\epsilon)\Big)\, dx_1 \wedge \cdots \wedge dx_k.\end{aligned}$$

The conclusion follows from

$$\hat{\mu}^p(T_pN \cap \widehat{B}_p) = \frac{1}{\epsilon^{Q_N}}\hat{\mu}^p(T_pN \cap B_{\widehat{d}_p}(0,\epsilon)) = \frac{1}{\epsilon^{Q_N}}\int_{T_pN \cap B_{\widehat{d}_p}(0,\epsilon)} \varpi_p(\partial_{x_1}, \dots, \partial_{x_k})\, dx_1 \wedge \cdots \wedge dx_k. \tag{23}$$

□

5.2 Hausdorff volume of equisingular submanifolds

Recall that $\mathrm{vol}_H^N = \mathcal{S}_N^{\dim_H N}$ (see Section 2.1).

Theorem 5.3. *Let $N \subset M$ be an oriented equisingular submanifold and let μ be a smooth volume on N. Then*

1. $\dim_H N = Q_N$ *and* $\mathrm{vol}_H^N = \mathcal{S}_N^{Q_N}$;
2. vol_H^N *is a Radon measure on* N, *i.e,* $\mathrm{vol}_H^N(K) < \infty$ *for every compact set* $K \subset N$;
3. $\mathrm{vol}_H^N \ll \mu$ *and* $\mu \ll \mathrm{vol}_H^N$;
4. *the Radon-Nikodym derivative of* vol_H^N *with respect to* μ *is the density* $\lim_{\epsilon\to 0} \frac{\mathrm{vol}_H^N(N\cap B(p,\epsilon))}{\mu(N\cap B(p,\epsilon))}$, *whose value is*

$$\lim_{\epsilon\to 0} \frac{\mathrm{vol}_H^N(N \cap B(p,\epsilon))}{\mu(N \cap B(p,\epsilon))} = \frac{\mathrm{diam}_{\widehat{d}_p}(T_pN \cap \widehat{B}_p)^{Q_N}}{\hat{\mu}^p(T_pN \cap \widehat{B}_p)}, \quad \forall\, p \in N. \tag{24}$$

Remark 5.4. When M is equiregular and connected, applying Theorem 5.3 to $N = M$ we deduce Theorem 3.1.

Remark 5.5. Point 4 together with points (iii) and (iv) of Proposition 5.1 shows that the Radon-Nikodym derivative $\frac{d\mathrm{vol}_H^N}{d\mu}(q)$ is continuous on N.

Before starting the proof of the theorem, we need to establish the following simple lemma.

Lemma 5.6. *Let* N *and* μ *be as in Theorem 5.3. Let* $p \in N$. *Assume there exists positive constants* ϵ_0 *and* $\mu_+ > \mu_-$ *such that, for every* $\epsilon < \epsilon_0$ *and every point* $q \in N \cap B(p,\epsilon_0)$, *there holds*

$$\mu_- \,\mathrm{diam}(N \cap B(q,\epsilon))^{Q_N} \le \mu(N \cap B(q,\epsilon)) \le \mu_+ \,\mathrm{diam}(N \cap B(q,\epsilon))^{Q_N}. \tag{25}$$

Then, for every $\epsilon < \epsilon_0$,

$$\frac{\mu(N \cap B(p,\epsilon))}{\mu_+} \le \mathcal{S}_N^{Q_N}(B(p,\epsilon)) \le \frac{\mu(N \cap B(p,\epsilon))}{\mu_-}. \tag{26}$$

Proof. Let $\bigcup_i B(q_i,r_i)$ be a covering of $N \cap B(p,\epsilon)$ with balls centered at points in N of radius smaller than $\delta < \epsilon_0$. If δ is small enough, every q_i belongs to $N \cap B(p,\epsilon_0)$ and, using (25), there holds

$$\mu(N \cap B(p,\epsilon)) \le \sum_i \mu(N \cap B(q_i,r_i)) \le \mu_+ \sum_i \mathrm{diam}(N \cap B(q_i,r_i))^{Q_N}.$$

Hence, we have $\mathcal{S}_N^{Q_N}(B(p,\epsilon)) \ge \frac{\mu(N\cap B(p,\epsilon))}{\mu_+}$.

For the other inequality, let $\eta > 0$, $0 < \delta < \epsilon_0$ and let $\bigcup_i B(q_i,r_i)$ be a covering of $N \cap B(p,\epsilon)$ such that $q_i \in N \cap B(p,\epsilon)$, $r_i < \delta$ and $\sum_i \mu(N \cap B(q_i,r_i)) \le \mu(N \cap B(p,\epsilon)) + \eta$. Such a covering exists due to the Vitali covering lemma. Using as above (25), we obtain

$$\mu(N \cap B(p,\epsilon)) + \eta \ge \sum_i \mu(N \cap B(q_i,r_i)) \ge \mu_- \sum_i \mathrm{diam}(N \cap B(q_i,r_i))^{Q_N}.$$

We then have $\mathcal{S}_{N,\delta}^{Q_N}(B(p,\epsilon)) \le \frac{N\cap\mu(B(p,\epsilon))}{\mu_-} + \frac{\eta}{\mu_-}$. Letting η and δ tend to 0, we get the conclusion. □

Proof of Theorem 5.3. Let $p \in N$ and fix $\delta > 0$. Set

$$\mu_\pm(\delta) = \frac{\hat{\mu}^p(T_pN \cap \widehat{B}_p)}{\mathrm{diam}_{\widehat{d}_p}(T_pN \cap \widehat{B}_p)} \pm \delta$$

Let K be the intersection of N with the closed ball of center p and radius δ. By Proposition 5.1, the functions $q \mapsto \mathrm{diam}_{\widehat{d}_q}(T_qN \cap \widehat{B}_q)$ and $q \mapsto \hat{\mu}^q(T_qN \cap \widehat{B}_q)$ are uniformly continuous on K and convergences (19) and (20) are uniform on K. Therefore there exists $\epsilon_0 > 0$ (depending only on δ) such that $\epsilon_0 < \delta$ and, for every $\epsilon < \epsilon_0$, for every $q \in N \cap B(p,\epsilon)$, there holds

$$\mu_-(\delta)\,\mathrm{diam}(B(q,\epsilon) \cap N)^{Q_N} \le \mu(B(q,\epsilon) \cap N) \le \mu_+(\delta)\,\mathrm{diam}(B(q,\epsilon) \cap N)^{Q_N}.$$

