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SOME STABILITY CONDITIONS FOR SCALAR VOLTERRA DIFFERENCE EQUATIONS

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Abstract. New explicit stability results are obtained for the following scalar linear difference equation

$$x(n+1)-x(n)=-a(n)x(n)+\sum_{k=1}^{n}A(n,k)x(k)+f(n)$$

and for some nonlinear Volterra difference equations.

Keywords: linear and nonlinear Volterra difference equations, boundedness of solutions, exponential and asymptotic stability.

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1. INTRODUCTION

We consider a Volterra difference equation of the following form

$$x(n+1)-x(n)=-a(n)x(n)+\sum_{k=1}^{n}A(n,k)x(k)+f(n),\quad n\geq 1, \tag{1.1}$$

where $a,f\colon\mathbb{N}\to\mathbb{R}$ and $A\colon\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ with $A(n,i)=0$ for all $n<i$ are given functions, and $x\colon\mathbb{N}\to\mathbb{R}$ is an unknown sequence. This equation can be easily transformed into the more familiar form

$$x(n+1)=b(n)x(n)+\sum_{k=1}^{n}A(n,k)x(k)+f(n) \tag{1.2}$$

by the substitution $a(n)=1-b(n)$.

Notice that Volterra difference equations appeared as a discretization of Volterra integral and integro-differential equations. In particular, equation (1.1) is a discretization of the integro-differential equation

$$\dot{x}(t) = -a(t)x(t) + \int_{t_0}^{t} A(t,s)x(s)ds + f(t),$$

and a discretization of the linear integral equation

$$x(t) = b(t)x(t) + \int_{t_0}^{t} A(t,s)x(s)ds + f(t)$$

gives a Volterra difference equation (1.2).

Discrete Volterra equations also often occur during the mathematical modelling of some real life situations. Therefore, the qualitative theory of these types of equations is developed by many authors. For example, the boundedness of solutions of discrete Volterra equations was studied in [2, 5, 10] or [13]–[18], the periodicity was investigated in papers [6, 8, 15, 18]. A survey of the fundamental results on the stability of linear Volterra difference equations, of both convolution and non–convolution type, can be found in [7], see also [3, 4, 11, 12, 17] or [19]. In [3] and [4] the authors study the exponential stability of equation

$$x(n+1) = \sum_{k=1}^{n} A(n,k)x(k). \tag{1.3}$$

In particular, they obtained the following sufficient condition for the exponential stability of (1.3)

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |A(n,k)|\gamma^{k-n} \leq \gamma$$

for some $0 < \gamma < 1$. The aim of this paper is to present new explicit boundedness and stability results for equations (1.1) and (1.2), and also for some nonlinear Volterra difference equations.

For the sake of convenience, throughout this paper, we use the convention $\sum_j^k q(j) := 0$ and $\prod_j^k q(j) := 1$, whenever $j > k$.

2. PRELIMINARIES

Together with equation (1.1) we will also consider the following simple equation

$$x(n+1) - x(n) = -a(n)x(n) + f(n), \quad n \geq 1, \tag{2.1}$$

where $a, f \colon \mathbb{N} \to \mathbb{R}$ and $a(n) \not\equiv 1$.

By the variation of constants formula, the solution of equation (2.1) with the initial condition

$$x(1) = x_0 \tag{2.2}$$

can be presented as

$$x(n) = X(n,1)x_0 + \sum_{k=1}^{n-1} X(n,k+1)f(k), \tag{2.3}$$

where

$$X(n,k) := \prod_{j=k}^{n-1}(1 - a(j)). \tag{2.4}$$

Note that $X(k,k) = 1$. Obviously, if

$$0 \le a(n) \le 1 \text{ for any } n \ge k, \tag{2.5}$$

then $X(n,k)$ is nonnegative and bounded from above by 1.

