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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DISCRETE VOLTERRA EQUATIONS

Janusz Migda and Małgorzata Migda

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Abstract. We consider the nonlinear discrete Volterra equations of non-convolution type

$$\Delta^m x_n = b_n + \sum_{i=1}^{n} K(n,i) f\left(i, x_i\right), \quad n \geq 1.$$

We present sufficient conditions for the existence of solutions with prescribed asymptotic behavior, especially asymptotically polynomial and asymptotically periodic solutions. We use $\mathrm{o}(n^s)$, for a given nonpositive real s, as a measure of approximation. We also give conditions under which all solutions are asymptotically polynomial.

Keywords: Volterra difference equation, prescribed asymptotic behavior, asymptotically polynomial solution, asymptotically periodic solution, bounded solution.

Mathematics Subject Classification: 39A10, 39A22.

1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of positive integers, all integers and real numbers respectively. Let $m \in \mathbb{N}$. We consider the nonlinear discrete Volterra equations of non-convolution type

$$\Delta^m x_n = b_n + \sum_{i=1}^{n} K(n,i) f\left(i, x_i\right), \quad n \geq 1, \tag{E}$$

$f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, $K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, $K(n,i) = 0$ for $n < i$, and $b : \mathbb{N} \to \mathbb{R}$. We regard $\mathbb{N} \times \mathbb{R}$ as a metric subspace of the Euclidean plane $\mathbb{R}^2$.

By a solution of (E) we mean a sequence $x : \mathbb{N} \to \mathbb{R}$ satisfying (E) for all large n. We say that x is a full solution of (E) if (E) is satisfied for all n. Moreover, if $p \in \mathbb{N}$ and (E) is satisfied for all $n \geq p$, then we say that x is a p-solution. For the sake of convenience, throughout this paper, we use the convention $\sum_j^k q_j = 0$, whenever $j > k$.

Volterra difference equations describe processes whose current state is determined by the whole previous history. They are widely used in the process of modeling some real phenomena or by applying a numerical method to a Volterra integral equation. Asymptotic behavior of solutions of first order Volterra difference equations has been studied by many authors. In particular, the boundedness of solutions was studied by, i.e., Crisci *et al.* [2], Diblík and Schmeidel [6], Gronek and Schmeidel [7], Győri and Horvath [10], Győri and Awwad [8], Kolmanovskii and Shaikhet [12], Migda and Migda [19], Migda and Morchało [20] or Morchało [21]. Asymptotically periodic solutions were studied, for example, by Baker and Song [1], Diblík *et al.* [4, 5] or Győri and Reynolds [11]. To the best of our knowledge, there are a few papers dealing with the asymptotic behavior of solutions of higher order Volterra difference equations, for example, the second order difference equation of Volterra type was studied in Medina [13].

In this paper we study the nonlinear equation (E). We establish sufficient conditions for the existence of solutions with prescribed asymptotic behavior, especially asymptotically polynomial and asymptotically periodic solutions. As in [15], we use $\mathrm{o}(n^s)$ as a measure of approximation. For example, we give sufficient conditions under which, for real constants $c_0, c_1, \ldots, c_{m-1}$ and $s \in (-\infty, 0]$, the difference equation (E) has a solution x of the form

$$x_n = c_{m-1}n^{m-1} + c_{m-2}n^{m-2} + \cdots + c_1 n + c_0 + \mathrm{o}(n^s). \tag{1.1}$$

Next, we provide a procedure for finding a full solution of (E) with the above asymptotic behaviour. Finally, we give sufficient conditions under which all solutions are asymptotically polynomial.

The results devoted to the study of asymptotically polynomial solutions of ordinary difference equations can be found, for example, in [15, 17, 18, 22] or [23]. In 1986, Popenda [22] gave sufficient conditions under which for any polynomial of degree at most $m-1$, there exists a solution x of the form (1.1) with $s = 0$ for a difference equation

$$\Delta^m x_n = f(n, x_n).$$

In 1995, Zafer [23] studied a difference equation of the form

$$\Delta^m x_n = f(n, x_{\sigma(n)}) + b_n$$

and obtained a sufficient condition for the solution of the above equation to satisfy $x_n = \varphi(n) + \mathrm{o}(n^{m-1})$, where φ is a polynomial of degree $m-1$.

