TWISTED LEVI SEQUENCES AND EXPLICIT TYPES ON Sp_4

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Introduction

Let G be a connected reductive group over a field F. A twisted Levi subgroup G' of G is a reductive subgroup such that $G' \otimes_F \overline{F}$ is a Levi subgroup of $G \otimes_F \overline{F}$. Twisted Levi subgroups have been an important tool in studying the structure theory of representations of p-adic groups. For example, supercuspidal representations are built out of certain representations of twisted Levi subgroups ([20]), and Hecke algebra isomorphisms are established with Hecke algebras on twisted Levi subgroups, which suggests an inductive structure of representations (see [9] for example).

In this paper, we first classify rational conjugacy classes of twisted Levi sequences in a connected reductive group over an arbitrary field via Galois cohomology. When F is a p-adic field, M. Reeder ([15]) gives a classification of maximal tamely ramified tori in G up to $G(F)$-conjugacy using Galois cohomology and Kottwitz's isomorphisms. We generalize this to classify twisted Levi sequences up to rational conjugacy in p-adic groups.

In the second half of this paper, using the classification of twisted Levi sequences, when $G = Sp_4$, we explicate the structure of tame supercuspidal representations and types (in the sense of Bernstein, Bushnell and Kutzko [1, 3]). While the general structure of tame supercuspidal representations are well understood thanks to recent progress in the classification of supercuspidal representations ([20, 10, 4], see also [11] and its references), more explicit and specific informations are lost in this generality. However, often more fine structural information would be necessary in applications (e.g. explicit local Langlands correspondence, construction of L-packets, explicit Plancherel formula etc). Here, we give a list of generic G-data from which supercuspidal representations are constructed for $G = Sp_4$. This list is complete when F satisfies the hypotheses in [10]. When the residue characteristic is odd, we also give a complete list of G-data for types on Sp_4 (§3): starting from a cuspidal type σ on a Levi subgroup of Sp_4, we give a G-datum to construct a G-cover of σ. The construction of tame types in [12] is reviewed in §2.

In a sequel of this paper, we use these explicit data of types in a crucial way to establish Hecke algebra isomorphisms as in [2, 5].

Notation and Conventions. We use T, L, M, G etc to denote a connected reductive group over a field F. If there is no confusion, we will use the same notation for the group of F-points. That is, we may write G for $G(F)$. Therefore, we sometimes write $F^\times$ for the algebraic group $\mathbb{G}_\mathrm{m}$, and $E^\times$ for the algebraic group $R_{E/F}\mathbb{G}_\mathrm{m}$ for any finite separable extension E of F. When F is a nonarchimedean local field of residue characteristic p, we will freely use most notation from [20], in particular, those related to affine buildings $\mathcal{B}(G)$.

As usual, let $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ be the set of integers, rational numbers and real numbers respectively. Let $\mathbb{Z}_+$ denote the set of strictly positive integers.

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1. **Twisted Levi sequences**

1.1. **Classifying Levi Sequences.**

In this subsection, we assume that G is a connected split reductive group over a field F. Let $\overline{F}$ be the algebraic closure of F. By a twisted Levi subgroup of G, we mean a F-subgroup G' of G such that $G' \otimes_F \overline{F}$ is a Levi subgroup of $G \otimes_F \overline{F}$.

1.1.1. Let $\mathrm{BRD}_G = (X^*, \Delta, X_*, \Delta^\vee)$ be the based root datum of G, defined as a projective limit following Kottwitz ([13]). We call X^* the weight lattice of G. Let $\mathcal{Z}$ be the center of G and put $G_{\mathrm{ad}} = G/\mathcal{Z}$.

There is a canonical split exact sequence

$$1 \to G_{\mathrm{ad}} \to \mathrm{Aut}(G) \to \mathrm{Aut}(\mathrm{BRD}_G) \to 1.$$

A splitting can be constructed from a pinning. Recall that $\mathrm{Aut}(\mathrm{BRD}_G)$ is the subgroup of $\mathrm{Aut}(X^*)$ stabilizing the subset

$$\{(a, a^\vee) : a \in \Delta\}$$

in $X^* \times X_*$. We can associate to each $a \in \Delta$ a simple reflection in $\mathrm{Aut}(X^*)$, and $W_G \subset \mathrm{Aut}(X^*)$ is generated by these simple reflections. Let A_G be the subgroup of $\mathrm{Aut}(X^*)$ which stabilizes the subset

$$\{(a, a^\vee) : a \in R\},$$

where $R = \{w.a : w \in W_G, a \in \Delta\}$ is the set of roots of G. Then A_G normalizes W_G and $\mathrm{Aut}(\mathrm{BRD}_G) = \mathrm{Stab}_{A_G}(\Delta)$.

Lemma 1.1.2. *We have*

$$A_G = W_G \rtimes \mathrm{Stab}_{A_G}(\Delta) = W_G \rtimes \mathrm{Aut}(\mathrm{BRD}_G).$$

More generally, for any subgroup H such that $W_G \subset H \subset A_G$, we have

$$H = W_G \rtimes \mathrm{Stab}_H(\Delta).$$

Proof. It is well known that $\{w.\Delta : w \in W_G\}$ is a principal homogeneous space of W_G. Clearly H acts on this set. It follows that every element of H is uniquely a product of an element of W_G and an element of $\mathrm{Stab}_H(\Delta)$. ∎

It follows that if we choose a maximal split torus T, and a Borel subgroup $B \supset T$, then X^* can be identified with $X^*(T)$ (this identification doesn't depend on B). W_G can be identified with $N_{G_{\mathrm{ad}}}(T_{\mathrm{ad}})/T_{\mathrm{ad}}$, and A_G with N/T_{ad}, where N is the normalizer of $T_{\mathrm{ad}} := T/\mathcal{Z}$ in $\mathrm{Aut}(G)$.

1.1.3. Automorphisms of (G, L). Let L be a connected reductive subgroup of G containing a maximal split torus T of G. Then we can identify the weight lattice of G with that of L, since both are identified with $X^*(T)$. Write BRD_L as $(X^*, \Delta_L, X_*, \Delta_L^\vee)$. If L is a Levi subgroup, we may choose a Borel subgroup B of G, use (G, B, T) to form BRD_G, and $(L, B \cap L, T)$ to form BRD_L, then we get an inclusion $\Delta_L \subset \Delta$. However, this inclusion depends on our choice of (B, T).

Let $\mathrm{Aut}(G, L)$ be the subgroup of $\mathrm{Aut}(G)$ stabilizing L. Clearly, $L/\mathcal{Z}$ is a subgroup of $G_{\mathrm{ad}} \cap \mathrm{Aut}(G, L)$, and there is a group homomorphism $\mathrm{Aut}(G, L) \to \mathrm{Aut}(L) \to \mathrm{Aut}(\mathrm{BRD}_L)$.

Proposition 1.1.4. *We have*

$$A_G \cap A_L = N_{A_G}(W_L) = W_L \rtimes \mathrm{Stab}_{A_G}(\Delta_L).$$

The image of the composition $\mathrm{Aut}(G, L) \to \mathrm{Aut}(L) \to \mathrm{Aut}(\mathrm{BRD}_L)$ is $\mathrm{Stab}_{A_G}(\Delta_L)$ and the kernel is $L/\mathcal{Z}$. Therefore, we have a canonical exact sequence

$$1 \to L/\mathcal{Z} \to \mathrm{Aut}(G, L) \to \mathrm{Stab}_{A_G}(\Delta_L) \to 1.$$

PROOF. It is clear $A_G \cap A_L \subset N_{A_G}(W_L)$. Let $R_L \subset X^*$ be the set of roots of L. We have $R_L \subset R$. Consider $w \in N_{A_G}(W_L)$ and $a \in R_L$. Then $(w.a, w.a^\vee) = (b, b^\vee)$ for some $b \in R$. We have $wr_aw^{-1} = r_b \in W_L$, so $b = cb'$ for some $b' \in R_L$, $c \in \mathbb{Q}^\times$. But the root system R is reduced, so $b = \pm b' \in R_L$. This shows that w permutes R_L and hence $w \in A_L$. We have proved the first equality in the first equation. The second equality follows from the preceding lemma.

Let N' be the inverse image of $N_{A_G}(W_L)$ under $N \to A_G$. We observe that the diagram

$$\begin{tikzcd} N' \arrow[r] \arrow[d, phantom, "\bigcap" rotate=0] & \mathrm{Stab}_{A_G}(\Delta_L) \arrow[d, phantom, "\bigcap"] \\ \mathrm{Aut}(G,L) \arrow[r] & \mathrm{Aut}(\mathrm{BRD}_L) \end{tikzcd}$$

is commutative, where the top arrow is defined by $N' \to N_{A_G}(W_L) \to \mathrm{Stab}_{A_G}(\Delta_L)$ using the semidirect product decomposition we just proved. This shows that the image of $\mathrm{Aut}(G,L) \to \mathrm{Aut}(\mathrm{BRD}_L)$ contains $\mathrm{Stab}_{A_G}(\Delta_L)$.

Let $g \in \mathrm{Aut}(G,L)$. Then we can find a representative n in the coset $g(L/\mathcal{Z})$ such that n acts on L by a pinned automorphism (relative to $(B \cap L, T, X)$ for some X). In particular, n stabilizes T, so $n \in N$. It is clear that $n.\Delta_L = \Delta_L$. This shows that the image of $\mathrm{Aut}(G,L) \to \mathrm{Aut}(\mathrm{BRD}_L)$ lies in $\mathrm{Stab}_{A_G}(\Delta_L)$, and we have $n \in N'$. The image of g under $\mathrm{Aut}(G,L) \to \mathrm{Aut}(\mathrm{BRD}_L)$ is the same as that of n. If it is trivial, then $n \in T_{\mathrm{ad}}$ and hence $g \in L/\mathcal{Z}$. This completes the proof of the proposition. ∎

Remark. The above sequence splits when $L = G$, but not in general: the case of $L = T$ was analyzed by Tits.