Applying Lemma 5.6 we deduce that for every $\epsilon < \epsilon_0$ there holds

$$\mu_-(\delta) \leq \frac{\mathcal{S}_N^{Q_N}(B(p,\epsilon))}{\mu(B(p,\epsilon)\cap N)} \leq \mu_+(\delta). \tag{27}$$

Therefore we infer that $\mathcal{S}_N^{Q_N}$ is a Radon measure and, using a covering argument, that $\mathcal{S}^{Q_N}$ and μ are mutually absolutely continuous, i.e., property 3. As δ goes to 0 we obtain that

$$\lim_{\epsilon\downarrow 0} \frac{\mathcal{S}_N^{Q_N}(B(p,\epsilon))}{\mu(B(p,\epsilon)\cap N)} = \frac{\operatorname{diam}_{\widehat{d}_p}(T_pN\cap \widehat{B}_p)}{\hat{\mu}^p(T_pN\cap \widehat{B}_p)}. \tag{28}$$

Since the right-hand side of (28) is positive and continuous on N, for $\epsilon > 0$ small enough,

$$0 < \mathcal{S}^{Q_N}(N\cap B(p,\epsilon)) < \infty,$$

whence $\dim_H N = Q_N$ and $\mathrm{vol}_H^N = \mathcal{S}_N^{Q_N}$. As a consequence, (28) gives (24). Finally, we apply the differentiation theorem for Radon measures [26, Theorem 4.7] (with $X = N$, $\mu_1 = \mu$ and $\mu_2 = \mathrm{vol}_H^N$) to obtain that $\lim_{\epsilon\to 0} \frac{\mathrm{vol}_H^N(N\cap B(p,\epsilon))}{\mu(N\cap B(p,\epsilon))}$ coincides with the Radon-Nikodym derivative of vol_H^N with respect to μ. □

5.3 Weak equivalent of $q \mapsto \hat{\mu}^q(T_qN \cap \widehat{B}_q)$

We end this section by stating a result which gives a weak equivalent of the function $q \mapsto \hat{\mu}^q(T_qN \cap \widehat{B}_q)$ appearing in Theorem 5.3. This is instrumental to determine whether the Radon-Nikodym derivative of vol_H^N with respect to μ is integrable or not. This result stems from the uniform Ball-Box Theorem, [18] and [20, Theorem 4.7]. A related formula computing explicitly Popp's measure in equiregular manifolds has been given in [5].

Proposition 5.7. *Let M be an oriented manifold with a volume form ω, and $N \subset M$ be an oriented submanifold with a volume form ϖ. We denote by μ the associated smooth volume on N and we set $k = \dim N$. Finally, let $X_1, \dots, X_m$ be a generating family for a sub-Riemannian structure on M.*

If N is equisingular, then for any compact subset $K \subset M$ there exists a constant $C > 0$ such that, for every $q \in N \cap K$,

$$\hat{\mu}^q(T_qN \cap \widehat{B}_q) \asymp_C \bar{\nu}(q),$$

where $\bar{\nu}(q) = \max\{\big|\left(\varpi \wedge dX_{I_{k+1}} \wedge \cdots \wedge dX_{I_n}\right)_q (X_{I_1}(q), \dots, X_{I_n}(q))\big|\}$, the maximum being taken among all n-tuples $(X_{I_1}, \dots, X_{I_n})$ in $\arg\max\{\omega_q(X_{I'_1}(q), \dots, X_{I'_n}(q)) \mid \sum_i |I'_i| = Q(q)\}$.

Remark 5.8. As a particular case of the proposition, if N is an open equisingular submanifold of M and $\varpi = \omega$,

$$\hat{\mu}^q(\widehat{B}_q) \asymp_C \max\left\{|\omega_q(X_{I'_1}, \dots, X_{I'_n})| \;\middle|\; \sum_i |I'_i| = Q(q)\right\}, \quad \text{for every } q \in N \cap K.$$

Remark 5.9. Proposition 5.7, together with Theorem 5.3, allows to give an estimate of the Radon-Nikodym derivative of vol_H^N with respect to μ. Indeed, there exists a constant $C > 0$ such that, for every $q \in K \cap N$,

$$\frac{d\mathrm{vol}_H^N}{d\mu}(q) \asymp_C \frac{1}{\bar{\nu}(q)}. \tag{29}$$

Proof. Let q be a point in $N\cap K$, and $(X_{I_1}, \dots, X_{I_n})$ be a n-tuple of brackets in $\arg\max\{\omega_q(X_{I'_1}(q), \dots, X_{I'_n}(q)) \mid \sum_i |I'_i| = Q(q)\}$ such that

$$\bar{\nu}(q) = \left(\varpi \wedge dX_{I_{k+1}} \wedge \cdots \wedge dX_{I_n}\right)_q (X_{I_1}(q), \dots, X_{I_n}(q)).$$

Let x be the local coordinates defined by the diffeomorphism

$$x \mapsto \exp(x_n X_{I_n}) \circ \cdots \circ \exp(x_1 X_{I_1})(q),$$

and denote by $\mathrm{Box}^2_{\mathcal{X}}(q,\epsilon)$ the set of points whose coordinates satisfy $|x_i| \leq \epsilon^{|I_i|}$, $i = 1, \ldots, n$. From Proposition A.1 (see Appendix), for ϵ small enough,

$$\mathrm{Box}^2_{\mathcal{X}}(q, \epsilon/C) \subset B(q,\epsilon) \subset \mathrm{Box}^2_{\mathcal{X}}(q, C\epsilon),$$

where the positive constant C depends only on ω, K, and $X_1, \ldots, X_m$.