Recently, J. Migda obtained in [15] and [17] various results concerning solutions of the form (1.1) with arbitrary $s \in (\infty, 0]$ of the equation

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n.$$

In [9], Győri and Hartung considered asymptotically quasi-polynomial solutions.

2. PRELIMINARIES

For $p, k \in \mathbb{Z}$, let

$$\mathbb{N}(p) = \{p, p+1, \dots\}, \qquad \mathbb{N}(p,k) = \{p, p+1, \dots, k\}.$$

The space of all sequences $x : \mathbb{N} \to \mathbb{R}$ we denote by SQ. For $m \in \mathbb{N}(0)$ we define

$$\mathrm{Pol}(m-1) = \mathrm{Ker}\Delta^m = \{x \in \mathrm{SQ} : \Delta^m x = 0\}.$$

Then $\mathrm{Pol}(m-1)$ is the space of all polynomial sequences of degree less than m. Let

$$\mathrm{S}(0) = \{x \in \mathrm{SQ} : \ \lim x_n = 0\},$$

$$\mathrm{S}(1) = \{x \in \mathrm{SQ} : \ \text{the series } \ \sum x_n \ \text{ is convergent}\}.$$

We define the remainder operator $r : \mathrm{S}(1) \to \mathrm{S}(0)$ by

$$r(x)(n) = \sum_{j=n}^{\infty} x_j.$$

For $m \in \mathbb{N}$ we define, by induction, the linear space $\mathrm{S}(m+1)$ and the linear operator $r^{m+1} : \mathrm{S}(m+1) \to \mathrm{S}(0)$ by

$$\mathrm{S}(m+1) = \{x \in \mathrm{S}(m) : \ r^m(x) \in \mathrm{S}(1)\}, \qquad r^{m+1}(x) = r(r^m(x)).$$

The value $r^m(x)(n)$ we denote also by $r^m_n(x)$ or simply $r^m_n x$.

Some basic properties of the iterations of the remainder operator are presented in the following lemma. This lemma will be useful in the proofs of our main results.

Lemma 2.1. *Assume $x, y \in \mathrm{SQ}$, $m, p \in \mathbb{N}$ and $s \in (-\infty, 0]$. Then*

(a) *if $|x| \in \mathrm{S}(m)$, then $x \in \mathrm{S}(m)$ and $|r^m x| \le r^m|x|$,*
(b) *$|x| \in \mathrm{S}(m)$ if and only if $\sum_{n=1}^{\infty} n^{m-1}|x_n| < \infty$,*
(c) *if $|x| \in \mathrm{S}(m)$, then $r^m_p|x| \le \sum_{n=p}^{\infty} n^{m-1}|x_n|$,*
(d) *if $x \in \mathrm{S}(m)$, then $\Delta^m r^m x = (-1)^m x$,*
(e) *if $x = \mathrm{o}(1)$, then $\Delta^m x \in \mathrm{S}(m)$ and $r^m \Delta^m x = (-1)^m x$,*
(f) *if $x, y \in \mathrm{S}(m)$ and $x_n \le y_n$ for $n \ge p$, then $r^m_n x \le r^m_n y$ for $n \ge p$,*
(g) *if $\sum_{n=1}^{\infty} n^{m-1-s}|x_n| < \infty$, then $x \in \mathrm{S}(m)$ and $r^m x = \mathrm{o}(n^s)$.*

For the proof see [16, Lemma 4.1]. The next lemma is a consequence of [16, Lemma 4.7].

Lemma 2.2. *Assume $y, \rho \in \mathrm{SQ}$ and $\lim \rho_n = 0$. In the set*

$$S = \{x \in \mathrm{SQ} : \ |x - y| \le |\rho|\}$$

we define the metric by the formula $d(x,z) = \sup_{n \in \mathbb{N}} |x_n - z_n|$. Then any continuous map $H : S \to S$ has a fixed point.