1.1.5. The automorphisms of a Levi sequence. Let $\vec{G} = (G^0, G^1, \ldots, G^d)$ be a Levi sequence in G. That is, G^i is a Levi subgroup of G^{i+1} for $i = 0, \ldots, d-1$, and $G^d = G$. We define $\mathrm{Aut}(\vec{G})$ to be the subset of $\mathrm{Aut}(G)$ stabilizing each G^i, $i = 0, \ldots, d$.

We choose a maximal split torus T in G^0 and a Borel subgroup $B \supset T$ of G. Using these, we can identify the weight lattice of each G^i is with that of G. If we write $\mathrm{BRD}_{G^i} = (X^*, \Delta_i, X_*, \Delta_i^\vee)$, then each Δ_i is a subset of Δ, and we have $\Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_d = \Delta$.

Obviously, $G^0/\mathcal{Z} \subset \mathrm{Aut}(\vec{G}) \subset \mathrm{Aut}(G, G^0)$, and hence $\mathrm{Aut}(\vec{G})/(G^0/\mathcal{Z})$ maps injectively to $\mathrm{Stab}_{A_G}(\Delta_0)$.

Proposition 1.1.6. *Let*

$$A_{\vec{G}} := \bigcap_{i=0}^{d} A_{G^i} = \bigcap_{i=0}^{d} N_{A_G}(W_{G^i}) = \bigcap_{i=0}^{d} \Big(W_{G^i} \rtimes \mathrm{Stab}_{A_G}(\Delta_i) \Big).$$

There are canonical exact sequences

$$1 \to W_{G^0} \to A_{\vec{G}} \to \mathrm{Stab}_{A_{\vec{G}}}(\Delta_0) \to 1,$$
$$1 \to G^0/\mathcal{Z} \to \mathrm{Aut}(\vec{G}) \to \mathrm{Stab}_{A_{\vec{G}}}(\Delta_0) \to 1.$$

The first one is split.

This reduces to the preceding proposition when $d = 1$. The proof remains the same.

Remark. For each $i = 0, \ldots, d$, there is a canonical commutative diagram

$$\begin{tikzcd} \mathrm{Aut}(\vec{G}) \arrow[r] \arrow[d] & \mathrm{Stab}_{A_{\vec{G}}}(\Delta_0) \arrow[d] \\ \mathrm{Aut}(G^i) \arrow[r] & \mathrm{Aut}(\mathrm{BRD}_{G^i}), \end{tikzcd}$$

where the vertical arrow on the right is the composition of

$$\mathrm{Stab}_{A_{\vec{G}}}(\Delta_0) \subset N_{A_G}(W_{G^i}) \to \mathrm{Stab}_{A_G}(\Delta_i) \subset \mathrm{Aut}(\mathrm{BRD}_{G_i}).$$

Variant. Let $N(\vec{G}) = \{g \in G : gG^i g^{-1} = G^i, \text{ for } i = 0, \dots, d\}$ be the normalizer of $\vec{G}$ in G. Let

$$W_{\vec{G}} := \bigcap_{i=0}^{d} N_{W_G}(W_{G^i}) = \bigcap_{i=0}^{d} \Big(W_{G^i} \rtimes \mathrm{Stab}_{W_G}(\Delta_i) \Big).$$

There are canonical exact sequences

$$1 \to W_{G^0} \to W_{\vec{G}} \to \mathrm{Stab}_{W_{\vec{G}}}(\Delta_0) \to 1,$$
$$1 \to G^0 \to N(\vec{G}) \to \mathrm{Stab}_{W_{\vec{G}}}(\Delta_0) \to 1.$$

The first one is split.

Remark. If $1 \to T_{\mathrm{ad}} \to N \to A_G \to 1$ splits, then $1 \to G^0/\mathcal{Z} \to \mathrm{Aut}(\vec{G}) \to \mathrm{Stab}_{A_{\vec{G}}}(\Delta_0) \to 1$ splits for any $\vec{G}$ in G. Similarly, if $1 \to T \to N_G(T) \to W \to 1$ splits, then $1 \to G^0 \to N(\vec{G}) \to \mathrm{Stab}_{W_{\vec{G}}}(\Delta_0) \to 1$ splits for any $\vec{G}$.

Remark. Let $\vec{G}'$ be another Levi sequence in G, corresponding to $\Delta_0' \subset \Delta_1' \subset \cdots \subset \Delta_{d'}'$. Then $\vec{G}'$ is conjugate to $\vec{G}$ by an element of $\mathrm{Aut}(G)$ (resp. of G) if and only if $d = d'$ and there exists $w \in A_G$ (resp. $w \in W_G$) such that $w.\Delta_i = \Delta_i'$ for $i = 0, \dots, d$.

1.1.7. Example. Let $G = Sp_4$. We have 3 Levi subgroups up to conjugacy. Choose a system of simple roots consisting of a long root a_{long} and a short root a_{short}. Let M^{long} (resp. M^{short}) be the centralizer of the kernel of a_{long} (resp. a_{short}). Then T, M^{long}, M^{short} represent the three classes of Levi subgroups. Note that $M^{\mathrm{long}} \simeq F^\times \times SL_2$ and $M^{\mathrm{short}} \simeq GL_2$.

We now enumerate the Levi sequences with $d \geq 1$ (up to conjugacy) and the exact sequences for their normalizer groups, as given in the preceding proposition.

(1) $\vec{G} = (T, G)$. We have $1 \to T \to N(\vec{G}) \to W \to 1$.
(2) $\vec{G} = (M^{\mathrm{long}}, G)$. We have $1 \to M^{\mathrm{long}} \to N(\vec{G}) \to D_1^{\mathrm{long}\perp} \to 1$, where $D_1^{\mathrm{long}\perp}$ is the subgroup generated by the reflection associated to the root $2a_{\mathrm{short}} + a_{\mathrm{long}}$.
(3) $\vec{G} = (M^{\mathrm{short}}, G)$. We have $1 \to M^{\mathrm{short}} \to N(\vec{G}) \to D_1^{\mathrm{short}\perp} \to 1$, where $D_1^{\mathrm{short}\perp}$ is the subgroup generated by the reflection associated to $a_{\mathrm{short}} + a_{\mathrm{long}}$.
(4) $\vec{G} = (T, M^{\mathrm{long}}, G)$. We have $1 \to T \to N(\vec{G}) \to D_1^{\mathrm{long}} \times D_1^{\mathrm{long}\perp} \to 1$.
(5) $\vec{G} = (T, M^{\mathrm{short}}, G)$. We have $1 \to T \to N(\vec{G}) \to D_1^{\mathrm{short}} \times D_1^{\mathrm{short}\perp} \to 1$.

1.2. **Classifying Twisted Levi Sequences.**

So far we have assumed that G and all the Levi subgroups in the preceding discussion are split. We now drop that assumption. Hence G may be non-split and $\vec{G}$ is a twisted Levi sequence in G.

We would like to consider two problems:

- Classify all twisted Levi sequences $\vec{G}'$ over F such that $\vec{G}' \otimes_F \bar{F} \simeq \vec{G} \otimes_F \bar{F}$, up to F-isomorphisms, i.e., to classify the F-forms of $\vec{G}$. Here an isomorphism of a twisted Levi sequence $\vec{G}$ in G to a twisted Levi sequence $\vec{G}'$ in G' means an isomorphism $G \to G'$ inducing an isomorphism $G^i \to (G')^i$ for each i and that $\vec{G}$ and $\vec{G}'$ have the same length. In particular G' is an F-form of G if $\vec{G}'$ is an F-form of $\vec{G}$.
- Classify all Levi sequences $\vec{G}'$ in G, such that $\vec{G}'$ is conjugate to $\vec{G}$ by an element of $G(\bar{F})$, up to $G(F)$-conjugation.

By a well-known principle in Galois cohomology, the first problem is to compute $H^1(F, \mathrm{Aut}(\vec{G}))$, and the second problem is to compute $\ker(H^1(F, N(\vec{G})) \to H^1(F, G))$. If G is an adjoint group such that all automorphisms of G are inner, and $H^1(F, G) = 1$ (e.g. if G is of type G_2 and F is local nonarchimedean), then the two problems are the same.

Galois cohomology of an algebraic group B is much better understood when the algebraic group is connected. Here the main problem is to handle the disconnection. Let $\pi_0 = B/B^0$ be the component group of B. Then we have a canonical map $\phi : H^1(F, B) \to H^1(F, \pi_0)$. One can approach the problem of computing $H^1(F, B)$ as follows:

- Identify the image of ϕ.

- For each c in the image of ϕ, form a twist ${}_bB$ of B corresponding to $b \in C^1(F, B)$ such that $\phi(b) = c$. Then the fiber $\phi^{-1}(c)$ can be identified with $H^1(F, {}_bB^0)/({}_b\pi_0)(F)$, where ${}_b\pi_0$ is the component group of ${}_bB$ and is a twist of π_0 ([18], page 52, Corollary 2). If $\Gamma = \mathrm{Gal}(\bar{F}/F)$ acts on π_0 trivially, then $({}_b\pi_0)(F)$ is the just centralizer of $c(\Gamma)$ in π_0.

Remark. When F is locally compact non-archimedean, and B^0 is a reductive group with root datum $(X^*, \Delta, X_*, \Delta^\vee)$, Kottwitz showed that $H^1(F, B^0)$ is isomorphic to the torsion subgroup of $\left(X_*/(\sum_{a^\vee \in \Delta^\vee} \mathbb{Z}a^\vee)\right)_\Gamma$.

Remark. The group $({}_b\pi_0)(F)$ naturally acts on the right of $H^1(F, B^0)$ ([18], page 52), which is the one used above. When B^0 is abelian, there is also a left action of $({}_b\pi_0)(F)$ on $H^1(F, B^0)$ ([18] page 53). The left action is compatible with the group structure of $H^1(F, B^0)$ and easier to compute. If B^0 is a torus, π_0 acts on $X_* = X_*(B^0)$, and hence $Z_{\pi_0}(c(\Gamma))$ acts on $(X_*)_{c(\Gamma)}$. This agrees with the left action of $Z_{\pi_0}(c(\Gamma))$ on $H^1(F, B^0)$ when we identify $H^1(F, B^0)$ with $(X_*)_{c(\Gamma)}$ by Kottwitz's isomorphism (assuming F local nonarchimedean ([13])).