Let us write $\bar{\nu}(q)$ as a limit,

$$\bar{\nu}(q) = \limsup_{\epsilon \to 0} \frac{1}{(2\epsilon)^{Q(q)}} \int_{\mathrm{Box}^2_{\mathcal{X}}(q,\epsilon)} (\varpi \wedge dX_{I_{k+1}} \wedge \cdots \wedge dX_{I_n})(\partial_{x_1}, \ldots, \partial_{x_n}) dx_1 \ldots dx_n.$$

The inclusions above imply

$$\bar{\nu}(q) \asymp_C \limsup_{\epsilon \to 0} \frac{1}{(2\epsilon)^{Q(q)}} \int_{B(q,\epsilon)} \varpi \wedge dX_{I_{k+1}} \wedge \cdots \wedge dX_{I_n}. \tag{30}$$

On the other hand, since N is equisingular, we can construct an adapted basis as in Lemma 4.3: choose vector fields $Y_1, \ldots, Y_k$ defined in a neighborhood of q such that Y_i is tangent to N, $Y_i \in \mathrm{span}\{X_{I_j} \mid |I_j| \leq \ell(Y_i)\}$ and $Y_1, \ldots, Y_k$ is a basis adapted to the flag (12) restricted to N at q. Note that the vectors $X_{I_{k+1}}(q), \ldots, X_{I_n}(q)$ in T_qM are transverse to T_qN since $\bar{\nu}(q) > 0$. Then, up to a rescaling of $Y_1, \ldots, Y_k$, we can assume that the n-tuple $\mathcal{Y} = (Y_1, \ldots, Y_k, X_{I_{k+1}}, \ldots, X_{I_n})$ satisfies condition (33) of Proposition A.1. Denote by y the coordinates defined by

$$y \mapsto \exp(y_n X_{I_n}) \circ \cdots \circ \exp(y_{k+1} X_{I_{k+1}}) \circ \exp(y_k Y_k) \circ \cdots \circ \exp(y_1 Y_1)(q),$$

and by $\mathrm{Box}^2_{\mathcal{Y}}(q,\epsilon)$ the set of points whose coordinates satisfy $|y_i| \leq \epsilon^{|I_i|}$, $i = 1, \ldots, n$. From Proposition A.1, for ϵ small enough,

$$\mathrm{Box}^2_{\mathcal{Y}}(q, \epsilon/C) \subset B(q,\epsilon) \subset \mathrm{Box}^2_{\mathcal{Y}}(q, C\epsilon). \tag{31}$$

Coordinates y are a particular kind of coordinates φ_q constructed in Lemma 4.3. In particular, the submanifold N lies in the set $\{y_{k+1} = \cdots = y_n = 0\}$ and, with a little abuse of notations,

$$\mathrm{Box}^2_{\mathcal{Y}}(q, \epsilon/C) \cap \{y_{k+1} = \cdots = y_n = 0\} \subset B(q,\epsilon) \cap N \subset \mathrm{Box}^2_{\mathcal{Y}}(q, C\epsilon) \cap \{y_{k+1} = \cdots = y_n = 0\}. \tag{32}$$

As in the proof of Proposition 5.1, we use coordinates y to identify locally M with T_qM and N with T_qN. In particular, inclusions (32) imply

$$\mathrm{Box}^2_{\mathcal{Y}}(q, \epsilon/C) \cap \mathbb{R}^k \subset T_qN \cap B_{\widehat{d}_p}(0,\epsilon) \subset \mathrm{Box}^2_{\mathcal{Y}}(q, C\epsilon) \cap \mathbb{R}^k.$$

Using this inclusion in (23), we get, up to increasing C,

$$\begin{aligned} \hat{\mu}^q(T_qN \cap \widehat{B}_q) &\asymp_C \frac{1}{\epsilon^{Q_N}} \int_{\mathrm{Box}^2_{\mathcal{Y}}(q,\epsilon) \cap \mathbb{R}^k} \varpi_q(\partial_{y_1}, \ldots, \partial_{y_k}) dy_1 \ldots dy_k \\ &\asymp_C \varpi_q(\partial_{y_1}, \ldots, \partial_{y_k}). \end{aligned}$$

Since $\partial_{y_i}(q) = Y_i(q)$, $i = 1, \ldots, k$, and since $X_{I_{k+1}}(q), \ldots, X_{I_n}(q)$ are transverse to T_qN, we have

$$\begin{aligned} \varpi_q(\partial_{y_1}, \ldots, \partial_{y_k}) &= \varpi_q(Y_1(q), \ldots, Y_k(q)) \\ &= (\varpi \wedge dX_{I_{k+1}} \wedge \cdots \wedge dX_{I_n})_q(Y_1(q), \ldots, Y_k(q), X_{I_{k+1}}(q), \ldots, X_{I_n}(q)). \end{aligned}$$

The latter term can be written as a limit,

$$\begin{aligned}(\varpi \wedge dX_{I_{k+1}} \wedge \cdots \wedge dX_{I_n})_q(Y_1(q), \ldots, Y_k(q), X_{I_{k+1}}(q), \ldots, X_{I_n}(q)) \\ = \limsup_{\epsilon \to 0} \frac{1}{(2\epsilon)^{Q(q)}} \int_{\mathrm{Box}^2_{\mathcal{Y}}(q,\epsilon)} (\varpi \wedge dX_{I_{k+1}} \wedge \cdots \wedge dX_{I_n})(\partial_{y_1}, \ldots, \partial_{y_n}) dy_1 \ldots dy_n \\ \asymp_C \limsup_{\epsilon \to 0} \frac{1}{(2\epsilon)^{Q(q)}} \int_{B(q,\epsilon)} \varpi \wedge dX_{I_{k+1}} \wedge \cdots \wedge dX_{I_n},\end{aligned}$$

where we have used (31). The conclusion then follows from (30). □

6 Applications

In this section we use the ideas of Section 4 to characterize local integrability of $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$ in the generic smooth case. As it turns out, this property depends on the placement of the dimension of the manifold with respect to dimensions of free Lie algebras and on the Hausdorff dimension of certain equisingular submanifolds. We end by listing a number of examples illustrating several possible cases.

6.1 Generic smooth case

Let M be a n-dimensional smooth oriented manifold and let $m \in \mathbb{N}$. We consider the set $\mathcal{U}_m$ of sub-Riemannian structures $(\mathbf{U}, \langle \cdot, \cdot \rangle, f)$ of rank m on M, i.e. such that rank $\mathbf{U} = m$. Using local generating families we endow this set with the $\mathcal{C}^\infty$-Whitney topology and we say that a sub-Riemannian structure of rank m is generic if it belongs to some residual subset of $\mathcal{U}_m$. Thus the analysis of the singular set of generic m-tuples of vector fields given in [28] provides the following description of the singular set of generic sub-Riemannian structures.