3. MAIN RESULTS

In this section, in Theorems 3.1, 3.2 and 3.5, we present results on the existence of solutions of equation (E) with prescribed asymptotic behavior. As in [15, 16], and [17] we use $\mathrm{o}(n^s)$ as a measure of approximation. The idea of the proofs of Theorems 3.1, 3.2 and 3.5 is based on the results obtained in [17]. In the proof of Theorem 3.7, we give a procedure for finding a full solution of (E) which is asymptotically polynomial. In the last theorem, we establish conditions under which all solutions are asymptotically polynomial.

We will use the following conditions:

(A1) $s \in (-\infty, 0]$, $k \in \mathbb{N}(0, m-1)$, $g : [0, \infty) \to [0, \infty)$,

(A2) f is continuous and $|f(n,t)| \le g\left(\dfrac{|t|}{n^k}\right)$ for $(n,t) \in \mathbb{N} \times \mathbb{R}$,

(A3) $\sum_{n=1}^{\infty} n^{m-1-s} \sum_{i=1}^{n} |K(n,i)| < \infty$.

Now, we state and prove our main result. We establish conditions under which a given solution y of the equation $\Delta^m y = b$ is asymptotic to a certain p-solution of (E).

Theorem 3.1. *Let conditions* (A1)*,* (A2) *and* (A3) *be satisfied. Assume* $p \in \mathbb{N}$*,* $Q, L, M > 0$*,*

$$g(t) \le M \quad \text{for} \quad t \le L \quad \text{and} \quad M \sum_{n=p}^{\infty} n^{m-1} \sum_{i=1}^{n} |K(n,i)| \le Q. \tag{3.1}$$

Then for every solution y *of the equation* $\Delta^m y = b$*, such that*

$$|y_n| \le Ln^k - Q$$

for any n*, there exists a* p*-solution* x *of* (E) *such that*

$$x = y + \mathrm{o}(n^s).$$

Proof. Define sequences U, a and a subset T of SQ by

$$U_j = \sum_{n=j}^{\infty} n^{m-1} \sum_{i=1}^{n} |K(n,i)|, \tag{3.2}$$

$$a_n = \sum_{i=1}^{n} |K(n,i)|, \qquad T = \{x \in \mathrm{SQ} : \ |x - y| \le Q\}.$$

If $x \in T$ and $n \in \mathbb{N}$, then

$$\left|\frac{x_n}{n^k}\right| = \left|\frac{x_n}{n^k} - \frac{y_n}{n^k} + \frac{y_n}{n^k}\right| \le \frac{1}{n^k}|x_n - y_n| + \left|\frac{y_n}{n^k}\right| \le \frac{Q}{n^k} + \frac{|y_n|}{n^k} \le L.$$

Hence

$$|f(n,x_n)| \le g\left(\frac{|x_n|}{n^k}\right) \le M. \tag{3.3}$$

By (A3), we have $|a| \in \mathrm{S}(m)$. Let $\rho = r^m a$ and

$$S = \{x \in \mathrm{SQ} : \ |x-y| \le M\rho \ \text{ and } x_n = y_n \text{ for } n < p\}.$$

We define a metric d on S by

$$d(x,z) = \|x-z\| = \sup_{n\in\mathbb{N}} |x_n - z_n|.$$

By Lemma 2.2, any continuous map $A : S \to S$ has a fixed point. If $x \in S$ and $n \ge p$, then, using Lemma 2.1 (c), we have

$$|x_n - y_n| \le M\rho_n = Mr_n^m a \le MU_n \le MU_p \le Q.$$

Hence $S \subset T$. For $x \in \mathrm{SQ}$ and $n \in \mathbb{N}$, let

$$\bar{x}_n = f(n,x_n), \qquad x_n^* = \sum_{i=1}^{n} K(n,i)\bar{x}_i.$$

If $x \in S$ and $n \in \mathbb{N}$, then, by (3.3),

$$|x_n^*| \le \sum_{i=1}^{n} |K(n,i)||f(i,x_i)| \le Ma_n. \tag{3.4}$$

Hence, $|x^*| \le Ma$. Therefore, $x^* \in \mathrm{S}(m)$. For $x \in S$ we define a sequence $A(x)$ by

$$A(x)(n) = \begin{cases} y_n & \text{for} \quad n < p, \\ y_n + (-1)^m r_n^m x^* & \text{for} \quad n \ge p. \end{cases}$$