Lemma 2.1. *If condition* (2.5) *is satisfied, then*

$$0 \le \sum_{k=1}^{n-1} X(n,k+1)a(k) \le 1, \tag{2.6}$$

where $X(n,k)$ *is defined by* (2.4)*.*

Proof. Consider equation (2.1), where $f(n) = a(n)$ and $x_0 = 1$. Then $x(n) \equiv 1$ is a solution of problem (2.1), (2.2). By the above and (2.3), we have

$$X(n,1) + \sum_{k=1}^{n-1} X(n,k+1)a(k) = 1.$$

Since assumption (2.5), by (2.4), we have that $X(n,1)$ is nonnegative for $n \in \mathbb{N}$. Hence inequality (2.6) holds. □

Note, that the solution of equation (1.1) satisfies the following equation

$$x(n) = X(n,1)x_0 + \sum_{k=1}^{n-1} X(n,k+1)\sum_{i=1}^{k} A(k,i)x(i) + \sum_{k=1}^{n-1} X(n,k+1)f(k), \tag{2.7}$$

where $x(1) = x_0$ and $X(n,k)$ is defined by (2.4).

Definition 2.2. Equation (1.1) is said to be exponentially stable if there exists a positive constant M and $\lambda \in (0,1)$ such that for any solution of the corresponding homogeneous equation with the initial condition (2.2) the following inequality holds:

$$|x(n)| \le M\,|x_0|\,\lambda^n.$$

The following lemma will be used in the sequel.

Lemma 2.3 ([1])**.** *Assume that there exists a positive constant* L *and* $\mu \in (0,1)$ *such that*

$$|A(n,m)| \le L\mu^{(n-m)} \text{ for } n \ge m \ge 1. \tag{2.8}$$

If for any bounded function f *the solution of problem* (1.1), (2.2) *is bounded, then equation* (1.1) *is exponentially stable.*

3. MAIN RESULTS

In this section we consider only equation (1.1). All results for equation (1.2) one can obtain by a substitution $b(n) = 1 - a(n)$.

Theorem 3.1. *Assume that there exist constants α_0, $\alpha_1 \in (0,1)$ such that*

$$a(n) \in [\alpha_0, 1],\ a(n) \not\equiv 1 \tag{3.1}$$

and

$$\sum_{k=1}^{n} |A(n,k)| \le \alpha_1 a(n) \quad \text{for} \quad n \in \mathbb{N}. \tag{3.2}$$

(i) *If the function f is bounded, then all solutions of (1.1) are bounded.*
(ii) *If condition (2.8) holds, then equation (1.1) is exponentially stable.*

Proof. *Ad* (i). Let $X(n,k)$ be defined by (2.4). Set $\|f\| = \sup_{n\in\mathbb{N}} |f(n)|$. By (3.1), we have $0 \le X(n,k) \le 1$. Using (2.7), for solution of (1.1), we have

$$|x(n)| \le X(n,1)|x_0| + \sum_{k=1}^{n-1} X(n,k+1) \sum_{i=1}^{k} |A(k,i)|\,|x(i)| + \sum_{k=1}^{n-1} X(n,k+1)|f(k)|$$

$$\le X(n,1)|x_0| + \sum_{k=1}^{n-1} X(n,k+1)a(k)\frac{\sum_{i=1}^{k} |A(k,i)|}{a(k)} \max_{1\le i\le n} |x(i)|$$

$$+ \sum_{k=1}^{n-1} X(n,k+1)a(k)\frac{|f(k)|}{a(k)}.$$

Hence, by Lemma 2.1, (3.1) and (3.2), we obtain

$$|x(n)| \le |x_0| + \alpha_1 \max_{1\le i\le n} |x(i)| + \frac{\|f\|}{\alpha_0}.$$

Thus

$$\max_{1\le i\le n} |x(i)| \le |x_0| + \alpha_1 \max_{1\le i\le n} |x(i)| + \frac{\|f\|}{\alpha_0}.$$

Then

$$(1-\alpha_1) \max_{1\le i\le n} |x(i)| \le |x_0| + \frac{\|f\|}{\alpha_0}.$$

Therefore,

$$|x(n)| \le \frac{|x_0| + \frac{\|f\|}{\alpha_0}}{1-\alpha_1}.$$

It means that all solutions of (1.1) are bounded.