Introduce first some notations. Let $\mathcal{L}$ be the free Lie algebra with m generators. We use $\mathcal{L}^s$ to denote the subspace generated by elements of $\mathcal{L}$ of length not greater than s, and $\tilde{n}_s$ to denote the dimension of $\mathcal{L}^s$. Let r be the integer such that

$$\tilde{n}_{r-1} < n \leq \tilde{n}_r.$$

Then the singular set S of a generic sub-Riemannian structure admits a locally finite stratification $\mathsf{S} = \cup_{i \in \mathbb{N}} \mathsf{S}_i$ by equisingular submanifolds and

$$\min_{i \in \mathbb{N}} \operatorname{codim} \mathsf{S}_i = \tilde{n}_r - n + 1,$$

in particular, S is a μ-negligible set (for every smooth volume μ on M). Moreover, at regular points the growth vector is the maximal one, i.e., $(\tilde{n}_1, \ldots, \tilde{n}_{r-1}, n)$.

Proposition 6.1. *Let M be a n-dimensional smooth oriented manifold, μ a smooth volume on M, and let $(\mathbf{U}, \langle \cdot, \cdot \rangle, f)$ be a generic sub-Riemannian structure of rank m on M.*

(i) If $n = \tilde{n}_r$ then $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$ is not locally integrable on M.

(ii) If $n < \tilde{n}_r$ then $\frac{d\mathcal{S}^{Q_{\mathsf{R}}} \llcorner \mathsf{R}}{d\mu}$ is locally integrable near every point of strata S_i of minimal codimension, i.e., such that $\operatorname{codim} \mathsf{S}_i = \tilde{n}_r - n + 1$.

Proof. The case $n = \tilde{n}_r$ is a direct consequence of Corollary 4.11. Then, we consider the case $n < \tilde{n}_r$. Let N be a stratum of minimal codimension, i.e., $\operatorname{codim} N = \tilde{n}_r - n + 1 \geq 2$. We set $k = \dim N = n - (\tilde{n}_r - n + 1)$. Let ω be a non degenerate n-form such that $\mu = \int \omega$. The construction in [12] allows to characterize N in the following way.

Denote by $\alpha_1, \ldots, \alpha_m$ a set of generators for the free Lie algebra $\mathcal{L}$. Consider a sequence of multi-indices $\{I_j\}_{j \in \mathbb{N}}$ such that, for every $s \in \mathbb{N}$, the family $\alpha_{I_1}, \ldots, \alpha_{I_{\tilde{n}_s}}$ generates $\mathcal{L}^s$. Fix a point $p \in N$ and a local

generating[4] family $(X_1, \dots, X_m)$ of $(\mathbf{U}, \langle\cdot,\cdot\rangle, f)$ near p. Possibly reordering multi-indices of length r, the vectors $X_{I_1}(p), \dots, X_{I_{n-1}}(p)$ are linearly independent. Then, one obtains adapted bases at regular points near p by taking $X_{I_1}, \dots, X_{I_{n-1}}$ and a last vector field chosen among $X_{I_n}, \dots, X_{I_{\tilde{n}_r}}$. There are $n-k = \tilde{n}_r - n + 1$ such bases, which we denote by $\mathcal{X}_{\mathcal{I}^{k+1}}, \dots, \mathcal{X}_{\mathcal{I}^n}$ (for the notation $\mathcal{X}_{\mathcal{I}}$ and $\omega(\mathcal{X}_{\mathcal{I}})$ see Section 3.3). Then the map

$$q \mapsto (\omega(\mathcal{X}_{\mathcal{I}^{k+1}})(q), \dots, \omega(\mathcal{X}_{\mathcal{I}^n})(q))$$

is a submersion at p and, locally,

$$N = \{q \mid \omega(\mathcal{X}_{\mathcal{I}^{k+1}})(q) = \cdots = \omega(\mathcal{X}_{\mathcal{I}^n})(q) = 0\}.$$

Note that the sub-Riemannian manifold satisfies assumption (A). By Proposition 3.12, it suffices to show that $1/\nu$ is locally integrable near p. Set $z_i = \omega(\mathcal{X}_{\mathcal{I}^i})$, $i = k+1, \dots, n$. There exists functions $y_1, \dots, y_k$ near p such that (y, z) is a system of local coordinates near p. In these coordinates, N is identified with the set $\{(y, z) \mid z = 0\}$ and $\nu(z) = \sqrt{z_{k+1}^2 + \cdots + z_n^2}$. Thus $1/\nu$ is locally integrable near p, which ends the proof. □

6.2 Examples

In this section we present several examples where assumption (A) is satisfied and one can directly tell whether $\mathcal{S}^{Q_\mathsf{R}}$ is integrable or not using criteria in Section 4.

Recall that a sub-Riemannian structure $(\mathbf{U}, \langle\cdot,\cdot\rangle, f)$ on an n-dimensional manifold M is called almost-Riemannian if the rank of $\mathbf{U}$ is n (see [3]).

Example 6.2 (Generic almost-Riemannian structures). Let $(\mathbf{U}, \langle\cdot,\cdot\rangle, f)$ be a generic almost-Riemannian structure on M. At regular points the structure is Riemannian, i.e., the growth vector is simply (n) and $Q_\mathsf{R} = n$. In particular, $n = \tilde{n}_1$ and thus the first case of Proposition 6.1 applies and $\frac{d\mathcal{S}^n \llcorner \mathsf{R}}{d\mu}$ is never locally integrable (with respect to a smooth volume) around singular points. As a consequence, $\mathcal{S}^n \llcorner \mathsf{R}$ is never a Radon measure. Also, an alternative proof of this fact comes as a consequence of Corollary 4.6 or Corollary 4.11.

We next build examples of almost-Riemannian structures which are not generic and for which $\mathcal{S}^n$ is or fails to be a Radon measure.

Example 6.3 (Non generic almost-Riemannian structures). We consider the almost-Riemannian structure on $\mathbb{R}^3$ for which a global generating family is

$$X_1(x_1, x_2, x_3) = \partial_1, \quad X_2(x_1, x_2, x_3) = \partial_2, \quad X_3(x_1, x_2, x_3) = (x_1^2 + x_2^2)\partial_3,$$

and the canonical volume form $\omega = dx_1 \wedge dx_2 \wedge dx_3$. The singular set S coincides with $\{(x_1, x_2, x_3) \mid x_1 = x_2 = 0\}$ and $Q_\mathsf{R} = 3$. The growth vector at a singular point p is $(2, 2, 3)$, $Q(p) = 5$, and the growth vector of the flag restricted to S is $(0, 0, 1)$, whence $Q_\mathsf{S} = 3$. Hence the Hausdorff dimension of $\mathbb{R}^3$ (endowed with the sub-Riemannian distance) is 3 and $\mathrm{vol}_H = \mathcal{S}^3$. Therefore conditions of Corollary 4.6 are satisfied and $\frac{d\mathcal{S}^3 \llcorner \mathsf{R}}{d\mu}$ is not integrable near a singular point. In other words, small neighborhoods of singular points have infinite Hausdorff volume.