If $x \in S$ and $n \in \mathbb{N}$, then, using Lemma 2.1 (a) and (3.4), we obtain

$$|A(x)(n) - y_n| = |r_n^m x^*| \le r_n^m |x^*| \le Mr_n^m a = M\rho_n.$$

Thus, $A(S) \subset S$. Let $\varepsilon > 0$. Choose $q \in \mathbb{N}$ and $\beta > 0$ such that

$$2M \sum_{n=q}^{\infty} n^{m-1} \sum_{i=1}^{n} |K(n,i)| < \varepsilon \qquad \text{and} \qquad \beta \sum_{n=1}^{q} n^{m-1} \sum_{j=1}^{n} |K(n,i)| < \varepsilon. \tag{3.5}$$

Let

$$Z = \{(n,t) \in \mathbb{N} \times \mathbb{R} : \ n \in \mathbb{N}(0,q) \quad \text{and} \quad |t - y_n| \le M\rho_n\}.$$

Z is a compact subset of the Euclidean plane $\mathbb{R}^2$. Hence, f is uniformly continuous on Z and there exists $\delta > 0$ such that if $(n,s),(n,t) \in Z$ and $|s-t| < \delta$, then

$$|f(n,s) - f(n,t)| < \beta. \tag{3.6}$$

Let $x, y \in S$, $\|x - y\| < \delta$. Then, using Lemma 2.1 (a) and (c), we have

$$\begin{aligned}\|Ax - Az\| &= \|r^m(x^* - z^*)\| = \sup_n |r_n^m(x^* - z^*)| \leq \sup_n r_n^m |x^* - z^*| \\ &= r_1^m |x^* - z^*| \leq \sum_{n=1}^{\infty} n^{m-1} |x_n^* - z_n^*| \\ &= \sum_{n=1}^{\infty} n^{m-1} \left| \sum_{i=1}^{n} K(n,i)(f(i,x_i) - f(i,z_i)) \right| \\ &\leq \sum_{n=1}^{q} n^{m-1} \sum_{i=1}^{n} |K(n,i)(f(i,x_i) - f(i,z_i))| \\ &\quad + \sum_{n=q}^{\infty} n^{m-1} \sum_{i=1}^{n} |K(n,i)(f(i,x_i) - f(i,z_i))|.\end{aligned}$$

Hence, using (3.3), (3.6) and (3.5), we obtain

$$d(Ax, Az) \leq \beta \sum_{n=1}^{q} n^{m-1} \sum_{i=1}^{n} |K(n,i)| + 2M \sum_{n=q}^{\infty} n^{m-1} \sum_{i=1}^{n} |K(n,i)| < 2\varepsilon.$$

Hence, the map $A : S \to S$ is continuous and there exists a sequence $x \in S$ such that $x = A(x)$. Then for $n \geq p$ we have

$$x_n = y_n + (-1)^m r_n^m x^*.$$

Using Lemma 2.1 (d), we obtain

$$\Delta^m x_n = \Delta^m y_n + (-1)^m \Delta^m r_n^m x^* = b_n + x_n^* = b_n + \sum_{i=1}^{n} K(n,i) f(i, x_i)$$

for $n \geq p$. Hence, x is a p-solution of (E). Moreover, using (A3) and Lemma 2.1 (g), we have $r_n^m x^* = \mathrm{o}(n^s)$. Hence, $x = y + \mathrm{o}(n^s)$. The proof is complete. $\square$

Theorem 3.1 generalizes the results obtained by J. Popenda [22] and M. Migda, J. Migda [18].

Let X be a metric space. A function $\varphi : X \to \mathbb{R}$ is called locally bounded if for any $x \in X$ there exists a neighborhood U of x such that $\varphi|U$ is bounded. Note that if X is a closed subset of $\mathbb{R}$, then a function $\varphi : X \to \mathbb{R}$ is locally bounded if and only if it is bounded on every bounded subset of X.