Ad (ii). In the part (i) of this theorem it was proved that for any bounded function f all solutions of (1.1) are bounded. Hence, by Lemma 2.3, equation (1.1) is exponentially stable. □

Example 3.2. Let us consider the linear Volterra difference equation

$$x(n+1)-x(n)=-\frac{2n+1}{3n}x(n)+\sum_{k=1}^{n}\frac{k}{n(n+1)}x(k)+(-1)^n,\quad n\in\mathbb{N}. \tag{3.3}$$

Here

$$a(n)=\frac{2n+1}{3n},\quad f(n)=(-1)^n\quad\text{and}\quad A(n,k)=\frac{k}{n(n+1)}.$$

Let us take

$$\alpha_0=\frac{2}{3},\quad \alpha_1=\frac{3}{4}.$$

Then

$$a(n)=\frac{2n+1}{3n}\in[\alpha_0,1],\quad \sum_{k=1}^{n}A(n,k)=\sum_{k=1}^{n}\frac{k}{n(n+1)}=\frac{1}{2}\le\frac{3}{4}\frac{2n+1}{3n}=\alpha_0 a(n).$$

So, by Theorem 3.1, all solutions of (3.3) are bounded.

Note, that for equation (3.3), Theorem 1 of [17] cannot be applied, since the assumption

$$\sum_{k=n_0}^{n-1}\prod_{j=k+1}^{n-1}|b(j)|\sum_{i=n_0}^{k}|A(k,i)|\le\alpha$$

is not satisfied.

Remark 3.3. If equation (1.1) is exponentially stable and $\lim_{n\to\infty}|f(n)|=0$, then for any solution of (1.1), (2.2) we also have $\lim_{n\to\infty}|x(n)|=0$.

Another asymptotic stability condition we will obtain by applying the first part of Theorem 3.1.

Corollary 3.4. *Assume that there exist positive constants λ and α_0, and $\beta_0\in(0,1)$, $\alpha_1\in(0,1)$ such that*

$$\alpha_0\le a(n)\le\beta_0,\quad \sum_{k=1}^{n}\left(\frac{n+1}{k}\right)^{\lambda}|A(n,k)|\le\alpha_1 a(n)\ \textit{for}\ n\in\mathbb{N}. \tag{3.4}$$

If the function $n^\lambda f(n)$ is bounded, then for any solution of (1.1)

$$\lim_{n\to\infty}x(n)=0.$$

Moreover,

$$|x(n)|\le\frac{M}{n^\lambda}$$

for some $M>0$.

Proof. After a substitution $x(n) = \frac{y(n)}{n^\lambda}$ equation (1.1) takes the form

$$y(n+1) - y(n) = -\left[\left(\frac{n+1}{n}\right)^\lambda (a(n)-1)+1\right] y(n) + \sum_{k=1}^{n} \left(\frac{n+1}{n}\right)^\lambda \left(\frac{n}{k}\right)^\lambda A(n,k)y(k) + (n+1)^\lambda f(n). \tag{3.5}$$

Let us introduce the following notation:

$$a_1(n) = \left(\frac{n+1}{n}\right)^\lambda (a(n)-1)+1,$$

$$A_1(n,k) = \left(\frac{n+1}{k}\right)^\lambda A(n,k),$$

and

$$f_1(n) = (n+1)^\lambda f(n).$$

Then equation (3.5) takes the form

$$y(n+1) - y(n) = -a_1(n)y(n) + \sum_{k=1}^{n} A_1(n,k)y(k) + f_1(n). \tag{3.6}$$

Theorem 3.1 is used for proving that all solutions of equation (3.6) are bounded. We checked it out that, for any $n \in \mathbb{N}$, assumptions of Theorem 3.1 are satisfied. By (3.4), since $\beta_0 \in (0,1)$, we get

$$a_1(n) \geq a(n) - 1 + 1 > \alpha_0 \quad \text{and} \quad a_1(n) \leq \left(\frac{n+1}{n}\right)^\lambda (\beta_0 - 1) + 1 < 1.$$

Again by (3.4), we have

$$\sum_{k=1}^{n} |A_1(n,k)| = \sum_{k=1}^{n} \left(\frac{n+1}{k}\right)^\lambda |A(n,k)| \leq \alpha_1 a_1(n).$$

The identity

$$\frac{f_1(n)}{(n+1)^\lambda f(n)} \equiv 1$$

implies that the function f_1 is bounded.

