A UNIFIED ANALYSIS OF BALANCING DOMAIN DECOMPOSITION BY CONSTRAINTS FOR DISCONTINUOUS GALERKIN DISCRETIZATIONS

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Abstract. The BDDC algorithm is extended to a large class of discontinuous Galerkin (DG) discretizations of second order elliptic problems. An estimate of $C(1 + \log(H/h))^2$ is obtained for the condition number of the preconditioned system where C is a constant independent of h or H or large jumps in the coefficient of the problem. Numerical simulations are presented which confirm the theoretical results. A key component for the development and analysis of the BDDC algorithm is a novel perspective presenting the DG discretization as the sum of element-wise "local" bilinear forms. The element-wise perspective allows for a simple unified analysis of a variety of DG methods and leads naturally to the appropriate choice for the subdomain-wise local bilinear forms. Additionally, this new perspective enables a connection to be drawn between the DG discretization and a related continuous finite element discretization to simplify the analysis of the BDDC algorithm.

Key words. Discontinuous Galerkin, Domain Decomposition, BDDC

AMS subject classifications. 65M55, 65M60, 65N30, 65N55

1. Introduction.

Domain decomposition (DD) methods provide efficient parallel preconditioners for solving large system of equations arising from the discretization of partial differential equations. The development of domain decompositions methods for the solution of elliptic problems using conforming finite element methods has matured significantly over the past 20 years. Toselli and Widlund provide a detailed overview of domain decomposition methods in [22]. In this paper we consider a class of non-overlapping domain decomposition methods based on the Neummann-Neumann methods originally introduced by Bourgat et al. [6]. These methods were improved by introducing a coarse space based on the null-space of the local Schur complement problems, leading to the Balancing Domain Decomposition (BDD) method of Mandel [17]. Dohrmann extended the BDD method by selecting a coarse space formed by enforcing continuity of a small set of primal degrees of freedom [11]. This Balancing Domain Decomposition by Constraints (BDDC) method was later proven by Mandel et al [18] to have a condition number bound of $\kappa \leq C(1 + \log(H/h))^2$ for preconditioned system of a continuous finite element discretization of second order elliptic problems. Further analysis of BDDC methods as well as the relationship between BDDC methods and dual-primal Finite Element Tearing and Interconnecting (FETI-DP) methods has been presented in [16, 7, 19].

In this paper we present a BDDC method for the solution of a discontinuous Galerkin (DG) discretization of a second-order elliptic problem. While domain decomposition methods have been widely studied for continuous finite element discretizations, relatively little work has been performed for discontinuous Galerkin discretizations. Feng and Karakashian presented a two-level Schwarz preconditioner for an interior penalty DG discretization of the Poisson problem [15]. Feng and Karakashian considered both overlapping and non-overlapping preconditioners and obtained condition number bounds of $O(H/\delta)$ and $O(H/h)$ respectively. Antonietti and Ayuso considered additive and multiplicative Schwarz preconditioners for a large class of DG discretizations of elliptic problems in [1, 2, 3]. Antonietti and Ayuso employed the unified framework of Arnold et al. [4] to analyze these DG methods and showed that condition number bounds of order $O(H/h)$ could be obtained with these preconditioners for symmetric DG schemes.

In the context of Neumann-Neumann type methods for DG discretizations, Dryja et al employed a conforming finite

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element discretization on each subdomain while using an interior penalty method across non-conforming subdomain boundaries [14, 12, 13]. Using this discretization Dryja et al were able to leverage results from the continuous finite element analysis to obtain condition number bounds of $\kappa \leq C(1+\log(H/h))^2$ for particular BDD and BDDC methods. In this work we present a BDDC method applied to a large class of DG methods considered in the unified analysis of Arnold et al. [4]. A key component for the development and analysis of the BDDC algorithm is a novel perspective presenting the DG discretization as the sum of element-wise "local" bilinear forms. The element-wise perspective leads naturally to the appropriate choice for the subdomain-wise local bilinear forms. Additionally, this new perspective enables a connection to be drawn between the DG discretization and a related continuous finite element discretization. By exploiting this connection, we prove a condition number bound of $\kappa \leq C(1+\log(H/h))^2$ for the BDDC preconditioned system for a large class of conservative and consistent DG methods.

