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A NOTE ON THE NECESSITY OF FILTERING MECHANISM FOR POLYNOMIAL OBSERVABILITY OF TIME-DISCRETE WAVE EQUATION

The problem of uniform polynomial observability was recently analyzed. It is shown that, when the continuous model is uniformly polynomially observable, it is sufficient to filter initial data to derive uniform polynomial observability inequalities for suitable time-discretization schemes. In this note, we prove that a filtering mechanism of high frequency modes is necessary to obtain uniform polynomial observability.

More precisely, we give a counterexample which proves that this latter fails without filtering the initial data for time semi-discrete approximations of the wave equation.

Key words and phrases: observability inequality, time discretization, filtering techniques.

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1 INTRODUCTION

We consider the following wave equation on interval of length 1

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & 0 < x < 1, \end{cases} \tag{1}$$

where $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$. It is easy to check (see [1]) that this system is well posed in the energy space $H_0^1(0,1) \times L^2(0,1)$. More precisely, for any $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ there exists a unique solution $u \in C((0,T), H_0^1) \cap C^1((0,T), L^2(0,1))$ of (1).

The energy of solutions of (1) is conserved in time, i.e.,

$$E(t) = \frac{1}{2} \int_0^1 \left(|u_t(x,t)|^2 + |u_x(x,t)|^2 \right) dx = E(0) \qquad \text{for all} \quad 0 \le t \le T.$$

Define the output function

$$y(t) = u_t(\xi, t), \quad \xi \in (0,1). \tag{2}$$

It was proved in [1] that system (1) is polynomially observable when $\xi \in \mathcal{S}$, where $\mathcal{S}$ is the set of all numbers $\rho \in (0,1)$ such that $\rho \notin \mathbb{Q}$ (the set of rational numbers) and if $[a_0, a_1, \dots, a_n, \dots]$ is the expansion of ρ as a continued fraction, then (a_n) is bounded. More precisely, we have the following assertion.

Proposition 1. *Let $T > 0$ be fixed. Then for all $\xi \in S$ the solution u of* (1) *satisfies*

$$C_\xi \int_0^T (u_t(\xi,t))^2 dt \geq \|u^0\|_{L^2(0,1)}^2 + \|u^1\|_{H^{-1}(0,1)}^2, \tag{3}$$

where $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$, C_ξ is a constant depending only on ξ.

In the remainder of this paper, ξ is fixed and belongs to $\mathcal{S}$. In this paper, we are interested in time discretization of system (1). The analysis of observability properties of numerical approximation schemes for the wave equation has been the object of intensive studies. However most analytical results concern the case of exact observability for discrete systems ([2, 7]). Recently in [3, 4], time semi-discretization of polynomial observability was analyzed. The author shows that a filtering technique allows to restore a uniform (with respect to the parameter of discretization) polynomial observability for the discrete model. But there is no result provided the necessity of this method. Consequently the main goal of our note is to give a counterexample which proves that uniform polynomial observability fails without filtering the initial data for time semi-discrete approximations of the wave equation.

2 NON UNIFORM POLYNOMIAL OBSERVABILITY

We set the time step Δt by $\Delta t = T/(N+1)$, where $N > 0$ is a given integer. Denote by u_k the approximation of the solution u of system (1) at time $t_k = k\Delta t$, for any $k = 0, \ldots, N+1$. We then introduce the following trapezoidal time semi-discretization of system (1)

$$\begin{cases} \frac{u_{k+1}+u_{k-1}-2u_k}{(\Delta t)^2} - \frac{\partial^2}{\partial x^2}\left(\frac{u_{k+1}+u_{k-1}}{2}\right) = 0, & k = 1, \ldots, N, \ 0 < x < 1, \\ u_k(0) = u_k(1) = 0, & k = 0, \ldots, N+1, \\ u_0 = u^0, \ u_1 = u^0 + (\Delta t)u^1, & 0 < x < 1. \end{cases} \tag{4}$$

Here $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ are the initial data given in system (1). As in the continuous case, we will check an observability inequality for system (4) which can be formulated as follows:

we must find positive constant C such that we have

$$C\Delta t \sum_{k=0}^N \left| \frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t} \right|^2 \geq \|(u_0, u_1)\|_{L^2(0,1) \times H^{-1}(0,1)}^2 \tag{5}$$

for all $(u_0, u_1) \in H_0^1(0,1) \times L^2(0,1)$. But there is not the case. Indeed, as in [6], we will choose a particular initial data which don't satisfy (5) uniformly with respect to the discretization parameter. The following theorem provides a quantitative statement of this negative result.