We continue to assume that B^0 is abelian. The right action of $({}_b\pi_0)(F)$ on $H^1(F, B^0)$ is related to the left one by the connection homomorphism $\delta : ({}_b\pi_0)(F) \to H^1(F, B^0)$ ([18], page 53, Proposition 40). When $1 \to B^0 \to B \to \pi_0 \to 1$ is a split exact sequence with B^0 abelian, we have $\delta = 0$.

Remark. When $1 \to B^0 \to B \to \pi_0 \to 1$ splits, ϕ is clearly surjective.

1.3. **Classification of Tamely Ramified Maximal Tori in Sp_4.**

A special case of twisted Levi sequences is of the form (T, G) where T is a tamely ramified maximal torus. Then, the results in the previous section specializes to a classification of embedded tori in G, which is identical to that in [15]. Reeder found additional features of this case by exploring the fact that $\mathrm{Aut}(\vec{G})^\circ$ is abelian. We summarize his results ([15, Section 6]), in view of what we established in the previous section, as follows. Fix a maximal split torus T in G. Let $N = N_G(T)$ and $W = N/T$. Let $\phi : H^1(F, N) \to H^1(F, W)$ be the map induced by the projection $N \to W$. We refer to [15, Section 6] for the definition of stably classes of tori.

Proposition 1.3.1. *Suppose $H^1(F, G) = 1$. Then, the stable classes of maximal tori in G are in bijection with $H^1(F, W)$. Moreover, for a given class $c \in H^1(F, W)$, the set of rational classes of maximal tori in the stable class corresponding to c is in bijection with $\phi^{-1}(c)$.*

Hence, to classify embedded tori in G, it suffices to compute $H^1(F, W)$, and for each $c \in H^1(F, W)$, to compute the fiber $\phi^{-1}(c)$. Let $U = c(\Gamma)$ and $Z_W(U)$ be the centralizer of U in W. Then, via Tate-Nakayama duality $\phi^{-1}(c)$ is in bijection with $(X_*)_{U,\mathrm{tors}}/Z_W(U)$, the $Z_W(U)$ orbits in the torsion subgroup of U covariants of X_*. A subtlety is that the action of $Z_W(U)$ on $(X_*)_{U,\mathrm{tors}}$ is what Reeder called the "affine action", which depends on choosing a cocycle $b \in C^1(F, N)$ lifting c (we refer to [15, §6] for details; it is the right action mentioned in the second Remark of 1.2). However, the size of $\phi^{-1}(c)$ does not depend of the choice of b. See [15] for some explicit computation.

When $G = Sp_4$, the following is Theorem 6.9-(2) in [15].

Theorem 1.3.2. *The W-conjugacy classes of continuous homomorphisms $c : \Gamma \to W$ are in bijection with the stable classes $\mathcal{T}_c$ of maximal tori in G. Denoting this correspondence by $c \to \mathcal{T}_c$, we have the rational classes in $\mathcal{T}_c$ are in bijection with the orbits of $Z_W(U)$ in $(X_*)_{U,\mathrm{tors}}$ under the affine action obtained by twisting the coinvariant representation by a cocycle belonging to the class of Δ_c in $H^1(Z_W(U), (X_*)_{U,\mathrm{tors}})$.*

In the rest of §1, let F be a nonarchimedean local field of odd residue characteristic and G denote Sp_4. Fix a maximal split torus T in G. We use T to denote a torus in G.

1.3.3. Subgroups of W and their coinvariants. We list subgroups U of $W(Sp_4) = D_4$ (dihedral group of order 8) up to conjugacy, and give $(X_*)_{U,\mathrm{tors}}$, $Z_W(U)$, and $N_W(U)$.

U	D_4	D_2^{long}	D_2^{short}	D_1^{long}	D_1^{short}	C_4	C_2	1
$(X_*)_{U,\text{tors}}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	0
$Z_W(U)$	C_2	D_2^{long}	D_2^{short}	D_2^{long}	D_2^{short}	C_4	D_4	D_4
$N_W(U)$	D_4	D_4	D_4	D_2^{long}	D_2^{short}	D_4	D_4	D_4

TABLE 1.3.3.

Here, C_n is the subgroup of order n in the subgroup of rotations in $W(Sp_4) = D_4$, and D_2^{long} (resp. D_2^{short}) is $C_2.D_1^{\text{long}}$ (resp. $C_2.D_1^{\text{short}}$).

1.3.4. The set $H^1(F,W)$ and $H^1(F,N)$. Let $I \subset \Gamma$ be the inertia subgroup. Suppose that $c : \Gamma \to W$ is a homomorphism with image U. Then $U_0 := c(I)$ is a cyclic normal subgroup of U such that U/U_0 is cyclic. For each pair of (U, U_0) (up to W-conjugacy) with these properties, we compute $H^1_{U,U^0} = \{c \in \mathrm{Hom}(\Gamma, W) : c(\Gamma) = U, c(I) = U_0\}/N_W(U, U_0)$, where $N_W(U, U_0) = N_W(U) \cap N_W(U_0)$. Then $H^1(F,W)$ is the disjoint union of these H^1_{U,U_0}. For each $c \in H^1_{U,U_0}$, the size of $\phi^{-1}(c)$ is given by Theorem 1.3.2.

Each $b \in \phi^{-1}(c)$ corresponds to an embedded torus $T_b \subset G$. The torus T_b is elliptic $\iff X_*^{c(\Gamma)} = 0 \iff c(\Gamma) \neq 1, D_1^{\text{long}}, D_1^{\text{short}}$. In that case, $\mathcal{B}(T_b)$ is a singleton $\{x_b\}$, and $x_b \in \mathcal{B}(G)$. We give the Kac coordinates ([7]) of x_b up to conjugacy. For a $x \in \mathcal{B}(G)$ with $a_{\text{short}}(x) = y_1$ and $a_{\text{long}}(x) = y_2$ with $y_i \in \mathbb{Q}$, one can find a strictly positive integer $m \in \mathbb{Z}_+$ such that $m(1-2y_1-y_2)$, my_1, my_2 are relatively prime. Then, the Kac coordinates of x are given by $(m(1-2y_1-y_2), my_1, my_2)$.

Label	$U \supset U_0$	$\#H^1_{U,U_0}$	$\#\phi^{-1}(c)$	x_b
T[1]	$D_4 \supset C_4$	$\begin{cases} 0 & q \equiv 1 \pmod 4 \\ 2 & q \equiv -1 \pmod 4 \end{cases}$	1	$(1,1,1)$
T[2]	$C_4 \supset C_4$	$\begin{cases} 4 & q \equiv 1 \pmod 4 \\ 0 & q \equiv -1 \pmod 4 \end{cases}$	$\begin{cases} 2 & q \equiv 1 \pmod 8 \\ 1 & q \equiv 5 \pmod 8 \end{cases}$	$(1,1,1)$
T[3]	$C_4 \supset C_2$	1	2	$(1,0,1)$
T[4]	$C_4 \supset 1$	1	2	$(1,0,0)$, or $(0,0,1)$
T[5]	$D_2^{\text{long}} \supset C_2$	1	$\begin{cases} 4 & q \equiv 1 \pmod 4 \\ 1 & q \equiv -1 \pmod 4 \end{cases}$	$(1,0,1)$
T[6]	$D_2^{\text{long}} \supset D_1^{\text{long}}$	2	$\begin{cases} 4 & q \equiv 1 \pmod 4 \\ 2 & q \equiv -1 \pmod 4 \end{cases}$	$(2,1,0)$, or $(0,1,2)$
T[7]	$C_2 \supset C_2$	2	$\begin{cases} 3 & q \equiv 1 \pmod 4 \\ 1 & q \equiv -1 \pmod 4 \end{cases}$	$(1,0,1)$
T[8]	$C_2 \supset 1$	1	3	$(1,0,0)$, $(0,1,0)$ or $(0,0,1)$
T[9]	$D_2^{\text{short}} \supset C_2$	1	2	$(1,0,1)$
T[10]	$D_2^{\text{short}} \supset D_1^{\text{short}}$	2	2	$(1,0,1)$, or $(0,1,0)$
T[11]	$D_1^{\text{long}} \supset D_1^{\text{long}}$	2	$\begin{cases} 2 & q \equiv 1 \pmod 4 \\ 1 & q \equiv -1 \pmod 4 \end{cases}$	
T[12]	$D_1^{\text{long}} \supset 1$	1	2	
T[13]	$D_1^{\text{short}} \supset D_1^{\text{short}}$	2	1	
T[14]	$D_1^{\text{short}} \supset 1$	1	1	
T[15]	$1 \supset 1$	1	1	

TABLE 1.3.4.

We mention some facts underlying the calculation. There exists a surjection $\Gamma \to C_n$ such that the image of inertia is of order e if and only if $q \equiv 1 \pmod e$, in that case, the number of such homomorphism is $e\varphi(e)\varphi(n/e)$. There exists a extension E/F with Galois group D_n, and ramification index $e = n$, residue degree $f = 2$, exactly when $q \equiv -1 \pmod e$. In that case the extension is unique.

The number of isomorphisms $\mathrm{Gal}(E/F) \xrightarrow{\sim} D_n$ sending the inertia subgroup to C_n is $n \cdot \varphi(n)$. Finally, the action of $N_W(U, U_0)/Z_W(U)$ on $\{c \in \mathrm{Hom}(\Gamma, W) : c(\Gamma) = U, c(I) = U^0\}$ is faithful.

The most laborious part of the calculation is the determination of $\#\phi^{-1}(c)$. To carry out the method outlined in Theorem 1.3.2, one may start with an explicit torus in each stable class. Such an explicit torus is given in 1.3.5 and 1.3.6.