In Section 2 we gives a classical presentation of the DG discretization. In Section 3 we present our new perspective on the DG discretization. In Sections 4 and 5, respectively, we discuss our domain decomposition strategy and present the BDDC algorithm. The analysis of the BDDC algorithm in presented in Section 6, while in Section 7 we present numerical results confirming the analysis.

2. DG Discretization. We consider the following second order elliptic equation in a domain $\Omega \subset \mathcal{R}^n$, $n = 2, 3$.

$$\begin{aligned} -\nabla\cdot(\rho\nabla u) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

with positive $\rho > 0 \in L^\infty(\Omega)$, $f \in L^2(\Omega)$. Following [4] we may rewrite (2.1) in mixed form in order to motivate the DG formulation. In practice, the fluxes are locally eliminated to obtain the DG discretization in primal form. The mixed form of (2.1) is given by:

$$\begin{aligned} \rho^{-1}\boldsymbol{q} + \nabla u &= 0 && \\ \nabla\cdot\boldsymbol{q} &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

Prior to introducing the exact form of the discrete equations, we introduce the functional setting and notation. Denote by $\mathcal{T}_h$ the family of triangulations obtained by partitioning Ω into triangles or quadrilaterals (if $n = 2$) or tetrahedra or hexahedra (if $n = 3$), with characteristic element size h. We make the usual assumption that the family of triangulations $\mathcal{T}_h$ is shape-regular, and quasi-uniform [22]. Define $\mathcal{E}$ to be the union of edges (if $n = 2$) or faces (if $n = 3$) of elements κ. Additionally, define $\mathcal{E}^i \subset \mathcal{E}$ and $\mathcal{E}^\partial \subset \mathcal{E}$ to be the set of interior, respectively boundary edges. We note that any edge $e \in \mathcal{E}^i$ is shared by two adjacent elements κ^+ and κ^- with corresponding outward pointing normal vectors $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$.

Let $\mathcal{P}^p(\kappa)$ denote the space of polynomials of order at most p on κ and define $\mathbf{P}^p(\kappa) := [\mathcal{P}^p(\kappa)]^n$. Given the triangulation $\mathcal{T}_h$ define the following finite element spaces

$$W_h^p := \{w_h \in \mathbf{L}^2(\Omega) : w_h|_\kappa \in \mathcal{P}^p(\kappa) \quad \forall \kappa \in \mathcal{T}_h\} \tag{2.3}$$

$$\mathbf{V}_h^p := \{\mathbf{v}_h \in \mathbf{L}^2(\Omega) : \mathbf{v}_h|_\kappa \in \mathbf{P}^p(\kappa) \quad \forall \kappa \in \mathcal{T}_h\} \tag{2.4}$$

Note that traces of functions $u_h \in W_h^p$ are in general double valued on each edge, $e \in \mathcal{E}^i$, with values u_h^+ and u_h^- corresponding to traces from elements κ^+ and κ^- respectively. On $e \in \mathcal{E}^\partial$, associate u_h^+ with the trace taken from the element, $\kappa^+ \in \mathcal{T}_h$, neighbouring e. The DG discretization of (2.1) obtains a solution $u_h \in W_h^p$ such that for all $\kappa \in \mathcal{T}_h$,

$$(\rho\nabla u_h, \nabla w_h)_\kappa - \left\langle \rho^+(u_h^+ - \hat{u}_h)\boldsymbol{n}^+, \nabla w_h^+ \right\rangle_{\partial\kappa} + \left\langle \hat{\boldsymbol{q}}_h, w_h^+\boldsymbol{n}^+ \right\rangle_{\partial\kappa} = (f, w_h)_\kappa \qquad \forall w_h \in \mathcal{P}^p(\kappa) \tag{2.5}$$