The next theorem is a consequence of Theorem 3.1. We show that if starting point p of the solution is not fixed, then the strong condition (3.1) may by replaced by the condition: g is locally bounded. On the other hand, if we assume that g is bounded, then we can take $p = 1$.

Theorem 3.2. *Assume* (A1), (A2) *and* (A3) *are satisfied. If g is locally bounded, then for every solution y of the equation $\Delta^m y = b$ such that $y = \mathrm{O}(n^k)$ there exists a solution x of* (E) *such that*

$$x = y + \mathrm{o}(n^s).$$

Moreover, if g is bounded, then we may assume that x is full.

Proof. Choose $Q > 0$ and $L > 0$ such that

$$|y_n| \leq Ln^k - Q \tag{3.7}$$

for any $n \in \mathbb{N}$. Choose $M > 0$ such that $g(t) \leq M$ for $t \leq L$. Let U be a sequence defined by (3.2). Then $U_n = \mathrm{o}(1)$ and there exists an index p such that $MU_p \leq Q$. Hence, by Theorem 3.1, there exists a p-solution x of (E) such that $x = y + \mathrm{o}(n^s)$. Now, assume g is bounded. Choose $M > 0$ such that $g(t) \leq M$ for any t. Let U be defined by (3.2). Choose $Q \geq U_1 M$ and $L > 0$ such that inequality (3.7) is satisfied for any n. By Theorem 3.1, there exists a 1-solution x of (E) such that $x = y + \mathrm{o}(n^s)$. The proof is complete. □

Remark 3.3. Assume the conditions of Theorem 3.2 are satisfied, $q \in \mathbb{N}$ and a sequence $w : \mathbb{N} \to \mathbb{R}$ is q-periodic. Let $b = \Delta^m w$. Then, by Theorem 3.2 for any constant c there exists a solution x of (E) such that

$$x = c + w + \mathrm{o}(n^s).$$

Note that all such x are asymptotically q-periodic.

Example 3.4. Let us consider the third order nonlinear Volterra equation

$$\Delta^3 x_n = 8(-1)^{n+1} + \sum_{i=1}^{n} \frac{1}{(-2)^i n^5}\, x_i^2. \tag{3.8}$$

Here $f(n,t) = t^2$, $g(t) = t^2$ and $b_n = 8(-1)^{n+1}$. Let $k = 0$, $s \in (-2, 0]$ and $w_n = (-1)^n$. Then $\Delta^3 w_n = 8(-1)^{n+1} = b_n$. Moreover,

$$\sum_{n=1}^{\infty} n^{m-1-s} \sum_{i=1}^{n} |K(n,i)| = \sum_{n=1}^{\infty} \frac{1}{n^{3+s}} \sum_{i=1}^{n} \frac{1}{2^i} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3+s}} < \infty.$$

Hence, all assumptions of Remark 3.3 hold. So, for any constant c there exists a solution x of (3.8) such that $x_n = c + (-1)^n + \mathrm{o}(n^s)$. This solution is asymptotically 2-periodic.

In the next theorem, which is a consequence of Theorem 3.2, we establish conditions under which any polynomial $\varphi \in \mathrm{Pol}(k)$ is an approximative solution of (E).

Theorem 3.5. *Assume* (A1), (A2), (A3) *are satisfied, g is locally bounded and*

$$\sum_{n=1}^{\infty} n^{m-1-s} |b_n| < \infty, \tag{3.9}$$

Then for every polynomial $\varphi \in \mathrm{Pol}(k)$ there exists a solution x of (E) *such that*

$$x = \varphi + \mathrm{o}(n^s).$$

Moreover, if g is bounded, then we may assume that x is full.