All assumptions of Theorem 3.1 are satisfied for equation (3.6). Hence solutions of this equation are bounded and the theorem is proved. □

Finally, let us consider the nonlinear Volterra equation

$$x(n+1) - x(n) = -F(n, x(n)) + \sum_{k=1}^{n} H(n, k, x(k)) + G(n, x(n)), \tag{3.7}$$

where $F\colon \mathbb{N}\times\mathbb{R} \to \mathbb{R}_+$, $G\colon \mathbb{N}\times\mathbb{R} \to \mathbb{R}$, $H\colon \mathbb{N}\times\mathbb{N}\times\mathbb{R} \to \mathbb{R}$ are given functions. From Theorem 3.1 we get for the above equation the following result.

Corollary 3.5. *If there exist constants α_0, $\alpha_1 \in (0,1)$, functions $A\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}_+$, $c\colon \mathbb{N} \to \mathbb{R}_+$, and a bounded sequence $g\colon \mathbb{N} \to \mathbb{R}_+$ such that for any $u \neq 0$*

$$\alpha_0 \le c(n) \le \frac{F(n,u)}{u} \le 1, \quad \left|\frac{H(n,k,u)}{u}\right| \le A(n,k) \tag{3.8}$$

and

$$\sum_{k=1}^{n} A(n,k) \le \alpha_1 c(n), \quad |G(n,u)| \le g(n) \quad \textit{for all} \quad n,k \in \mathbb{N}, \tag{3.9}$$

then all solutions of equation (3.7) *are bounded. If, in addition, there exist positive constants L and $\mu < 1$ such that*

$$A(n,k) \le L\mu^{(n-k)} \quad \textit{for} \quad n \ge k \ge 1$$

and $\lim_{n\to\infty} g(n) = 0$*, then any solution of equation* (3.7) *tends to zero.*

Proof. Suppose $\bar{x}$ is a fixed solution of equation (3.7). Let us denote

$$a(n) = \begin{cases} \frac{F(n,\bar{x}(n))}{\bar{x}(n)}, & \bar{x}(n) \neq 0, \\ 1, & \bar{x}(n) = 0, \end{cases}$$

$$A(n,k) = \begin{cases} \left|\frac{H(n,k,\bar{x}(n)}{\bar{x}(n)}\right|, & \bar{x}(n) \neq 0, \\ 0, & \bar{x}(n) = 0, \end{cases} \tag{3.10}$$

and

$$f(n) = \begin{cases} G(n,\bar{x}(n)), & \bar{x}(n) \neq 0, \\ 0, & \bar{x}(n) = 0. \end{cases}$$

Hence $\bar{x}$ is a solution of the linear equation of the form (1.1). By (3.8), we have

$$0 < \alpha_0 \le c(n) \le \frac{F(n,\bar{x}(n))}{\bar{x}(n)} = a(n) \le 1. \tag{3.11}$$

From the above and by (3.8), (3.9) and (3.10), we get

$$\sum_{k=1}^{n} \left|\frac{H(n,k,\bar{x}(k))}{\bar{x}(k)}\right| = \sum_{k=1}^{n} A(n,k) \le \alpha_1 c(n) \le \alpha_1 a(n).$$

Condition (3.9) implies that

$$|f(n)| \le g(n),$$

where g is bounded. Hence for linear equation (1.1) all conditions of Theorem 3.1 are satisfied. Then all solutions of this equation are bounded. Therefore the solution $\bar{x}$ of equation (3.7) is also bounded.