We conclude

$$\#H^1(F,W) = \begin{cases} 22 & \text{if } q \equiv 1 \pmod 4 \\ 20 & \text{if } q \equiv -1 \pmod 4 \end{cases} \qquad \#H^1(F,N) = \begin{cases} 49 & \text{if } q \equiv 1 \pmod 8 \\ 45 & \text{if } q \equiv 5 \pmod 8 \\ 32 & \text{if } q \equiv -1 \pmod 4. \end{cases}$$

1.3.5. Compact tori. Although we have a classification of embedded tori in Sp_4 up to rational conjugacy as above in terms of Galois cohomology, using another description given in [8, 14], we can give a more explicit description of each tori. Stating the result in loc.cit., let $\langle\ ,\ \rangle$ be the symplectic form on $V = F^4$ to realize Sp_4.

Theorem. *Let* T *be a tamely ramified compact maximal torus in* $Sp_4(F)$*. Then, we have one of the following:*

(1) *There is a tower* $F \subset E \subset E'$ *with* $(E' : E) = (E : F) = 2$*, a unitary form* $(\ ,\ )$ *on* E' *over* E *and an* F*-linear isomorphism* $j : E' \to F^4$ *so that*

$$\langle j(v), j(w)\rangle = Tr_{E'/F}(\alpha(v,w))$$

for a nonzero $\alpha \in \ker(Tr_{E'/E})$*. Moreover,* j *induces an embedding from the unitary group of* $(\ ,\ )$ *on* E' *onto* T*.*

(2) *There are quadratic extensions* E_1*,* E_2 *equipped with a Hermitian form* $(\ ,\ )_i$ *on* E_i *over* F*, and an* F*-linear isomorphism* $j : E_1 \oplus E_2 \to F^4$ *such that*

$$\langle j(v_1, v_2), j(w_1, w_2)\rangle = Tr_{E_1/F}(\alpha_1(v_1, w_1)_1) + Tr_{E_2/F}(\alpha_2(v_2, w_2)_2)$$

for nonzero $\alpha_i \in \ker(Tr_{E_i/F})$*,* $i = 1, 2$*. Moreover,* j *induces an embedding of the unitary group of* $(\ ,\ )_1 \oplus (\ ,\ )_2$ *on* $E_1 \oplus E_2$ *onto* T*.*

Conversely, any unitary group in (1) *and* (2) *embedds onto a maximal anisotropic torus in* Sp_4*.*

In the above cases, we will say that T is the "isometric image" of the unitary group U and write $\mathrm{T} \overset{i}{\simeq} U$. From now on, we write $F^\times/F^{\times 2} = \{1, \varepsilon, \varpi, \varepsilon\varpi\}$ where $\varepsilon \in \mathcal{O}_F^\times$ is a nonsquare and ϖ is a uniformizer in F.

Analyzing U and U_0 in Table 1.3.4., we see that $\mathrm{T}[5]$, $\mathrm{T}[6]$, $\mathrm{T}[7]$, $\mathrm{T}[8]$ belong to cases (2) and we can find E_1, E_2 in each case. To be more explicit, for $a, b \in F^\times/F^{\times 2}$, let $U_{a,b}$ be the unitary group of one variable in $F[\sqrt{a}]$ with respect to the unitary form $(v, w) = bv\overline{w}$ where $\overline{w}$ is the Galois conjugate in $F[\sqrt{a}]$ over F. We can list all possible unitary groups (up to isometry) in one variable as follows:

$$U_{\varepsilon,1},\ U_{\varepsilon,\varpi},\ U_{\varpi,1},\ U_{\varpi,\varepsilon},\ U_{\varepsilon\varpi,1},\ U_{\varepsilon\varpi,\varepsilon}.$$

These embed in $SL_2(F)$. If $q \equiv 1 \pmod 4$, they are not rationally conjugate. However, if $q \equiv 3 \pmod 4$, $U_{\varpi,1}$ and $U_{\varpi,\varepsilon}$ are rationally conjugate, and so are $U_{\varepsilon\varpi,1}$ and $U_{\varepsilon\varpi,\varepsilon}$ (see §6.4 in [15]).

	$E_1,\ E_2$	$\mathrm{T} \overset{i}{\simeq}$	parameters	x_b
T[5]	$F[\sqrt{\varpi}],\ F[\sqrt{\varepsilon\varpi}]$	$U_{\varpi,a} \times U_{\varpi\varepsilon,b}$	(a,b) : $a,b \in \{1,\varepsilon\}$	$(1,0,1)$
T[6]	$F[\sqrt{\varpi'}],\ F[\sqrt{\varepsilon}]$	$U_{\varpi',a} \times U_{\varepsilon,b}$	$(\varpi',a,b): \begin{array}{ll} \varpi' & \in \{\varpi,\varepsilon\varpi\} \\ a & \in \{1,\varepsilon\} \\ b & \in \{1,\varpi\} \end{array}$	$\begin{cases} (2,1,0) & \text{if } b=1 \\ (0,1,2) & \text{if } b=\varpi \end{cases}$
T[7]	$F[\sqrt{\varpi'}],\ F[\sqrt{\varpi'}]$	$U_{\varpi',a} \times U_{\varpi',b}$	(ϖ',a,b) : $(a,b) \in \{(1,1),(1,\varepsilon),(\varepsilon,\varepsilon)\}$ $\varpi' \in \{\varpi,\varepsilon\varpi\}$	$(1,0,1)$
T[8]	$F[\sqrt{\varepsilon}],\ F[\sqrt{\varepsilon}]$	$U_{\varepsilon,a} \times U_{\varepsilon,b}$	$(a,b) \in$ $\{(1,1),(1,\varpi),(\varpi,\varpi)\}$	$\begin{cases} (1,0,0) & \text{if } (a,b)=(1,1) \\ (0,1,0) & \text{if } (a,b)=(1,\varpi) \\ (0,0,1) & \text{if } (a,b)=(\varpi,\varpi) \end{cases}$

TABLE 1.3.5-I.

The parameters in the above table will label rational conjugacy classes of embedded tori with same U and U_0 in TABLE 1.3.4. For example, T[5](a,b) labels the torus in T[5] which is an isometric image of $U_{\varpi,a} \times U_{\varpi\varepsilon,b}$.

Remark. If $q \equiv 3 \pmod 4$, T[5](a,b) are all rationally conjugate to each other. Similarly, T[7](ω',a,b) are all rationally conjugate. Likewise, the labeling of T[6], T[1], T[2] and T[11] is redundant (see TABLES 1.3.5-II and 1.3.6). For a uniform description incorporating cases both cases $q \equiv 1$ and $q \equiv 3 \pmod 4$, we keep the redundant labeling. Moreover, this redundancy is necessary in describing $\vec{G}^s[4]$ (see TABLE 1.4.4), since two rationally conjugate T[7]$(\omega',1,1)$ and T[7]$(\omega',1,\varepsilon)$ give rise to non conjugate twisted Levi sequences.

Comparing the above with TABLE 1.3.4., T[1], T[2], T[3], T[4] belong to case (1). In each case, E' associated to the torus satisfies $e(E'/E) = \#(U_0)$, $f(E'/E) = \#(U/U_0)$. Moreover, E' has a unique subextension E of degree 2.

T[9] and T[10] also belong to case (1) with $E' = F[\sqrt{\varepsilon},\sqrt{\varpi}]$, the abelian extension of degree 4 which contains all quadratic extensions of F. In this cases, E' contains three quadratic extensions $F[\sqrt{a}]$, $a \in F^\times/F^{\times 2} - \{1\}$ and each $E'/F[\sqrt{a}]$ has two unitary forms (up to equivalence) of 1 variable, which accounts for all 6 tori in T[9] and T[10]. For $\alpha \in E^\times/N_{E'/E}(E'^\times)$, let $U_{E'/E}(\alpha)$ denote the isometry class of the unitary group on E' over E with respect to $(v,w) = \alpha v\overline{w}$. In the following table, α runs over $E^\times/N_{E'/E}(E'^\times)$.

Label	$E \subset E'$	$\mathrm{T} \overset{i}{\simeq}$	parameters	x_b
T[1], T[2]	$E = F[(c\omega)^{\frac12}],\ E' = F[(c\omega)^{\frac14}]$	$U_{E'/E}(\alpha)$	(c,α) : $c \in \mathbb{F}_q^\times/\mathbb{F}_q^{\times 4}$	(1,1,1)
T[3]	$E = F[\sqrt{\varepsilon}],\ E' \neq F[\sqrt{\varepsilon},\sqrt{\varpi}]$	$U_{E'/E}(\alpha)$	α	$(1,0,1)$
T[4]	$f(E'/F) = 4$	$U_{E'/E}(\alpha)$	α	$\begin{cases} (1,0,0) & \text{if } \alpha=1 \\ (0,0,1) & \text{if } \alpha\neq 1 \end{cases}$
T[9]	$E = F[\sqrt{\varepsilon}],\ E' = F[\sqrt{\varepsilon},\sqrt{\varpi}]$	$U_{E'/E}(\alpha)$	α	$(1,0,1)$
T[10]	$E = F[\sqrt{\varpi'}],\ E' = F[\sqrt{\varepsilon},\sqrt{\varpi}]$	$U_{E'/E}(\alpha)$	(ϖ',α) : $\varpi' \in \{\varpi,\varepsilon\varpi\}$	$\begin{cases} (1,0,1) & \text{if } \alpha=1 \\ (0,1,0) & \text{if } \alpha\neq 1 \end{cases}$

TABLE 1.3.5-II.

1.3.6. Non compact tori The rest of the tori in T[11]–T[15] are non compact and they are either embedded in $M^{\mathrm{long}} \simeq F^\times \times SL_2(F)$ or $M^{\mathrm{short}} \simeq GL_2(F)$. Although the tori in $SL_2(F)$ and $GL_2(F)$ are well known, we will make a list here for completeness. In the following, let E denote the splitting field of T.

Label	E	$\mathrm{T} \overset{i}{\simeq}$	parameters
T[11]	$F[\sqrt{\varpi'}]$	$F^\times \times U_{\varpi',a}$	(ϖ', a) : $\varpi' \in \{\varpi, \varepsilon\varpi\}$ $a \in \{1, \varepsilon\}$
T[12]	$F[\sqrt{\varepsilon}]$	$F^\times \times U_{\varepsilon,a}$	$a \in \{1, \varpi\}$
T[13]	$F[\sqrt{\varpi'}]$	$E^\times$	$\varpi' \in \{\varpi, \varepsilon\varpi\}$
T[14]	$F[\sqrt{\varepsilon}]$	$E^\times$	
T[15]	F	$F^\times \times F^\times$	

TABLE 1.3.6.