One could also obtain the conclusion by applying Proposition 4.10 since the only adapted basis at regular points is X_1, X_2, X_3. Indeed,

$$\nu(x_1, x_2, x_3) \asymp x_1^2 + x_2^2,$$

and ν is homogeneous of degree 2. Note that $\varrho_{\min}(p, \mathsf{S}) = \varrho_{\max}(p, \mathsf{S}) = 2 = Q(p) - Q_\mathsf{S}$.

Consider now the analogue structure in higher dimension, that is, the almost-Riemannian structure on $\mathbb{R}^4$ for which a global generating family is

$$X_1(x) = \partial_1, \quad X_2(x) = \partial_2, \quad X_3(x) = \partial_3, \quad X_4(x) = (x_1^2 + x_2^2 + x_3^2)\partial_4,$$

[4] For instance, taking a local orthonormal frame $\sigma_1, \dots, \sigma_k$ on $\mathbf{U}$, set $X_i = f \circ \sigma_i$.

where $x = (x_1, x_2, x_3, x_4)$, and the canonical volume form $\omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$. The singular set is $\mathsf{S} = \{x \in \mathbb{R}^4 \mid x_1 = x_2 = x_3 = 0\}$ and $Q_{\mathsf{R}} = 4$. The growth vector at a singular point p is $(3, 3, 4)$, $Q(p) = 6$, and the growth vector of the flag restricted to S is $(0, 0, 1)$, whence $Q_{\mathsf{S}} = 3$. Hence the Hausdorff dimension of $\mathbb{R}^4$ (endowed with the sub-Riemannian distance) is 4 and $\mathrm{vol}_H = \mathcal{S}^4$. The only adapted basis at regular points is X_1, X_2, X_3, X_4, which gives

$$\nu(x) \asymp x_1^2 + x_2^2 + x_3^2,$$

i.e., ν is homogeneous of degree 2 and $\varrho_{\min}(p, \mathsf{S}) = \varrho_{\max}(p, \mathsf{S}) = 2 < Q(p) - Q_{\mathsf{S}}$. Then by Proposition 4.9 $\frac{d\mathcal{S}^4}{d\mu}$ is integrable near any singular point. Equivalently, $\mathcal{S}^4$ is a Radon measure.

Example 6.4 (the Martinet space). Recall the sub-Riemannian structure of Example 3.14, where $M = \mathbb{R}^3$ and a global generating family is

$$X_1 = \partial_1, \quad X_2 = \partial_2 + \frac{x_1^2}{2}\partial_3.$$

The singular set is the plane $\mathsf{S} = \{x \in \mathbb{R}^3 \mid x_1 = 0\}$ which is equisingular and has codimension 1. The growth vector at any singular point p is $(2, 2, 3)$, $Q(p) = 5$ and the growth vector restricted to S is $(1, 1, 1)$, whence $Q_{\mathsf{S}} = 4$. Since $Q_{\mathsf{R}} = 4$, the Hausdorff dimension of the Martinet space is 4 and the Hausdorff volume is $\mathrm{vol}_H = \mathcal{S}^4$. The only adapted basis at regular points is $X_1, X_2, [X_1, X_2]$, whence $\nu(x) \asymp |x_1|$, in particular ν is homogeneous and $\varrho_{\min}(p, \mathsf{S}) = \varrho_{\max}(p, \mathsf{S}) = 1$. Either integrating directly or applying Corollary 4.11 one infers that $\frac{d\mathcal{S}^4 \llcorner \mathsf{R}}{d\mu}$ is not integrable near singular points.

Recall that the parameter $\varrho_{\min}$ used to characterize non integrability admits a lower bound, namely $\varrho_{\min} \geq Q(p) - Q_{\mathsf{R}}$ (see Remark 4.5). We next see an example showing that $\varrho_{\min}$ can be greater than this bound.

Example 6.5. Consider the sub-Riemannian structure on $\mathbb{R}^5$ for which a global generating family is

$$X_1 = \partial_1, \quad X_2 = \partial_2 + x_1\partial_3 + x_1^2\partial_5, \quad X_3 = \partial_4 + (x_1^k + x_2^k)\partial_5,$$

with $k \in \mathbb{N}$, and the canonical volume form $\omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5$. Computing Lie brackets one obtains

$$X_{12} = [X_1, X_2] = \partial_3 + 2x_1\partial_5, \quad X_{13} = kx_1^{k-1}\partial_5, \; X_{23} = kx_2^{k-1}\partial_5, \; X_{112} = 2\partial_5.$$

Hence at regular points the growth vector is $(3, 5)$, $Q_{\mathsf{R}} = 7$, and the singular set is $\mathsf{S} = \{x_1 = x_2 = 0\}$. At a singular point p the growth vector is $(3, 4, 5)$, $Q(p) = 8$ and the growth vector of the flag restricted to S is $(1, 2, 3)$, which gives $Q_{\mathsf{S}} = 6$. Hence the Hausdorff dimension of $\mathbb{R}^5$ (endowed with the sub-Riemannian distance) is 7 and the Hausdorff volume is $\mathrm{vol}_H = \mathcal{S}^7$. At regular points there are two adapted bases $X_1, X_2, X_3, X_{12}, X_{13}$ and $X_1, X_2, X_3, X_{12}, X_{23}$, which implies that

$$\nu(x_1, x_2, x_3, x_4, x_5) \asymp \sqrt{x_1^{2(k-1)} + x_2^{2(k-1)}}.$$

Thus ν is homogeneous of order $k-1$ and $\varrho_{\min}(p, \mathsf{S}) = \varrho_{\max}(p, \mathsf{S}) = k-1$. In particular $\varrho_{\min}(p, \mathsf{S}) > Q(p) - Q_{\mathsf{R}}$ if and only if $k > 2$. Applying Proposition 4.4 and Proposition 4.9, $\frac{d\mathcal{S}^7}{d\mu}$ is locally integrable around singular points if and only if $k \leq 2$.

We end by illustrating a case where $\varrho_{\min} < Q(p) - Q_{\mathsf{S}} < \varrho_{\max} < \infty$ and vol_H is locally integrable, showing that the condition on $\varrho_{\max}$ given in Proposition 4.9 is not sharp.