The second part of this theorem follows from the second part of Theorem 3.1. □

Example 3.6. Consider the equation

$$x(n+1) - x(n) = -\left(\frac{1}{2} - \frac{1}{4}\cos x(n)\right)x(n) + \sum_{k=1}^{n} \frac{1}{k^2} x^2(k) e^{-\lambda x(k)} + \sin x(n) \qquad (3.12)$$

with initial condition $x(1) = x_0 \ge 0$, where $\lambda \in \mathbb{R}$. Here

$$F(n,u) = \left(\frac{1}{2} - \frac{1}{4}\cos u\right)u, \quad H(n,k,u) = \frac{1}{k^2} u^2 e^{-\lambda u} = A(n,k), \quad G(n,u) = \sin u.$$

It is easy to see that for any $n \in \mathbb{N}$ we have $x(n) \ge 0$, where x is the solution of equation (3.12). Let us take

$$\alpha_0 = \frac{1}{4}, \quad c(n) \equiv \frac{1}{4}, \quad \alpha_1 = \frac{2\pi^2}{3\lambda e}, \quad A(n,k) = \frac{1}{\lambda e k^2} \quad \text{and} \quad g(n) \equiv 1.$$

For $u \ge 1$ we have

$$\alpha_0 = c(n) = \frac{1}{4} \le \left|\frac{F(n,u)}{u}\right| \le \frac{3}{4} < 1,$$

$$\left|\frac{H(n,k,u)}{u}\right| = \left|\frac{1}{k^2} u e^{-\lambda u}\right| \le \frac{1}{\lambda e k^2} = A(n,k),$$

$$|G(n,u)| \le 1,$$

and

$$\sum_{k=1}^{n} A(n,k) \le \frac{1}{\lambda e} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6\lambda e} = \alpha_1 c(n).$$

Note, that for

$$\lambda \ge \frac{2\pi^2}{3e} \approx 2.41$$

we have $\alpha_1 < 1$. Then all assumptions of Theorem 3.5 are satisfied and so all positive solutions of equation (3.12) are bounded.

4. FINAL REMARKS

In the Introduction we noted that there exists a close connection between the Volterra difference equation (1.1) and the scalar linear integro-differential equation

$$\dot{x}(t) = -a(t)x(t) + \int_{t_0}^{t} A(t,s)x(s)ds + f(t), \ t \geq t_0. \tag{4.1}$$

The connection between the two classes of equations can be used in another way – to apply the methods, approaches and ideas known for one class to obtain new results for the second class of equations. There are several papers, where the authors applied this connection. In particular, in [9] asymptotic stability conditions for the following linear delay difference equation

$$x(n+1) - x(n) = -a(n) + \sum_{k=1}^{m} a_k(n)x(h_k(n)), \ h_k(n) \leq n,$$

were obtained by applications of the known stability conditions for linear delay differential equations.

Similarly, using the idea of the proof of Theorem 3.1 we will get the following statement. Assume that in equation (4.1) the functions a and f are continuous on $[t_0, \infty)$ and $A(t,s)$ is continuous for $t \geq s \geq t_0$.

Theorem 4.1. *Assume that $a(t) \geq a_0$ for some positive number a_0 and*

$$\limsup_{t\to\infty} \frac{1}{a(t)} \int_{t_0}^{t} |A(t,s)|\, ds < 1.$$

Then for any function f bounded on $[t_0, \infty)$ all solutions of (4.1) are bounded. If in addition function a is bounded on $[t_0, \infty)$ and there exist $M > 0, \lambda > 0$ such that

$$|A(t,s)| \leq Me^{-\lambda(t-s)}, \tag{4.2}$$

then equation (4.1) is exponentially stable.

Note that in this case assumptions are weaker than in the discrete case.

Example 4.2. Consider the equation

$$\dot{x}(t) = -x(t) + \frac{1}{t+1} \int_{0}^{t} \sin(s)x(s)ds + f(t), \ t \geq 0. \tag{4.3}$$

By simple calculations we have

$$\limsup_{t\to\infty} \frac{1}{t+1} \int_{0}^{t} |\sin(s)|ds = \frac{2}{\pi} < 1.$$

Hence, for any function f bounded on $[0,\infty)$ all solutions of equation (4.3) are bounded.
Consider the next equation

$$\dot{x}(t) = -x(t) + \alpha \int_0^t e^{-(t-s)} \sin(ts)x(s)ds + f(t),\ t \geq 0. \tag{4.4}$$

For this equation condition (4.2) holds. Moreover, we have

$$\limsup_{t\to\infty} |\alpha| \int_0^t e^{-(t-s)} |\sin(ts)|ds \leq |\alpha|.$$

Hence, if $|\alpha| < 1$, then equation (4.4) is exponentially stable.

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