1.4. Classification of twisted Levi sequences in Sp_4.

1.4.1. Classifying twists of M^{long}. We compute $H^1(F, N(\vec{G}))$, where $\vec{G} = (M^{\text{long}}, G)$. Note that we have $M^{\text{long}} = T_{SL_2} \times SL_2(F) \subset SL_2(F) \times SL_2(F) \subset Sp_4(F)$ and $N(\vec{G}) \simeq N_{SL_2}(T_{SL_2}) \times SL_2$ where T_{SL_2} is a maximal split torus in SL_2. Hence, we have

$$H^1(F, N(\vec{G})) \simeq H^1(F, N_{SL_2}(T_{SL_2})) \times H^1(F, SL_2) \simeq H^1(F, N_{SL_2}(T_{SL_2})).$$

$H^1(F, N_{SL_2}(T_{SL_2}))$ classifies the embdded twists of T_{SL_2} in SL_2 and it is known that $\#(H^1(F, N_{SL_2}(T_{SL_2}))) = 7$ if $q \equiv 1 \pmod 4$ and 5 if $q \equiv 3 \pmod 4$ (see §6.4 in [15]). Hence, we have 7 embedded twists of M^{long}. We can list them as follows:

$$M^{\text{long}},\ U_{\varepsilon,1} \times SL_2,\ U_{\varepsilon,\varpi} \times SL_2,\ U_{\varpi,1} \times SL_2,$$
$$U_{\varpi,\varepsilon} \times SL_2,\ U_{\varepsilon\varpi,1} \times SL_2,\ U_{\varepsilon\varpi,\varepsilon} \times SL_2.$$

Similarly as in SL_2 case, if $q \equiv 3 \pmod 4$, $U_{\varpi,1} \times SL_2$, and $U_{\varpi,\varepsilon} \times SL_2$ are rationally conjugate and so are $U_{\varepsilon\varpi,1} \times SL_2$ and $U_{\varepsilon\varpi,\varepsilon} \times SL_2$.

1.4.2. Classifying twists of M^{short}. We now compute $H^1(F, N(\vec{G}))$, where $\vec{G} = (M^{\text{short}}, G)$. Since $1 \to M^{\text{short}} \to N(\vec{G}) \to D_1^{\text{short}\perp} \to 1$ splits, we have a surjection

$$H^1(F, N(\vec{G})) \to H^1(F, D_1^{\text{short}\perp}) = F^\times / F^{\times 2}.$$

The fiber at $a \in F^\times/(F^\times)^2$ can be identified with $H^1(F, U_2)$, where U_2 is the quasi-split unitary group in 2 variables for the quadratic extension $F(\sqrt{a})/F$ (which may be a split étale algebra). When $a = 1$, $H^1(F, U_2) = H^1(F, GL_2) = 1$.

If $a \in F^\times$ is not a square, Kottwitz's formula gives $\#H^1(F, U_2) = 2$. Hence, $\#H^1(F, N(\vec{G})) = 7$ and there are at most 7 embedded twists of M^{short}.

It is easy to see that every unitary group in 2 variable occurs as a twisted M^{short} in Sp_4. More precisely, let $E = F(\sqrt{a})$ be a nontrivial quadratic extension of F. Let $V = E \oplus E$ be a E-vector space equipped with a Hermitian form $(\ ,\ )_E$ with respect to the Galois involution on E. Let U_2 be the group of isometries of $(V, (\ ,\ )_E)$. Regarding V as a four dimensional F-vector space, define a skew-symmetric form $(\ ,\ )_F$ on V as follows ([8]):

$$(v, w)_F = Tr_{E/F}(\sqrt{a}(v, w)_E).$$

Then, U_2 preserves $(v, w)_F$ and it is embedded in the group of isometries of $(V, (\ ,\ )_F)$, which is isomorphic to $Sp_4(F)$.

There are 6 such unitary groups up to isometry and each is unique up to $G(F)$-conjugacy. Together with M^{short}, we have 7 embedded twists of M^{short}, up to $G(F)$-conjugacy.

For $a \in F^\times/F^{\times 2}$, let $U_a(1,1)$ be the quasi split unitary group and $U_a(2)$ be the compact unitary group in two variables in $F(\sqrt{a})$. Writing as $F^\times/F^{\times 2} = \{1, \varepsilon, \varpi, \varepsilon\varpi\}$ as before, we may list the twists of M^{short} as follows:

$$M^{\text{short}},\ U_\varepsilon(1,1),\ U_\varepsilon(2),\ U_\varpi(1,1),\ U_\varpi(2),\ U_{\varepsilon\varpi}(1,1),\ U_{\varepsilon\varpi}(2).$$

1.4.3. Classifying twists of $\vec{G} = (T, M^{\mathrm{long}}, G)$. The exact sequence for $N(\vec{G})$ is $1 \to T \to N(\vec{G}) \to D_2^{\mathrm{long}} \to 1$. In particular, we have a homomorphism $N(\vec{G}) \to N = N_G(T)$. We can compute $H^1(F, N(\vec{G}))$ in the same way we compute $H^1(F, N)$.

U	D_2^{long}	C_2	D_1^{long}	$D_1^{\mathrm{long}\perp}$	1
$(X_*)_{U,\mathrm{tors}}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
$Z_{D_2^{\mathrm{long}}}(U)$	D_2^{long}	D_2^{long}	D_2^{long}	D_2^{long}	D_2^{long}
$N_{D_2^{\mathrm{long}}}(U)$	D_2^{long}	D_2^{long}	D_2^{long}	D_2^{long}	D_2^{long}

In the following the parameters run over those in Table 1.3.5-I and 1.3.6.

Label	$U \supset U_0$	$\#H^1_{U,U_0}$	$\#\phi^{-1}(c)$	G^0	G^1
$\vec{G}^\ell[1]$	$D_2^{\mathrm{long}} \supset C_2$	2	$\begin{cases} 4 & q \equiv 1 \pmod 4 \\ 2 & q \equiv -1 \pmod 4 \end{cases}$	$\mathrm{T}[5](a,b)$	$U_{\varpi',a'} \times SL_2$ $(\varpi', a') = (\varpi, a)$ or $(\varepsilon\varpi, b)$
$\vec{G}^\ell[2]$	$D_2^{\mathrm{long}} \supset D_1^{\mathrm{long}}$	2	$\begin{cases} 4 & q \equiv 1 \pmod 4 \\ 2 & q \equiv -1 \pmod 4 \end{cases}$	$\mathrm{T}[6](\varpi', a, b)$	$U_{\varepsilon,b} \times SL_2$
$\vec{G}^\ell[3]$	$D_2^{\mathrm{long}} \supset D_1^{\mathrm{long}\perp}$	2	$\begin{cases} 4 & q \equiv 1 \pmod 4 \\ 2 & q \equiv -1 \pmod 4 \end{cases}$	$\mathrm{T}[6](\varpi', a, b)$	$U_{\varpi',a} \times SL_2$
$\vec{G}^\ell[4]$	$C_2 \supset C_2$	2	$\begin{cases} 4 & q \equiv 1 \pmod 4 \\ 1 & q \equiv -1 \pmod 4 \end{cases}$	$\mathrm{T}[7](\varpi', a, b)$	$U_{\varpi',a'} \times SL_2$ $a' = a$ or b
$\vec{G}^\ell[5]$	$C_2 \supset 1$	1	4	$\mathrm{T}[8](a,b)$	$U_{\varepsilon,a'} \times SL_2$ $a' = a$ or b
$\vec{G}^\ell[6]$	$D_1^{\mathrm{long}} \supset D_1^{\mathrm{long}}$	2	$\begin{cases} 2 & q \equiv 1 \pmod 4 \\ 1 & q \equiv -1 \pmod 4 \end{cases}$	$\mathrm{T}[11](\varpi', a)$	M^{long}
$\vec{G}^\ell[7]$	$D_1^{\mathrm{long}} \supset 1$	1	2	$\mathrm{T}[12](a)$	M^{long}
$\vec{G}^\ell[8]$	$D_1^{\mathrm{long}\perp} \supset D_1^{\mathrm{long}\perp}$	2	$\begin{cases} 2 & q \equiv 1 \pmod 4 \\ 1 & q \equiv -1 \pmod 4 \end{cases}$	$\mathrm{T}[11](\varpi', a)$	$U_{\varpi',a} \times SL_2$
$\vec{G}^\ell[9]$	$D_1^{\mathrm{long}\perp} \supset 1$	1	2	$\mathrm{T}[12](a)$	$U_{\varepsilon,a} \times SL_2$
$\vec{G}^\ell[10]$	$1 \supset 1$	1	1	$\mathrm{T}[15]$	M^{long}

Table 1.4.3.

1.4.4. Classifying twists of $\vec{G} = (T, M^{\mathrm{short}}, G)$. The exact sequence for $N(\vec{G})$ is $1 \to T \to N(\vec{G}) \to D_2^{\mathrm{short}} \to 1$.