Proof. By Lemma 2.1 (g), we have $b \in \mathrm{S}(m)$ and $r^m b = \mathrm{o}(n^s)$. Let

$$z = (-1)^m r^m b, \qquad y = \varphi + z.$$

Then $y = \mathrm{O}(n^k)$, $z = \mathrm{o}(n^s)$ and

$$\Delta^m y = \Delta^m \varphi + \Delta^m (-1)^m r^m b = b.$$

Hence, by Theorem 3.2, there exists a solution x of (E) such that $x = y + \mathrm{o}(n^s)$. Then

$$x = \varphi + z + \mathrm{o}(n^s) = \varphi + \mathrm{o}(n^s).$$

If g is bounded, then, by Theorem 3.2, we may assume that x is full. □

Example 3.6. Let $m = 2$, $k = 1$, $s \in (-1, 0]$,

$$b_n = \frac{2}{(n+2)^{\underline{3}}} + \frac{1 - (n+3)^{\underline{3}}}{n^6}, \qquad K(n,i) = \frac{i}{n^5}, \qquad f(n,t) = \frac{t}{n}, \qquad g(t) = t.$$

Then equation (E) takes the form

$$\Delta^2 x_n = \frac{2}{(n+2)^{\underline{3}}} + \frac{1 - (n+3)^{\underline{3}}}{n^6} + \sum_{i=1}^{n} \frac{i}{n^6} x_i, \quad n \geq 1. \tag{3.10}$$

It is easy to check that

$$\sum_{n=1}^{\infty} n\,|b_n| = \sum_{n=1}^{\infty} n \left| \frac{2}{(n+2)^{\underline{3}}} + \frac{1-(n+3)^{\underline{3}}}{n^6} \right| < \infty$$

and

$$\sum_{n=1}^{\infty} n^{m-1-s} \sum_{i=1}^{n} |K(n,i)| = \sum_{n=1}^{\infty} n^{1-s} \sum_{i=1}^{n} \frac{i}{n^5} \leq \sum_{n=1}^{\infty} n^{1-s-5} n^2 = \sum_{n=1}^{\infty} \frac{1}{n^{2+s}} < \infty.$$

Thus, by Theorem 3.5, for any real c, d, equation (3.10) has a solution x such that

$$x_n = cn + d + \mathrm{o}(n^s).$$

One such solution is $x_n = n + \frac{1}{n}$.

Let $k \in \mathbb{N}(0, m-1)$. Let us consider the $n \times n$ matrix $A(n)$ of the form

$$A(n) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \tag{3.11}$$

where

$$\begin{aligned}
a_{ii} &= i^{-k}K(i,i) + (-1)^{m-1} \quad \text{for} \quad i \in \mathbb{N}(1,n), \\
a_{ij} &= j^{-k}K(i,j) \quad \text{for} \quad j < i, \quad i \in \mathbb{N}(2,n), \\
a_{ij} &= (-1)^{m+i-j-1}\binom{m}{m+i-j} \quad \text{for} \quad j \in \mathbb{N}(i+1, m+i), \\
a_{ij} &= 0 \quad \text{for} \quad j \in \mathbb{N}(m+i+1, n).
\end{aligned}$$

Now, we establish conditions which allow us to change the first p terms in the p-solution so that a full solution is obtained.

Theorem 3.7. *Let conditions* (A1), (A3), (3.9) *be satisfied,* $g(t) = t$ *and*

$$f(n,t) = \frac{t}{n^k}. \tag{3.12}$$

Assume the matrix $A(n)$ *defined by* (3.11) *is such that*

$$\det A(n) \neq 0 \quad \textit{for all} \quad n \in \mathbb{N}. \tag{3.13}$$

Then, for every polynomial $\varphi \in \mathrm{Pol}(k)$ *there exists a full solution* x *of* (E) *such that* $x = \varphi + \mathrm{o}(n^s)$.

Proof. Note that, by (3.12), the condition (A2) is also satisfied. Let us choose a polynomial $\varphi \in \mathrm{Pol}(k)$. From Theorem 3.5 it follows that there exists a number $p \in \mathbb{N}$ and a p-solution x' of (E) such that $x' = \varphi + \mathrm{o}(n^s)$. Using this x', we will construct a full solution x such that $x = \varphi + \mathrm{o}(n^s)$. Let

$$x_n = x'_n \quad \text{for} \quad n > p. \tag{3.14}$$

We can find the p first terms of the sequence x. These terms are the solutions of the following system of p linear equations