Example 6.6. Consider the sub-Riemannian structure on $\mathbb{R}^4$ for which a global orthonormal frame is

$$X_1(x) = \partial_1, \; X_2(x) = \partial_2 + x_1\partial_3 + (x_1^2x_3^2 - x_1x_2^2)\partial_4,$$

and the canonical volume form $\omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$. Computing Lie brackets one obtains

$$X_{12}(x) = \partial_3 + (2x_1x_3^2 - x_2^2)\partial_4, \quad X_{112}(x) = 2x_3^2\partial_4, \quad X_{212}(x) = (4x_1^2x_3 - 2x_2)\partial_4.$$

Hence, the singular set is $\mathsf{S} = \{x \in \mathbb{R}^4 \mid x_2 = x_3 = 0\}$, at regular points the growth vector is $(2,3,4)$ and $Q_{\mathsf{R}} = 7$. At a singular point p the growth vector is $(2,3,3,4)$, $Q(p) = 8$, and the growth vector of the flag restricted to S is $(1,1,1,2)$, which gives $Q_{\mathsf{S}} = 5$. Thus the Hausdorff dimension of $\mathbb{R}^4$ (endowed with the sub-Riemannian distance) is 7 and the Hausdorff volume is $\mathrm{vol}_H = \mathcal{S}^7$. At regular points there are two adapted bases $X_1, X_2, X_{12}, X_{112}$ and $X_1, X_2, X_{12}, X_{212}$, therefore

$$\nu(x) \asymp \sqrt{\omega_1(x)^2 + \omega_2(x)^2},$$

where $\omega_1(x) = \det(X_1, X_2, X_{12}, X_{112}) = 2x_3^2$ and $\omega_2(x) = \det(X_1, X_2, X_{12}, X_{212}) = 3x_1^2x_3^2 - 2x_2$. According to the notation in Proposition 4.1, coordinates $y = (x_1, x_4)$ (of weights $1, 4$ at $p = 0$ respectively) parameterize the singular set, whereas coordinates $z = (x_2, x_3)$ (of weights $1, 2$ at p respectively) are transversal to S. Apply the coordinate change $(y, z) \mapsto (y, \tilde{z})$ where $\tilde{z} = (\omega_2(x), x_3) = (\tilde{x}_2, x_3)$. Then,

$$\nu(y, \delta_\lambda \tilde{z})^2 = \nu(x_1, \lambda\tilde{x}_2, \lambda^2 x_3, x_4) = 4\lambda^8 x_3^4 + \lambda^2 \tilde{x}_2^2,$$

and one easily gets $\varrho_{\min}(p, \mathsf{S}) = 1$, $\varrho_{\max}(p, \mathsf{S}) = 4$. Thus $\varrho_{\min}(p, \mathsf{S}) < Q(p) - Q_{\mathsf{S}} = 3 < \varrho_{\max}(p, \mathsf{S})$ and neither the condition of Proposition 4.4 nor the one of Proposition 4.9 is satisfied. However, taking a sufficiently small neighborhood U of p we have

$$\mathcal{S}^7 \llcorner_{\mathsf{R}}(U) \asymp \int_U \frac{1}{\nu(x)} dx \asymp \int_{[-1,1]^4} \frac{1}{\sqrt{\omega_1(x)^2 + \omega_2(x)^2}} dx \asymp \int_{[-1,1]^2} \frac{1}{\sqrt{\tilde{x}_2^2 + x_3^4}} d\tilde{x}_2 dx_3 < \infty.$$

Thus $\mathcal{S}^7 = \mathcal{S}^7 \llcorner_{\mathsf{R}}$ is a Radon measure.

A Appendix

We show here a technical result which is needed in Section 5.3 to provide the local equivalent of $\frac{d\mathcal{S}_N^{Q_N}}{d\mu}$.

Proposition A.1. *Let M be oriented, ω be a volume form on M, K be a compact subset of M, and $X_1, \dots, X_m$ be a generating family for $(\mathcal{D}, g)$. There exist a constant $C > 0$ and a function $\eta : K \to (0, +\infty)$ such that the following holds for every $p \in K$.*

Let $\mathcal{X} = (X_{I_1}, \dots, X_{I_n})$ be a n-tuple in $\arg\max\{\omega_p(X_{I'_1}(p), \dots, X_{I'_n}(p)) \mid X_{I'_1}, \dots, X_{I'_n}$ s.t. $\sum_i |I'_i| = Q(p)\}$. Consider another n-tuple of vector fields $\mathcal{Y} = (Y_1, \dots, Y_n)$ such that, for $i = 1, \dots, n$,

$$Y_i \in \mathrm{span}\{X_{I_j} \ : \ |I_j| \le \ell(Y_i)\}, \quad \textit{i.e.} \quad Y_i = \sum_{|I_j| \le \ell(Y_i)} Y_i^j X_{I_j},$$

where all components Y_i^j are smooth functions satisfying, for $s \in \mathbb{N}$,

$$\|Y^{(s)}(p)\| \quad \textit{and} \quad \|(Y^{(s)})^{-1}(p)\| \le 2, \tag{33}$$

$Y^{(s)}(p)$ being the matrix

$$Y^{(s)}(p) = \begin{cases} \left(Y_i^j(p)\right)_{\{i,j \ \mid \ \ell(Y_i) = |I_j| = s\}} & \textit{if } \{i,j \ \mid \ \ell(Y_i) = |I_j| = s\} \neq \emptyset \\ 1 & \textit{otherwise.} \end{cases}$$

Then, for any $\epsilon \le \eta(p)$ there holds:

$$\mathrm{Box}^i_{\mathcal{Y}}(p, \epsilon/C) \subset B(p, \epsilon) \subset \mathrm{Box}^i_{\mathcal{Y}}(p, C\epsilon), \quad i = 1, 2, \tag{34}$$

where

$$\mathrm{Box}^1_{\mathcal{Y}}(p, \epsilon) = \left\{ \exp\left(\sum_i y_i Y_i\right)(p) \ \mid \ |y_i| \le \epsilon^{\ell(Y_i)}, \ i = 1, \dots, n \right\},$$
$$\mathrm{Box}^2_{\mathcal{Y}}(p, \epsilon) = \{\exp(z_1 Y_1) \circ \cdots \circ \exp(z_n Y_n)(p) \ \mid \ |z_i| \le \epsilon^{\ell(Y_i)}, \ i = 1, \dots, n\}.$$

Remark A.2. Notice that any n-tuples $\mathcal{X}$ and $\mathcal{Y}$ satisfying the hypothesis of the proposition at a point p form a basis adapted to the flag (8) of the distribution at p. Also, $\mathcal{Y} = \mathcal{X}$ satisfies the hypothesis and in this case the inclusions (34) for $i = 2$ result from the uniform Ball-Box Theorem, [18] and [20, Theorem 4.7].