Theorem 1. *For all $T > 0$, there exist a positive constant $C(T, \Delta t)$ and initial data $(u_0, u_1) \in H_0^1(0,1) \times L^2(0,1)$, such that the solution u_k of* (4) *satisfies*

$$C(T, \Delta t)\Delta t \sum_{k=0}^N \left| \frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t} \right|^2 \leq \|(u_0, u_1)\|_{L^2(0,1) \times H^{-1}(0,1)}^2.$$

Proof. We denote by $(\mu_j^2)_{j \geq 1}$ the eigenvalues of the Dirichlet Laplacian and $(\varphi_j)_{j \geq 1}$ the corresponding eigenvectors. Assume that

$$u_0 = \sum_{j=1}^\infty a_j \varphi_j, \quad u_1 = \sum_{j=1}^\infty (a_j + b_j \Delta t) \varphi_j.$$

Then, by proceeding as in Lemma 2.2 of [6], we easily show that the solution of system (4) is given by

$$u_k = \sum_{j=1}^{\infty} r_j^k \varphi_j, \tag{6}$$

where

$$r_j^k = e^{-iw_j k} \frac{(e^{iw_j} - 1)a_j - \Delta t b_j}{2i \sin(w_j)} + e^{iw_j k} \frac{(1 - e^{iw_j})a_j + \Delta t b_j}{2i \sin(w_j)},$$

and

$$w_j = \arccos\left(\frac{1}{1 + \Delta t^2 \mu_j^2/2}\right).$$

If a_j and b_j are chosen so that $(e^{iw_j} - 1)a_j = \Delta t b_j$ for $j = 1, 2, \ldots$, then

$$u_k = \sum_{j=1}^{\infty} a_j e^{iw_j k} \varphi_j.$$

Now, by using continuous fractions (see [5] and references therein for details) we construct a sequence $(q_m) \subset \mathbb{N}$ such that $q_m \to \infty$ and

$$|\sin(q_m \pi \xi)| \leq \frac{\pi}{q_m} \qquad \text{for all} \quad m \geq 1. \tag{7}$$

Since $q_m \to +\infty$ as $m \to +\infty$, one can choose a $m_0 = m_0(\Delta t)$ such that

$$\frac{1}{(\Delta t)^{\frac{3}{2}}} \leq q_{m_0}, \tag{8}$$

which leads to

$$q_{m_0}\Delta t \to +\infty, \quad \text{as} \quad \Delta t \to 0. \tag{9}$$

We choose $u_0 = a_{q_{m_0}} \varphi_{q_{m_0}}$, $u_1 = a_{q_{m_0}} e^{iw_{q_{m_0}}} \varphi_{q_{m_0}}$, then $u_k = a_{q_{m_0}} e^{ikw_{q_{m_0}}} \varphi_{q_{m_0}}$, $k \geq 0$. A simple calculations give $\|u_0\|_{L^2(0,1)}^2 = a_{q_{m_0}}^2/2$ and $\|u_1\|_{H^{-1}(0,1)}^2 = a_{q_{m_0}}^2/2\mu_{q_{m_0}}^2$. On the other hand, one has

$$\left|\frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t}\right|^2 = \frac{2a_{q_{m_0}}^2}{(\Delta t)^2} \varphi_{q_{m_0}}^2(\xi)(1 - \cos(w_{q_{m_0}})),$$

and then, since $(N+1) = T/\Delta t$,

$$\Delta t \sum_{k=0}^{N} \left|\frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t}\right|^2 = \frac{2Ta_{q_{m_0}}^2 \mu_{q_{m_0}}^2 \varphi_{q_{m_0}}^2(\xi)}{2 + (\mu_{q_{m_0}}\Delta t)^2}.$$