$$\begin{cases}
\Delta^m x_1 &= b_1 + K(1,1)x_1, \\
\Delta^m x_2 &= b_2 + \sum\limits_{i=1}^{2} i^{-k}K(2,i)x_i, \\
&\vdots \\
\Delta^m x_p &= b_p + \sum\limits_{i=1}^{p} i^{-k}K(p,i)x_i,
\end{cases} \tag{3.15}$$

where $x_1, x_2, \ldots, x_p$ are unknowns. Using the formula

$$\Delta^m x_n = \sum_{i=0}^{m} (-1)^{m-i}\binom{m}{m-i} x_{n+i},$$

we can write system (3.15) in the form

$$A(p)X(p) = C(p), \tag{3.16}$$

where $A(p)$ is defined by (3.11),

$$X(p) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}, \qquad C(p) = \begin{pmatrix} \sum\limits_{i=p}^{m} (-1)^{m-i} \binom{m}{m-i} x_{i+1} - b_1 \\ \sum\limits_{i=p-1}^{m} (-1)^{m-i} \binom{m}{m-i} x_{i+2} - b_2 \\ \vdots \\ \sum\limits_{i=2}^{m} (-1)^{m-i} \binom{m}{m-i} x_{i+p-1} - b_{p-1} \\ \sum\limits_{i=1}^{m} (-1)^{m-i} \binom{m}{m-i} x_{i+p} - b_p \end{pmatrix}.$$

By (3.13), we have $\det A(p) \neq 0$. Thus, by Cramer's rule, we obtain a solution

$$x_1, x_2, \ldots, x_p$$

of system (3.15). The proof is complete. □

Example 3.8. Let us consider the second order linear Volterra equation

$$\Delta^2 x_n = b_n + \sum_{i=0}^{n} K(n,i)\, x_i, \quad n \geq 1. \tag{3.17}$$

Let $g : [0,\infty) \to [0,\infty)$, $g(t) = t$, $f(n,t) = t$, $k = 0$, $s \in (-1, 0]$. Assume

$$\sum_{n=1}^{\infty} n\,|b_n| < \infty, \qquad K(n,n) = \frac{1}{n^2(n+1)},$$

$K(n,i) = 0$ for $i \neq n$ and $A(n)$ is defined by (3.11). Then

$$\sum_{n=1}^{\infty} n^{m-1-s} \sum_{i=1}^{n} |K(n,i)| = \sum_{n=1}^{\infty} n^{1-s} |K(n,n)| = \sum_{n=1}^{\infty} \frac{n^{1-s}}{n^3+n^2} < \sum_{n=1}^{\infty} \frac{1}{n^{2+s}} < \infty,$$

$$A(n) = \begin{pmatrix} K(1,1)-1 & 2 & -1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & K(2,2)-1 & 2 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & K(3,3)-1 & 2 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & K(n,n)-1 \end{pmatrix}$$

and

$$\det A(n) = \prod_{i=1}^{n} \left(\frac{1}{i^2(i+1)} - 1 \right) \neq 0$$

for all n. Hence, by Theorem 3.7, for any real constant c there exists a full solution x of (3.17) such that $x_n = c + \mathrm{o}(n^s)$.

Finally, we establish conditions under which all solutions of equation (E) are asymptotically polynomial. We need two lemmas. Lemma 3.9 is a consequence of Theorem 1 in [3] and Lemma 3.10 is a consequence of Theorem 2.1 in [15].

Lemma 3.9. *Assume that $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\varphi(t) > 0$ for $t \geq 1$ and $\int_1^\infty \frac{1}{\varphi(s)} ds = \infty$. Let $c \geq 0$ be a nonnegative constant and let h, u be sequences of nonnegative real numbers such that*

$$\sum_{n=1}^{\infty} h_n < \infty \qquad \text{and} \qquad u_n \leq c + \sum_{i=1}^{n-1} h_i \varphi(u_i)$$

for all $n \geq 1$. Then the sequence u is bounded.

Lemma 3.10. *Assume $s \in (-\infty, m-1]$, $x \in \mathrm{SQ}$ and*

$$\sum_{n=1}^{\infty} n^{m-1-s} |\Delta^m x_n| < \infty.$$

Then there exists a polynomial $\varphi \in \mathrm{Pol}(m-1)$ such that $x = \varphi + \mathrm{o}(n^s)$.