Let us introduce some notations. Fix $p \in M$. Recall that the weights $w_i = w_i(p)$, $i = 1, \dots, n$, are defined by setting $w_i = s$ if $\dim \mathcal{D}_p^{s-1} < i \leq \dim \mathcal{D}_p^s$. The largest of these integers, i.e. $w_n(p)$, is an upper semi-continuous function of p, and hence it admits a maximum $\bar{w} \in \mathbb{N}$ on the compact set K. Thus $w_i(p) \leq \bar{w}$ for every $p \in K$ and $i = 1, \dots, n$. Using these weights w_i we also define the pseudo-norm $\| \cdot \|_p$ on $\mathbb{R}^n$ by

$$\|x\|_p = \max(|x_1|^{1/w_1}, \dots, |x_n|^{1/w_n}).$$

Notice that any n-tuple of vector fields $\mathcal{Y} = (Y_1, \dots, Y_n)$ verifying the hypothesis of Proposition A.1 satisfies, up to reordering, $\ell(Y_i) = w_i$, for every i. We will always suppose in this section that $\mathcal{Y}$ and $\mathcal{X}$ have been ordered in that way. In particular $|I_i| = w_i$, for every i.

Let $x = (x_1, \dots, x_n)$ be a system of privileged coordinates at p and let us denote by $\widehat{d}_p^x$ the distance of the nilpotent approximation at p defined by means of the coordinates x. Then

$$d(p, q) = \widehat{d}_p^x(0, x(q)) + o(\widehat{d}_p^x(0, x(q))), \tag{35}$$

where $x(q)$ are the coordinates of the point q (see for instance [6]).

Fix now a n-tuple $\mathcal{Y} = (Y_1, \dots, Y_n)$ verifying the hypothesis of Proposition A.1 and denote by y and z the coordinates associated with $\mathrm{Box}_{\mathcal{Y}}^1$ and $\mathrm{Box}_{\mathcal{Y}}^2$ respectively: $y = y(q)$ are the coordinates of the point q satisfying

$$q = \exp\left(\sum_i y_i Y_i\right)(p),$$

and $z = z(q)$ are the coordinates of the point q satisfying

$$q = \exp(z_1 Y_1) \circ \cdots \circ \exp(z_n Y_n)(p).$$

The inclusions (34) are equivalent to

$$\frac{1}{C}\|y(q)\|_p \leq d(p, q) \leq C\|y(q)\|_p \quad \text{and} \quad \frac{1}{C}\|z(q)\|_p \leq d(p, q) \leq C\|z(q)\|_p \quad \text{for } d(p, q) \text{ small enough.}$$

Since the coordinates y and z are privileged coordinates at p, using (35) we get that the inclusions (34) are also equivalent to

$$\frac{1}{C}\|y\|_p \leq \widehat{d}_p^y(0, y) \leq C\|y\|_p \quad \text{and} \quad \frac{1}{C}\|z\|_p \leq \widehat{d}_p^z(0, z) \leq C\|z\|_p \quad \text{for any } y, z \in \mathbb{R}^n. \tag{36}$$

Lemma A.3. *Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two systems of privileged coordinates at p. Assume that the change of coordinates formulas are*

$$x_i = \psi_i(y), \quad i = 1, \dots, n,$$

and let $\widehat{\psi}_i(y)$ be the homogeneous part of $\psi_i(y)$ of degree w_i (that is, the sum of monomials of weighted degree w_i in the Taylor expansion of $\psi_i(y)$). Then

$$\widehat{d}_p^y(0, y) = \widehat{d}_p^x(0, \widehat{\psi}(y)),$$

where $\widehat{\psi}(y) = (\widehat{\psi}_1(y), \dots, \widehat{\psi}_n(y))$.

Proof. This lemma may be seen as a consequence of [6, Proposition 5.20]. We give here a direct proof. Let us use the system of coordinates x to identify a neighborhood of p in M with a neighborhood of 0 in $\mathbb{R}^n$. We have

$$\widehat{d}_p^x(0, x) = \lim_{s \to 0^+} \frac{1}{s} d(0, \delta_s x) \quad \text{and} \quad \widehat{d}_p^y(0, y) = \lim_{s \to 0^+} \frac{1}{s} d(0, \psi(\delta_s y)),$$

where as usual $\delta_s x = (s^{w_1} x_1, \dots, s^{w_n} x_n)$.

Hence $\widehat{d}_p^y(0, y) = \lim_{s \to 0^+} d(0, \delta_{1/s} \psi(\delta_s y))$. The conclusion follows from $\lim_{s \to 0^+} \delta_{1/s} \psi(\delta_s y) = \widehat{\psi}(y)$. □

Proof of Proposition A.1. We already know that (34) holds with $\mathcal{Y} = \mathcal{X}$ and $i = 2$. Let $\mathcal{Y}$ be a n-tuple of vector fields satisfying the hypothesis of Proposition A.1. We will prove that (34) holds for $\mathcal{Y}$ with $i = 1$, the proof with $i = 2$ being similar.