Using (7), we get

$$C(T, \Delta t)\Delta t \sum_{k=0}^{N} \left|\frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t}\right|^2 \leq \|(u_0, u_1)\|_{L^2(0,1) \times H^{-1}(0,1)}^2,$$

where $C(T, \Delta t) = (2 + (\mu_{q_{m_0}}\Delta t)^2)/4T\pi^4$.   □

The above inequality and (9) claim that (5) fails uniformly with respect to the discretization parameter. Indeed, it is clear that $C(T, \Delta t) \to +\infty$ as $\Delta t \to 0$, and then

$$\frac{\|(u_0, u_1)\|_{L^2(0,1) \times H^{-1}(0,1)}^2}{\Delta t \sum_{k=0}^{N} \left|\frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t}\right|^2} \to +\infty \quad \text{as} \quad \Delta t \to 0.$$

Consequently, filtering the initial data is needed to obtain (5) uniformly with respect to the discretization parameter.

3 FILTERING MECHANISM

We first transform system (4) into a first order time-discrete scheme as in [2]. For simplicity, we denote $A_0 = -\partial^2/\partial x^2$. We have

$$(I + \frac{\Delta t^2}{2} A_0)(u_{k+1} + u_{k-1}) - 2u_k = 0,$$

then

$$(I + \frac{\Delta t^2}{2} A_0)(u_{k+1} + u_{k-1} - 2u_k) = -\Delta t^2 A_0 u_k,$$

which gives

$$(I + \frac{\Delta t^2}{4} A_0)(u_{k+1} + u_{k-1} - 2u_k) = -\Delta t^2 A_0 (\frac{u_{k+1} + u_{k-1} + 2u_k}{4}).$$

Consequently (4) can be rewritten as

$$\frac{u_{k+1} + u_{k-1} - 2u_k}{(\Delta t)^2} + A_1 (\frac{u_{k+1} + u_{k-1} + 2u_k}{4}), \tag{10}$$

with $A_1 = A_0(I + \frac{\Delta t^2}{4} A_0)^{-1}$. Now using the following change of variables

$$\begin{cases} y_{k+1}^1 = \frac{u_{k+1} - u_k}{\Delta t} + iA_1^{1/2}(\frac{u_{k+1} + u_k}{2}), \\[2mm] y_{k+1}^2 = \frac{u_{k+1} - u_k}{\Delta t} - iA_1^{1/2}(\frac{u_{k+1} + u_k}{2}), \end{cases}$$

we obtain

$$\begin{cases} \frac{y_{k+1} - y_k}{\Delta t} = A(\frac{y_{k+1} + y_k}{2}), \\[3mm] y_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{cases} \tag{11}$$

with

$$A = \begin{pmatrix} iA_1^{1/2} & 0 \\ 0 & -iA_1^{1/2} \end{pmatrix}, \quad y^{k+1} = \begin{pmatrix} y_{k+1}^1 \\ y_{k+1}^2 \end{pmatrix}. \tag{12}$$

Note that the spectrum of A is explicitly given by the spectrum of A_0. More precisely, the eigenvalues of A are $i\lambda_j$ with corresponding eigenvectors

$$\varphi_j = \begin{pmatrix} \varphi_j \\ 0 \end{pmatrix}, \quad \varphi_{-j} = \begin{pmatrix} 0 \\ \varphi_j \end{pmatrix}, \quad j \in \mathbb{N}^*,$$

where $\lambda_j = \mu_j / \sqrt{1 + \Delta t^2 \mu_j^2/4}$. Moreover we define

$$C_s = \text{span}\{\varphi_j \text{ such that } \mu_j \leq s\}.$$

We are ready to prove the following uniform boundary polynomial observability of the time discrete wave equation.