The idea of the proof of our last theorem is based on the proof of Theorem 4.1 in [15]. Note that we extend the range of s from $(-\infty, 0]$ to $(-\infty, m-1]$.

Theorem 3.11. *Assume $s \in (-\infty, m-1]$, $g : [0,\infty) \to [0,\infty)$ is nondecreasing,*

$$\sum_{n=1}^{\infty} n^{m-1-s} \sum_{i=1}^{n} |K(n,i)| < \infty, \qquad \sum_{n=1}^{\infty} n^{m-1-s} |b_n| < \infty, \tag{3.18}$$

$$|f(n,t)| \leq g\left(\frac{|t|}{n^{m-1}}\right) \quad \text{for} \quad (n,t) \in \mathbb{N} \times \mathbb{R}, \qquad \int_1^\infty \frac{dt}{g(t)} = \infty$$

and x is a solution of (E)*. Then there exists a polynomial $\varphi \in \mathrm{Pol}(m-1)$ such that*

$$x = \varphi + \mathrm{o}(n^s).$$

Proof. As in the proof ot Theorem 4.1 in [15] one can see that there exists a constant A such that

$$\frac{|x_n|}{n^{m-1}} \leq A + \sum_{j=1}^{n-1} |\Delta^m x_j|. \tag{3.19}$$

Moreover, by (E), we have

$$\Delta^m x_j = b_j + \sum_{i=1}^{j} K(j,i) f(i, x_i) \tag{3.20}$$

for any j. Let

$$a_n = \sum_{i=1}^{n} |K(n,i)|, \qquad z_n = a_n + |b_n|, \qquad c = A + \sum_{j=1}^{\infty} |b_j|.$$

Note that, by (3.18), we have

$$\sum_{n=1}^{\infty} n^{m-1-s}|z_n| < \infty. \tag{3.21}$$

Let

$$u_i = \frac{|x_i|}{i^{m-1}}, \qquad h_i = \sum_{j=i}^{\infty} |K(j,i)|.$$

Using the condition: $K(j,i) = 0$ for $i > j$ and (3.18), we obtain

$$\begin{aligned}\sum_{i=1}^{\infty} h_i &= \sum_{i=1}^{\infty}\sum_{j=i}^{\infty} |K(j,i)| \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |K(j,i)| \\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |K(j,i)| = \sum_{j=1}^{\infty}\sum_{i=1}^{j} |K(j,i)| < \infty.\end{aligned}$$

Using (3.19) and (3.20) we obtain

$$\begin{aligned}u_n &\le A + \sum_{j=1}^{n-1} \left| b_j + \sum_{i=1}^{j} K(j,i) f(i,x_i) \right| \le c + \sum_{j=1}^{n-1}\sum_{i=1}^{j} |K(j,i)| g(u_i) \\ &= c + \sum_{i=1}^{n-1} \left(\sum_{j=i}^{n-1} |K(j,i)| \right) g(u_i) \le c + \sum_{i=1}^{n-1} h_i g(u_i).\end{aligned}$$

Hence, by Lemma 3.9, the sequence u is bounded. Therefore, there exists a constant $M > 1$ such that $g(u_i) \le M$ for any i and we obtain $|f(i,x_i)| \le M$ for any i. Hence

$$|\Delta^m x_n| = \left| b_n + \sum_{i=1}^{n} K(n,i) f(i,x_i) \right| \le |b_n| + M \sum_{i=1}^{n} |K(n,i)| = |b_n| + M a_n \le M z_n.$$

Therefore, by (3.21) and Lemma 3.10, there exists a polynomial $\varphi \in \operatorname{Pol}(m-1)$ such that

$$x = \varphi + \mathrm{o}(n^s). \qquad \square$$

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Janusz Migda
migda@amu.edu.pl

Adam Mickiewicz University
Faculty of Mathematics and Computer Science
Umultowska 87, 61-614 Poznań, Poland

Małgorzata Migda
malgorzata.migda@put.poznan.pl

Poznan University of Technology
Institute of Mathematics
Piotrowo 3A, 60-965 Poznań, Poland

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