Let x and x' be the coordinates associated with $\mathrm{Box}^1_{\mathcal{Y}}$ and $\mathrm{Box}^2_{\mathcal{X}}$ respectively, and assume that the change of coordinate formulas are $x' = \psi(x)$. From (36) and Lemma A.3, it suffices to show that there exists a constant $C' > 0$, independent of p and of $\mathcal{X}$, such that

$$\frac{1}{C'}\|x\|_p \leq \|\widehat{\psi}(x)\|_p \leq C'\|x\|_p \quad \text{for any } x \in \mathbb{R}^n. \tag{37}$$

By definition of $\psi(x)$, there holds

$$\exp\left(-\sum_i x_i Y_i\right) \circ \exp(\psi_1(x)X_{I_1}) \circ \cdots \circ \exp(\psi_n(x)X_{I_n})(p) = p.$$

Remind that $Y_i = \sum_{w_j \leq w_i} Y_i^j X_{I_j}$. Taking the homogeneous terms of weighted degree zero in the equality above, we obtain

$$\exp\left(-\sum_{j=1}^n \left(\sum_{\{i\,:\,w_i = w_j\}} x_i Y_i^j(p)\right) \widehat{X}_{I_j}\right) \circ \exp(\widehat{\psi}_1(x)\widehat{X}_{I_1}) \circ \cdots \circ \exp(\widehat{\psi}_n(x)\widehat{X}_{I_n})(p) = p.$$

Set $y_j = \sum_{\{i\,:\,w_i = w_j\}} x_i Y_i^j(p)$. The Campbell-Hausdorff formula allows to expand the product above as follows,

$$\sum_{j=1}^n (\widehat{\psi}_j(x) - y_j)\widehat{X}_{I_j}(p) + \sum_{|I| \leq w_n} \sum_{\alpha,\beta} c^I_{\alpha,\beta} y_{\alpha_1} \cdots y_{\alpha_s} \widehat{\psi}_{\beta_1}(x) \cdots \widehat{\psi}_{\beta_r}(x)\widehat{X}_I(p) = 0, \tag{38}$$

where the second sum is taken over all the multi-indices $\alpha = (\alpha_1, \ldots, \alpha_s)$ and $\beta = (\beta_1, \ldots, \beta_r)$ such that $s + r > 1$ and $w(\alpha) + w(\beta) = |I|$ (we use the notation $w(\alpha)$ for $w_{\alpha_1} + \cdots + w_{\alpha_s}$). Note that this implies that all w_{α_i} and w_{β_i} are smaller than $|I|$. The structural constants $c^I_{\alpha,\beta}$ do not depend on p but only on the coefficients of the Campbell-Hausdorff series. More precisely, there exists a constant $C_{\bar{w}}$ depending only on the maximum value $\bar{w}$ of w_n on K such that $|c^I_{\alpha,\beta}| \leq C_{\bar{w}}$ for every I, α, β appearing in (38) at any point $p \in K$.

Now, for every multi-index I there holds $\widehat{X}_I(p) = \sum_{|I_i| = |I|} X^i_I(p)\widehat{X}_{I_i}(p)$ where

$$X^i_I(p) = \frac{\omega(X_{I_1}, \ldots, X_{I_{i-1}}, X_I, X_{I_{i+1}}, \ldots, X_{I_n})(p)}{\omega(X_{I_1}, \ldots, X_{I_n})(p)}.$$

The choice of the family $\mathcal{X} = (X_{I_1}, \ldots, X_{I_n})$ ensures that, for every I and i such that $|I_i| = |I|$ we have $|X^i_I(p)| \leq 1$.

Plugging the expression of $\widehat{X}_I(p)$ into (38), we obtain

$$\sum_{i=1}^n \left(\widehat{\psi}_i(x) - y_i + \sum_{|I| = w_i} \sum_{\alpha,\beta} c^I_{\alpha,\beta} X^i_I(p) y_{\alpha_1} \cdots y_{\alpha_s} \widehat{\psi}_{\beta_1}(x) \cdots \widehat{\psi}_{\beta_r}(x)\right) \widehat{X}_{I_i}(p) = 0. \tag{39}$$

Note that these equalities are in triangular form: all the indices α_j and β_j appearing in the ith equality are smaller than i. Since moreover all the coefficients in the above equality are bounded independently of p, we get the following change of coordinates formulas, for $i = 1, \ldots, n$,

$$\widehat{\psi}_i(x) = \sum_{\alpha \text{ s.t. } w(\alpha) = w_i} a'_\alpha(p) y_{\alpha_1} \cdots y_{\alpha_s} = \sum_{\alpha \text{ s.t. } w(\alpha) = w_i} a_\alpha(p) x_{\alpha_1} \cdots x_{\alpha_s},$$

where $a_\alpha(p) = \sum_{w(\beta)=w_i} a'_\beta(p) Y_{\beta_1}^{\alpha_1}(p) \cdots Y_{\beta_s}^{\alpha_s}(p)$. The coefficients $a'_\alpha(p)$ are bounded independently of p, therefore it follows from hypothesis (33) that it is also the case for $a_\alpha(p)$: there exists a constant $C'_{\bar{w}}$ depending only on $\bar{w}$ such that $|a_\alpha(p)| \leq C'_{\bar{w}}$ for any $p \in K$ and any α such that $w(\alpha) \leq \bar{w}$.

Let N be the number of multi-indices α such that $w(\alpha) \leq \bar{w}$ and set $C' = n(NC'_{\bar{w}})^{1/\bar{w}}$. There holds

$$\sup\{\|\widehat{\psi}(x)\|_p \mid p \in K,\ \|x\|_p = 1\} \leq C',$$

and since every $\widehat{\psi}_i(x)$ is a weighted homogeneous polynomial of degree w_i, we obtain

$$\|\widehat{\psi}(x)\|_p \leq C' \|x\|_p. \tag{40}$$

Now observe that hypothesis (33) implies that the vectors $\widehat{Y}_i(p) = \sum_{w_j=w_i} Y_i^j(p) \widehat{X}_{I_j}(p)$, $i = 1, \ldots, n$, form a basis of T_pM and that the change-of-basis matrix $M(p)$ between $(\widehat{X}_{I_1}(p), \ldots, \widehat{X}_{I_n}(p))$ and $(\widehat{Y}_1(p), \ldots, \widehat{Y}_n(p))$ has coefficients $M_{ij}(p)$ that are bounded independently of p. Identity (39) can then be rewritten as

$$\sum_{i=1}^{n} \left(\sum_{w_j=w_i} M_{ji}(p) \widehat{\psi}_j(x) - x_i + \sum_{w_j=w_i} \sum_{|I|=w_i} \sum_{\alpha,\beta} c_{\alpha,\beta}^I M_{ji}(p) X_I^j(p) x_{\alpha_1} \cdots x_{\alpha_s} \widehat{\psi}_{\beta_1}(x) \cdots \widehat{\psi}_{\beta_r}(x) \right) \widehat{Y}_i(p) = 0.$$

Using the above reasoning in which we exchange the role of x and $\widehat{\psi}$ coordinates, we obtain, up to enlarging the constant C',

$$\|x\|_p \leq C' \|\widehat{\psi}(x)\|_p.$$

This inequality together with (40) gives exactly (37), which ends the proof. □

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