U	D_2^{short}	C_2	D_1^{short}	$D_1^{\mathrm{short}\perp}$	1
$(X_*)_{U,\mathrm{tors}}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	0	0	0
$Z_{D_2^{\mathrm{short}}}(U)$	D_2^{short}	D_2^{short}	D_2^{short}	D_2^{short}	D_2^{short}
$N_{D_2^{\mathrm{short}}}(U)$	D_2^{short}	D_2^{short}	D_2^{short}	D_2^{short}	D_2^{short}

Label	$U \supset U_0$	$\#H^1_{U,U_0}$	$\#\phi^{-1}(c)$	G^0	G^1
$\vec{G}^s[1]$	$D_2^{\text{short}} \supset C_2$	2	2	T[9](1)	$U_{\varpi}(2)$
					$U_{\varepsilon\varpi}(1,1)$
				T[9](α), $\alpha \neq 1$	$U_{\varepsilon\varpi}(2)$
					$U_{\varpi}(1,1)$
$\vec{G}^s[2]$	$D_2^{\text{short}} \supset D_1^{\text{short}}$	2	2	T[10]($\varpi, 1$)	$U_{\varepsilon\varpi}(1,1)$
				T[10](ϖ, α), $\alpha \neq 1$	$U_{\varepsilon\varpi}(2)$
				T[10]($\varepsilon\varpi, 1$)	$U_{\varpi}(1,1)$
				T[10]($\varepsilon\varpi, \alpha$), $\alpha \neq 1$	$U_{\varpi}(2)$
$\vec{G}^s[3]$	$D_2^{\text{short}} \supset D_1^{\text{short}\perp}$	2	2	T[10]($\varpi, 1$)	$U_{\varepsilon}(2)$
				T[10](ϖ, α), $\alpha \neq 1$	$U_{\varepsilon}(1,1)$
				T[10]($\varepsilon\varpi, 1$)	$U_{\varepsilon}(2)$
				T[10]($\varepsilon\varpi, \alpha$), $\alpha \neq 1$	$U_{\varepsilon}(1,1)$
$\vec{G}^s[4]$	$C_2 \supset C_2$	2	$\begin{cases} 3 & q \equiv 1 \pmod 4 \\ 2 & q \equiv -1 \pmod 4 \end{cases}$	T[7](ϖ', a, b), $a = b$	$U_{\varpi'}(1,1)$
				T[7](ϖ', a, b), $a \neq b$	$U_{\varpi'}(2)$
$\vec{G}^s[5]$	$C_2 \supset 1$	1	3	T[8](a, b), $a = b$	$U_{\varepsilon}(1,1)$
				T[8](a, b), $a \neq b$	$U_{\varepsilon}(2)$
$\vec{G}^s[6]$	$D_1^{\text{short}} \supset D_1^{\text{short}}$	2	1	T[13](ϖ')	M^{short}
$\vec{G}^s[7]$	$D_1^{\text{short}} \supset 1$	1	1	T[14]	M^{short}
$\vec{G}^s[8]$	$D_1^{\text{short}\perp} \supset D_1^{\text{short}\perp}$	2	1	T[13](ϖ')	$U_{\varpi'}(1,1)$
$\vec{G}^s[9]$	$D_1^{\text{short}\perp} \supset 1$	1	1	T[14]	$U_{\varepsilon}(1,1)$
$\vec{G}^s[10]$	$1 \supset 1$	1	1	T[15]	M^{short}

Table 1.4.4.

2. Review of construction of types

2.1. Notation and Conventions.

2.1.1. From now on, let F be a fixed non-archimedean local field with residue characteristic p. Let G be a connected reductive group over F, split over a tamely ramified extension of F. We adopt all notation and conventions from [20]. For simplicity, we assume that p is not a torsion prime for $\psi(G)^\vee$, the root datum dual to the root datum $\psi(G)$ of $G \otimes_F \overline{F}$. See §7 in [20] for relevant notation. Then, p is not a torsion prime for any twisted Levi subgroup G' of G.

2.1.2. Let $\vec{G} = (G^0, G^1, \cdots, G^d)$ be a tamely ramified twisted Levi sequence in G. Let M^0 be a Levi subgroup of G^0 and $\mathcal{Z}_s(M^0)$ be the maximal F-split torus of the center Z_{M^0} of M^0. To $\vec{G}$, we associate a sequence of Levi subgroup $\vec{M} = (M^0, \cdots, M^d)$ where M^i is a Levi subgroup of G^i given as the centralizer of $\mathcal{Z}_s(M^0)$ in G^i.

2.2. **Generic embeddings of buildings.** Recall that if G' is a twisted Levi subgroup of G, then there exists a family of natural embeddings of buildings $\mathcal{B}(G') \hookrightarrow \mathcal{B}(G)$, which is an affine space under $X_*(\mathcal{Z}_s(G')) \otimes \mathbb{R}$.

Definition 2.2.1. Let M be a Levi subgroup of G, $y \in \mathcal{B}(M)$, and $s \in \mathbb{R}$. We say that the embedding $\iota : \mathcal{B}(M) \hookrightarrow \mathcal{B}(G)$ is (y, s)-*generic*, or *s-generic with respect to* y, if $U_{a,\iota(y),s} = U_{a,\iota(y),s+}$ for all $a \in \Phi(G, S, F) \smallsetminus \Phi(M, S, F)$, where S is any maximal F-split torus of M such that $y \in A(M, S, F)$ and $\Phi(G, S, F)$ and $\Phi(M, S, F)$ are the corresponding root systems.

Here, U_a is the root subgroup of G associated to a, and we are referring to [17] for the filtration $\{U_{a,\iota(y),r}\}_{r\in\mathbb{R}}$ on U_a. For $r \geq 0$, we have $U_{a,\iota(y),r} = U_a \cap G_{\iota(y),r}$. Given a twisted Levi sequence $\vec{G}$ and $\vec{M}$ as in (2.1.2), consider a commutative diagram of embeddings:

$$\{\iota\} : \quad \begin{array}{ccccccc} \mathcal{B}(G^0) & \longrightarrow & \mathcal{B}(G^1) & \longrightarrow & \cdots & \longrightarrow & \mathcal{B}(G^d) \\ \uparrow & & \uparrow & & & & \uparrow \\ \mathcal{B}(M^0) & \longrightarrow & \mathcal{B}(M^1) & \longrightarrow & \cdots & \longrightarrow & \mathcal{B}(M^d) \end{array} .$$

Definition 2.2.2. Let $\vec{s} = (s_0, \cdots, s_d)$ be a sequence of real numbers, and $y \in \mathcal{B}(M^0)$. We say that $\{\iota\}$ is *$\vec{s}$-generic (relative to y)* if $\iota : \mathcal{B}(M^i) \to \mathcal{B}(G^i)$ is s_i-generic relative to $i(y) \in \mathcal{B}(M^i)$ for $0 \le i \le d$.

From [12], given $\vec{G}$, $\vec{s}$-generic commutative diagrams of embeddings exist.

2.3. **G-datum and construction of types.**

Definition 2.3.1. A *depth-zero datum* is a triple $((G, M), (y, \iota), (K_M, \rho_M))$ such that

- G is a connected reductive group over F and M a Levi subgroup of G.
- $y \in \mathcal{B}(M)$ is such that $M_{y,0}$ is a maximal parahoric subgroup of M, and $\iota : \mathcal{B}(M) \hookrightarrow \mathcal{B}(G)$ is a 0-generic embedding relative to y.
- K_M is a compact open subgroup of M containing $M_{y,0}$, and ρ_M is an irreducible smooth representation of K_M such that $\rho_M | M_{y,0}$ contains a cuspidal representation of $M_{y,0}/M_{y,0^+}$.

Definition 2.3.2. The *G-datum* Σ consists of a 5-tuple

$$((\vec{G}, M^0), (y, \iota), \vec{r}, (K_{M^0}, \rho_{M^0}), \vec{\phi})$$

satisfying the following:

D1. $\vec{G} = (G^0, G^1, \cdots, G^d)$ is a tamely ramified twisted Levi sequence in G, and M^0 a Levi subgroup of G^0. Let $\vec{M}$ be associated to $\vec{G}$ as in (2.1.2).

D2. y is a point in $\mathcal{B}(M^0)$ and $\{\iota\}$ is a commutative diagram of $\vec{s}$ generic embeddings of buildings relative to y, where $\vec{s} = (0, r_0/2, \cdots, r_{d-1}/2)$.

D3. $\vec{r} = (r_0, r_1, \cdots, r_d)$ is a sequence of real numbers satisfying $0 < r_0 < r_1 < \cdots < r_{d-1} \le r_d$ if $d > 0$, $0 \le r_0$ if $d = 0$.

D4. (K_{M^0}, ρ_{M^0}) is such that $((G^0, M^0), (y, \iota : \mathcal{B}(M^0) \hookrightarrow \mathcal{B}(G)), (K_{M^0}, \rho_{M^0}))$ is a depth zero datum.

D5. $\vec{\phi} = (\phi_0, \phi_1, \cdots, \phi_d)$ is a sequence of quasi-characters, where ϕ_i is a quasi-character of G^i such that ϕ_i is G_{i+1}-generic of depth r_i relative to x for all $x \in \mathcal{B}(G_i)$.

2.3.3. The construction. For a given G-datum Σ as above, let $K^0 = K_{M^0} G^0_{y,0^+}$ and ρ the trivial extension of ρ_{M^0} to K^0. Following the recipe in [20], we can construct a pair of an open compact subgroup

$$K_\Sigma := K^d = K^0 G^1_{y,s_0} \cdots G^d_{y,s_{d-1}}$$

and the irreducible representation $\rho_\Sigma := \rho^d$ of K^d.

Theorem 2.3.4. *([12]) Let $K^d_M := K_\Sigma \cap M^d$ and $\rho^d_M := \rho_\Sigma | (K_\Sigma \cap M^d)$.*

1. *(K^d_M, ρ^d_M) is a supercuspidal type on M^d.*
2. *(K_Σ, ρ_Σ) is a G-cover of (K^d_M, ρ^d_M) and hence it is a type in the sense of Bushnell and Kutzko.*

Remark. When $\mathcal{Z}_{G^0}/\mathcal{Z}_G$ is F-anisotropic, the condition on ι is empty and the above G-datum reduces to a generic G-datum in [20]. In this case, our construction gives a supercuspidal type in [20].

3. **Types on Sp_4**

3.1. **Supercuspidal representations.**

Yu's construction of supercuspidal representations starts from a generic G-datum $\Sigma = (\vec{G}, x, \vec{r}, \vec{\phi}, \rho)$ (see [20] for details). Here, we give a list of all possible $(\vec{G}, x, \vec{r})$ to give a supercuspidal representations via Yu's construction. We define the length $\ell(\Sigma)$ of Σ to be d where $\vec{G} = (G^0, G^1, \cdots, G^d = G)$. In our case $G = Sp_4$, d is at most 2.

In the following, $d(\pi)(= r_d)$ denotes the depth of the supercuspidal representation constructed from Σ with given $(\vec{G}, x, \vec{r})$.