Theorem 2. *For any $\delta > 0$, there exists $T_\delta > 0$ such that for any $T > T_\delta$, there exists a positive constant $C = C_{T,\delta}$, independent of Δt, such that for Δt small enough, the solution u_k of (4) satisfies*

$$C\Delta t \sum_{k=0}^{N} \left| \frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t} \right|^2 \geq \|(u_0, u_1)\|^2_{L^2(0,1) \times H^{-1}(0,1)} \qquad \text{for all} \quad (u_0, u_1) \in \mathcal{C}^2_{\delta/\Delta t}. \quad (13)$$

Proof. We have, for all $k \neq l$, $|\lambda_k - \lambda_l| = |f(\mu_k) - f(\mu_l)|$, where f is defined by $f(t) = t/(\sqrt{1 + (t^2\Delta t^2)/4})$. Applying the mean value theorem to the function f, there exists a point c between μ_k and μ_l such that

$$|\lambda_k - \lambda_l| = |f'(c)||\mu_k - \mu_l|.$$

Simple calculations give that $f'(c) = 1/(1 + \frac{\Delta t^2 c^2}{4})^{3/2}$. It is easy to check that $|f'(c)| \geq 1/(1 + \frac{\delta^2}{4})^{3/2}$, and $|\mu_k - \mu_l| \geq \pi$ for all $k \neq l$. Consequently there exists $\gamma > 0$ such that, for all $k \neq l$ $|\lambda_k - \lambda_l| \geq \gamma$. Besides, we have (see [1]) $|\sin(j\pi\xi)| \geq \frac{\nu}{j}$, for all $j \geq 1$, for some $\nu > 0$, and then $|\sin(j\pi\xi)| \geq \frac{\theta}{\lambda_j}$, for all $j \geq 1$, with $\theta = \nu\pi/\sqrt{1 + \frac{\delta^2}{4}}$. Hence, applying Proposition 2.5 of [3], we obtain the desired result. $\qquad\square$

Remark 1. *In the last proof, we used Proposition 2.5 of [3] in which we assumed that the damping operator is bounded, but this assumption is not needed in the proof of Proposition 2.5, and the result still correct even if the dissipation is unbounded.*

4   OPEN PROBLEMS

1. In this paper we dealt with the polynomial observability of time discrete wave equation. The question of space semi-discrete polynomial observability for wave equation still open. Another interesting open problem is whether the fully discrete schemes have these properties of observability uniformly with respect to the discretization parameters.

2. At the continuous case, it is well-known that polynomial observability implies polynomial stability for associated dissipative system (see [1]). At the discreet level, the only result excitant, in this context, is [3] which deals with bounded dissipation. However the situation is complicated when the dissipation is unbounded, as for example the case of wave equation with punctual dissipation (which correspond to the associated dissipative system of (1)–(2)), and this issue requires further work.

3. Other question arise when discretizing in time and/or in space semilinear dissipative wave equations. It would be interesting to analyze the uniform (with respect to the steps) decay properties of solutions when the conservative system satisfies a polynomial observability inequality. Actually, this question is also open at the continuous level.

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Хаджеж З. *Про необхідність механізму фільтрації для поліноміального дослідження часово дискретних хвильових рівнянь* // Карпатські матем. публ. — 2017. — Т.9, №1. — С. 98–103.

У статті проаналізовано питання поліноміального дослідження. Показано, що якщо неперевні моделі є рівномірно поліноміально досліджувані, то достатньо відфільтрувати початкові дані для виокремлення поліноміально досліджувальних нерівностей у відповідних часово дискретизованих схемах. У зв'язку з цим ми доводимо, що механізм фільтрування частотних модулів є необхідним для існування рівномірного поліноміального дослідження.

А саме, побудовано контрприклад, який показує, що процедура дослідження пізніше не реалізується без початкового фільтрування даних у напівдискретній апроксимації хвильового рівняння.

Ключові слова і фрази: нерівність спостереження, часова дискретизація, техніки фільтрації.