3.1.1. Case 1: $d = 0$**.**

These are depth zero supercuspidal representations. Then, $\vec{r} = (0)$, $\vec{\phi} = (1)$ and the Kac coordinates of x are $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$. If $x = (0, 1, 0)$, ρ is inflated from a cuspidal representation of $SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)$. Otherwise, ρ is coming from a cuspidal representation of $Sp_4(\mathbb{F}_q)$.

3.1.2. Case 2: $d = 1$**.**

The second column in the table indicates where r_i should belong. To simplify writing, by $r_0 \in \frac{1}{4}\mathbb{Z}_+$, we mean that $r_0 \in \frac{1}{4}\mathbb{Z}_+ - \frac{1}{2}\mathbb{Z}$, and $r_0 \in \frac{1}{2}\mathbb{Z}_+$ means $r_0 \in \frac{1}{2}\mathbb{Z}_+ - \mathbb{Z}$. In each case, the parameters run over those in Tables 1.3.5-I and II.

G^0	$r_0 = r_1$ $= d(\pi)$	parameters		x
T[1](c, α)	$\frac{1}{4}\mathbb{Z}_+$	(c, α)		$(1, 1, 1)$
T[2](c, α)	$\frac{1}{4}\mathbb{Z}_+$	(c, α)		$(1, 1, 1)$
T[3](α)	$\frac{1}{2}\mathbb{Z}_+$	α		$(1, 0, 1)$
T[4](α)	$\mathbb{Z}_+$	α	$= 1$	$(1, 0, 0)$
			$\neq 1$	$(0, 0, 1)$
T[5](a, b)	$\frac{1}{2}\mathbb{Z}_+$	(a, b)		$(1, 0, 1)$
T[7](ϖ', a, b)	$\frac{1}{2}\mathbb{Z}_+$	(ϖ', a, b)		$(1, 0, 1)$
T[8](a, b)	$\mathbb{Z}_+$	$(a, b) =$	$(1, 1)$	$(1, 0, 0)$
			$(1, \varpi)$	$(0, 1, 0)$
			(ϖ, ϖ)	$(0, 0, 1)$
$U_\varepsilon(2)$	$\mathbb{Z}_+$			$(0, 1, 0)$
$U_{\varpi'}(2)$	$\frac{1}{2}\mathbb{Z}_+$	$\varpi' \in \{\varpi, \varepsilon\varpi\}$		$(1, 0, 1)$
$U_\varepsilon(1, 1)$	$\mathbb{Z}_+$			$(1, 0, 0)$ or $(0, 0, 1)$
$U_{\varpi'}(1, 1)$	$\frac{1}{2}\mathbb{Z}_+$	$\varpi' \in \{\varpi, \varepsilon\varpi\}$		(0,1,0)
$U_{\varpi', a} \times SL_2$	$\frac{1}{2}\mathbb{Z}_+$	(ϖ', a)	$a = 1$	(2,1,0)
			$a = \varpi$	(0,1,2)
$U_{\varepsilon, a} \times SL_2$	$\mathbb{Z}_+$	$a =$	1	$(1, 0, 0)$ or $(0, 1, 0)$
			ϖ	$(0, 1, 0)$ or $(0, 0, 1)$

Table 3.1.2.

3.1.3. Case 3: $d = 2$**.**

As before, the parameters in the table run over those in Tables 1.4.3 and 1.4.4.

$\vec{G}$	r_0	$r_1 = r_2 = d(\pi)$	parameters		x
$\vec{G}^\ell[1](a,b,\varpi'))$	$\frac{1}{2}\mathbb{Z}_+$	$\frac{1}{2}\mathbb{Z}_+$	(a,b)		$(1,0,1)$
$\vec{G}^\ell[2](\varpi',a,b)$	$\frac{1}{2}\mathbb{Z}_+$	$\mathbb{Z}_+$	(ϖ',a,b)	$b=1$	$(2,1,0)$
				$b=\varpi$	$(0,1,2)$
$\vec{G}^\ell[3](\varpi',a,b)$	$\mathbb{Z}_+$	$\frac{1}{2}\mathbb{Z}_+$	(ϖ',a,b)	$b=1$	$(2,1,0)$
				$b=\varpi$	$(0,1,2)$
$\vec{G}^\ell[4](\varpi',a,b)$	$\frac{1}{2}\mathbb{Z}_+$	$\frac{1}{2}\mathbb{Z}_+$	(ϖ',a,b)		$(1,0,1)$
$\vec{G}^\ell[5](a,b,a')$	$\mathbb{Z}_+$	$\mathbb{Z}_+$	(a,b,a')	$(1,1,a')$	$(1,0,0)$
				$(1,\varpi,a')$	$(0,1,0)$
				(ϖ,ϖ,a')	$(0,0,1)$
$\vec{G}^s[1](\alpha,\varpi')$	$\frac{1}{2}\mathbb{Z}_+$	$\frac{1}{2}\mathbb{Z}_+$	(α,ϖ')		$(1,0,1)$
$\vec{G}^s[2](\varpi',\alpha)$	$\mathbb{Z}_+$	$\frac{1}{2}\mathbb{Z}_+$	(ϖ',α)	$\alpha=1$	$(0,1,0)$
				$\alpha\neq 1$	$(1,0,1)$
$\vec{G}^s[3](\varpi',\alpha)$	$\frac{1}{2}\mathbb{Z}_+$	$\mathbb{Z}_+$	(ϖ',α)	$\alpha=1$	$(0,1,0)$
				$\alpha\neq 1$	$(1,0,1)$
$\vec{G}^s[4](\varpi',a,b)$	$\frac{1}{2}\mathbb{Z}_+$	$\frac{1}{2}\mathbb{Z}_+$	(ϖ',a,b)		$(1,0,1)$
$\vec{G}^s[5](a,b)$	$\mathbb{Z}_+$	$\mathbb{Z}_+$	(a,b)	$(1,1)$	$(1,0,0)$
				$(1,\varpi)$	$(0,1,0)$
				(ϖ,ϖ)	$(0,0,1)$

TABLE 3.1.3.

Remark. The above G-datums give inequivalent supercuspidal representations ([4]).

In the rest of the paper, we construct non supercuspidal types of Sp_4. Let M be a Levi subgroup of Sp_4. Suppose $(K_{\Sigma_M}, \rho_{\Sigma_M})$ is a supercuspidal type constructed from a generic M-datum Σ_M. The classification of supercuspidal representations (hence supercuspidal types) of all proper Levi subgroups in Sp_4 is well known. For each supercuspidal type on M with a generic M-datum Σ_M, we can construct a G-cover. In the rest of the paper, we give a G-datum for a G-cover in each case. The choice of ι is not unique. We will give one choice of ι satisfying genericity in each case. Once a G-datum is given, one can follow §2 or [12] to construct the G-cover.

In the following, we define the depth of a supercuspidal type as the depth of the supercuspidal representation with the same generic G-datum.

3.2. **Supercuspidal types on M^{long} and G-covers.**

To simplify notation in this section, we will write M for M^{long} if there is no confusion. Since $M \simeq F^\times \times SL_2$, we can write $\rho_{\Sigma_M} = \phi \otimes \rho'_{\Sigma_M}$ for a character ϕ of $F^\times$ and a supercuspidal type ρ'_{Σ_M} of SL_2. Note that we can extend ϕ trivially to a character of M. We will still use ϕ for the extended character.

3.2.1. Depth zero case.

Suppose ρ'_{Σ_M} is a depth zero supercuspidal type on SL_2. Then, Σ_M is of the form (M, y, ϕ, r, ρ_M) where $M_{y,0}$ is a maximal compact subgroup of $F^\times \times SL_2$ and $r = \mathrm{depth}(\phi)$ is an integer. Moreover, we have $(K_{\Sigma_M}, \rho_{\Sigma_M}) = (M_{y,0}, \phi \otimes \rho_M)$

Note that $M_{y,0}$, $y \in \mathcal{B}(M)$ is determined by $a_{\mathrm{long}}(y)$. In this case, we may assume that $a_{\mathrm{long}}(y) = 0$ or 1. Moreover, ι is uniquely determined by $\iota(y)$. We choose ι as follows:

y	$\iota(y)$
$a_{\mathrm{long}}(y) = 0$	$\begin{cases} (1,0,0) & \text{if } r \text{ is odd} \\ (2,1,0) & \text{if } r \text{ is even} \end{cases}$
$a_{\mathrm{long}}(y) = 1$	$\begin{cases} (0,0,1) & \text{if } r \text{ is odd} \\ (-2,1,4) & \text{if } r \text{ is even} \end{cases}$

Then, we can choose Σ as follows to construct a G-cover of $(M_{y,0}, \phi \otimes \rho_M)$.

Cases: Σ_M		$\iota(y)$	Σ
r	y		
$r = 0$	$a_{\text{long}}(y) = 0$	$(2, 1, 0)$	$((G, M), (y, \iota), (M_{y,0}, \phi \otimes \rho_M))$
	$a_{\text{long}}(y) = 1$	$(-2, 1, 4)$	
$r \neq 0$, even	$a_{\text{long}}(y) = 0$	$(2, 1, 0)$	$(((M^{\text{long}}, G), M^{\text{long}}), (y, \iota), (\phi \otimes 1, 1), (r, 0), (M_{y,0}, \rho_M))$
	$a_{\text{long}}(y) = 1$	$(-2, 1, 4)$	
r odd	$a_{\text{long}}(y) = 0$	$(1, 0, 0)$	
	$a_{\text{long}}(y) = 1$	$(0, 0, 1)$	

TABLE 3.2.1.

3.2.2. Positive depth cases.

Suppose ρ'_{Σ_M} is a supercuspidal type of positive depth on SL_2. Write $\Sigma_M = (\vec{M}, y, \vec{r}, \vec{\phi}, \rho_{M^0})$. Then, we have the following:

- $\ell(\Sigma_M) = 1$ and $\vec{M} = (M^0, M)$ where M^0 is either T[11](ϖ', a') or T[12](a) with $\varpi' \in \{\varpi, \varepsilon\varpi\}$, $a' \in \{1, \varepsilon\}$ and $a \in \{1, \varpi\}$ (see TABLE 1.3.6).
- Without loss of generality, one may assume that

$$a_{\text{long}}(y) = \begin{cases} \frac{1}{2} & \text{if } M^0 = \text{T}[11](\varpi', a') \\ 0 & \text{if } M^0 = \text{T}[12](1) \\ 1 & \text{if } M^0 = \text{T}[12](\varpi). \end{cases}$$

- Writing $\vec{\phi} = (\phi_0, \phi_1)$, ϕ_1 is a character which is trivial on SL_2. Without loss of generality, we may assume that either ϕ_1 is trivial or nontrivial of depth r_1.

In all cases, specifying $\vec{G}$ and ι as in the table below,

$$\Sigma = ((\vec{G}, M^0), (y, \iota), (\phi_0, \phi_1, 1), (r_0, r_1, r_1), (M^0_{y,0}, \rho_{M^0}))$$

gives a G-cover of $(K_{\Sigma_M}, \rho_{\Sigma_M})$.

Cases: Σ_M			$\vec{G}$	$\iota(y)$
M^0, y	ϕ_1	r_0, r_1		
T[11](ϖ', a') $a_{\text{long}}(y) = \frac{1}{2}$	$\phi_1 = 1$	$r_0 = r_1 \in \frac{1}{2}\mathbb{Z}$	$\vec{G}^\ell[8](\varpi', a')$	$(1, 0, 1)$
	$\phi_1 \neq 1$	r_1 even	$\vec{G}^\ell[6](\varpi', a')$	$(2, 1, 4)$
		r_1 odd	$\vec{G}^\ell[6](\varpi', a')$	$(0, 1, 2)$
T[12](1) $a_{\text{long}}(y) = 0$	$\phi_1 = 1$	$r_0 = r_1$ even	$\vec{G}^\ell[9](1)$	$(2, 1, 0)$
		$r_0 = r_1$ odd	$\vec{G}^\ell[9](1)$	$(1, 0, 0)$
	$\phi_1 \neq 1$	r_1 even	$\vec{G}^\ell[7](1)$	$(2, 1, 0)$
		r_1 odd	$\vec{G}^\ell[7](1)$	$(1, 0, 0)$
T[12](ϖ) $a_{\text{long}}(y) = 1$	$\phi_1 = 1$	$r_0 = r_1$ even	$\vec{G}^\ell[9](\varpi)$	$(-2, 1, 4)$
		$r_0 = r_1$ odd	$\vec{G}^\ell[9](\varpi)$	$(0, 0, 1)$
	$\phi_1 \neq 1$	r_1 even	$\vec{G}^\ell[7](\varpi)$	$(-2, 1, 4)$
		r_1 odd	$\vec{G}^\ell[7](\varpi)$	$(0, 0, 1)$

TABLE 3.2.2.

3.3. **Supercuspidal types on M^{short} and G-covers.**

In this section , to simplify the notation, write M for M^{short} if there is no confusion.

3.3.1. Essentially depth zero cases.

Suppose ρ_{Σ_M} is an essentially depth zero supercuspidal type on M, that is, it is a supercuspidal type up to twisting by a character of M. Then, Σ_M is of the form (M, y, ϕ, r, ρ_M) where $K_{\Sigma_M} = M_{y,0}$ is a maximal compact subgroup of GL_2 and $r = \mathrm{depth}(\rho_{\Sigma_M})$ is an integer. If $r = 0$, we may assume $\phi = 1$ without loss of generality.

Note that $M_{y,0}$, $y \in \mathcal{B}(M)$ is determined by $a_{\mathrm{short}}(y)$. In this case, we may assume that $a_{\mathrm{short}}(y) = 0$. Moreover, ι is completely determined by $\iota(y)$. Then, we can choose ι and Σ as follows to construct a G-cover.

r	$\iota(y)$	Σ
$r = 0$	$(1,0,1)$	$((G, M), (y, \iota), (M_{y,0}, \rho_M))$
$r \neq 0$ even	$(1,0,1)$	$(((M, G), M), (y, \iota), (r, 0), (\phi, 1), (M_{y,0}, \rho_M))$
r odd	$(1,0,0)$	

Table 3.3.1.

3.3.2. Positive depth cases.

Write $\Sigma_M = (\vec{M}, y, \vec{r}, \vec{\phi}, \rho_{M^0})$ as before. Then, we have the following:

- $\ell(\Sigma_M) = 1$ and $\vec{M} = (M^0, M)$ where M^0 is either T[13](ϖ'), $\varpi' \in \{\varpi, \varepsilon\varpi\}$, or T[14] (see Table 1.3.6).
- Without loss of generality, one may assume that $a_{\mathrm{short}}(y)$ is $\frac{1}{2}$ if $M^0 =$T[13](ϖ', a'), and 0 if $M^0 =$T[14].
- Write $\vec{r} = (r_0, r_1)$ and $\vec{\phi} = (\phi_0, \phi_1)$. If $r_0 = r_1$, we may assume that ϕ_1 is the trivial character.
- Let $\mathcal{Z}_M^\circ$ be the maximal compact subgroup of the center of M. If $\phi_0|\mathcal{Z}_M^\circ$ are trivial, ϕ_0 can be extended to a unitary group U containing M^0. That is, ϕ_0 can be extended to a character of $U_{\varpi'}(1,1)$ if $M^0 =$T[13](ϖ'), and to a character of $U_\varepsilon(1,1)$ if $M^0 =$T[14]. We use the same notation ϕ_0 for the extended character.

In all cases, for a given Σ_M as above, we take $\Sigma = ((\vec{G}, M^0), (y, \iota), \vec{r}, \vec{\phi}, (M^0_{y,0}, \rho_{M^0}))$ as in the following table:

Cases					Σ			
M^0	$\phi_0\vert\mathcal{Z}_M^\circ$	ϕ_1	r_0, r_1		$\iota(y)$	$\vec{G}$	$\vec{r}$	$\vec{\phi}$
T[13](ϖ')	$= 1$	$= 1$	$r_0 = r_1$		$(1,1,-1)$	$(U_{\varpi'}(1,1), G)$	(r_0, r_0)	$(\phi_0, 1)$
	$\neq 1$	$= 1$	$r_0 = r_1$		$(0,1,0)$	(M^0, G)	(r_0, r_0)	$(\phi_0, 1)$
		$\neq 1$	$r_0 < r_1$		$(1,2,-1)$	$\vec{G}^s[6](\varpi') = (M^0, M, G)$	(r_0, r_1, r_1)	$(\phi_0, \phi_1, 1)$
T[14]	$= 1$	$= 1$	$r_0 = r_1$	r_1 odd	$(1,0,0)$	$(U_\varepsilon(1,1), G)$	(r_0, r_0)	$(\phi_0, 1)$
				r_1 even	$(1,0,1)$			
	$\neq 1$	$= 1$	$r_0 = r_1$	r_1 odd	$(1,0,0)$	(M^0, G)	(r_0, r_0)	$(\phi_0, 1)$
				r_1 even	$(1,0,1)$			
		$\neq 1$	$r_0 < r_1$	r_1 odd	$(1,0,0)$	$\vec{G}^s[7] = (M^0, M, G)$	(r_0, r_1, r_1)	$(\phi_0, \phi_1, 1)$
				r_1 even	$(1,0,1)$			

Table 3.3.2.

3.4. *G*-covers of principal series.

The types for principal series are constructed in [16]. We will merely restate the result in loc. cit. in terms of the language in this paper. The supercuspidal representations of $M = T \simeq F^\times \times F^\times$ are of the form $\chi_1 \otimes \chi_2$ for characters χ_1 and χ_2 of $F^\times$. Without loss of generality, we may assume that $d(\chi_1) \geq d(\chi_2)$. and $\Sigma_M = (T, y, \chi_1 \otimes \chi_2, d(\chi_1), 1)$ for any $y \in \mathcal{B}(T)$. Let $r' = d(\chi_1\chi_2^{-1})$ and $r_1 = d(\chi_1)$. Fix ι so that

$$\iota(y) = \begin{cases} [-1,1,1] & \text{if } r',\ r_1 \in 2\mathbb{Z} \\ [1,0,0] & \text{if } r',\ r_1 \in 2\mathbb{Z}+1 \\ [1,0,1] & \text{if } r' \in 2\mathbb{Z}+1,\ r_1 \in 2\mathbb{Z} \\ [0,1,0] & \text{if } r' \in 2\mathbb{Z},\ r_1 \in 2\mathbb{Z}+1 \end{cases}$$

In each case, $\Sigma = ((\vec{G}, T), (y, \iota), \vec{\phi}, \vec{r}, (T_0, 1_{T_0}))$ with $(\vec{G}, \iota, \vec{\phi}, \vec{r})$ in the table gives rise to a cover of $(T_0, (\chi_1 \otimes \chi_2)|T_0)$.

<table>
<tr><th>cases</th><th>$\vec{G}$</th><th>$\vec{\phi}$</th><th>$\vec{r}$</th><th>$\iota(y)$</th></tr>
<tr><td>$r' = r_1$</td><td>$\vec{G}^{\ell}[10]$</td><td>$(1_{F^\times} \otimes \chi_2, \chi_1 \otimes 1_{SL_2}, 1)$</td><td>$(d(\chi_2), r_1, r_1)$</td><td rowspan="2">$\begin{cases} [-1,1,1] & \text{if } r',\ r_1 \in 2\mathbb{Z} \\ [1,0,0] & \text{if } r',\ r_1 \in 2\mathbb{Z}+1 \\ [1,0,1] & \text{if } r' \in 2\mathbb{Z}+1,\ r_1 \in 2\mathbb{Z} \\ [0,1,0] & \text{if } r' \in 2\mathbb{Z},\ r_1 \in 2\mathbb{Z}+1 \end{cases}$</td></tr>
<tr><td>$r' < r_1$</td><td>$\vec{G}^{s}[10]$</td><td>$(1 \otimes \chi_1^{-1}\chi_2, \chi_1 \circ \det, 1)$</td><td>$(r', r_1, r_1)$</td></tr>
</table>

TABLE 3.4.1.

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