where $(\cdot,\cdot)_\kappa := \int_\kappa$ and $\langle\cdot,\cdot\rangle_{\partial\kappa} := \int_{\partial\kappa}$. Superscript $^+$ is used to explicitly denote values on $\partial\kappa$, taken from κ. For all $w_h \in W_h^p$, $\hat{w}_h = \hat{w}_h(w_h^+, w_h^-)$ is a single valued numerical trace on $e \in \mathcal{E}^i$, while $\hat{w}_h = 0$ for $e \in \mathcal{E}^\partial$. Note that $\hat{u}_h = 0$ on $e \in \mathcal{E}^\partial$, corresponds to weakly enforced homogeneous boundary conditions on $\partial\Omega$. Similarly $\hat{\boldsymbol{q}}_h = \hat{\boldsymbol{q}}_h(\nabla u_h^+, \nabla u_h^-, u_h^+, u_h^-, \rho^+, \rho^-)$ is a single valued numerical flux on $e \in \mathcal{E}$. Summing (2.5) over all elements gives the complete DG discretization: Find $u_h \in W_h^p$ such that

$$a(u_h, w_h) = (f, w_h)_\Omega \qquad \forall w_h \in W_h^p \tag{2.6}$$

Following [4], a piecewise discontinuous numerical approximation of the flux, $\boldsymbol{q}_h$, may be evaluated locally as:

$$\boldsymbol{q}_h = -(\rho\nabla u_h - \rho^{\frac{1}{2}} r_\kappa(\rho^{\frac{1}{2}^+}(u_h^+ - \hat{u}_h)\boldsymbol{n}^+)), \tag{2.7}$$

where $r_\kappa(\phi) \in \mathbf{P}^p(\kappa)$ is defined by:

$$(r_\kappa(\phi), \mathbf{v}_h)_\kappa = \left\langle \phi, \mathbf{v}_h^+ \right\rangle_{\partial\kappa} \qquad \forall \mathbf{v}_h \in \mathbf{P}^p(\kappa) \tag{2.8}$$

We note that while ∇u_h and $r_\kappa(\rho^{\frac{1}{2}^+}(u_h^+ - \hat{u}_h)\boldsymbol{n}^+)$ lie in the polynomial space $\mathbf{P}^p(\kappa)$, $\boldsymbol{q}_h$, in general, does not when ρ varies within an element κ. The DG discretizations presented in this paper lift $\rho^{\frac{1}{2}}\nabla u$ (as opposed to ∇u or $\rho\nabla u$) to ensure that the discretization is symmetric for any $\rho \in L^\infty(\Omega)$. In the case of piecewise constant ρ the DG formulations lifting ∇u, $\rho^{\frac{1}{2}}\nabla u$ or $\rho\nabla u$ are identical.

The choice of the numerical trace $\hat{u}_h$ and flux $\hat{\boldsymbol{q}}_h$ define the particular DG method considered. Table 2.1 lists the numerical traces and fluxes for the DG methods considered in this paper. In the definition of the different DG methods, the following average and jump operators are used to define the numerical trace and flux on $e \in \mathcal{E}^i$:

$$\{u_h\} = \frac{1}{2}(u_h^+ + u_h^-) \qquad \text{and} \qquad [\![u_h]\!] = u_h^+\boldsymbol{n}^+ + u_h^-\boldsymbol{n}^- \tag{2.9}$$

Additionally we define a second set of jump operators involving the numerical trace $\hat{u}$:

$$[\![u_h]\!]^+ = u_h^+\boldsymbol{n}^+ + \hat{u}_h\boldsymbol{n}^- \qquad \text{and} \qquad [\![u_h]\!]^- = \hat{u}_h\boldsymbol{n}^+ + u_h^-\boldsymbol{n}^- \tag{2.10}$$

such that we may express $\boldsymbol{q}_h$ as:

$$\boldsymbol{q}_h = -(\rho\nabla u_h - \rho^{\frac{1}{2}} r_\kappa(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+)) \tag{2.11}$$

Method	$\hat{u}_h$	$\hat{\boldsymbol{q}}_h$
Interior Penalty	$\{u_h\}$	$-\{\rho\nabla u_h\} + \frac{\eta_e}{h}\left\{\rho\,[\![u_h]\!]^\pm\right\}$
Bassi and Rebay [5]	$\{u_h\}$	$-\{\rho\nabla u_h\} + \eta_e\left\{\rho^{\frac{1}{2}} r_e(\rho^{\frac{1}{2}}\,[\![u_h]\!]^\pm)\right\}$
Brezzi et al. [8]	$\{u_h\}$	$\{\boldsymbol{q}_h\} + \eta_e\left\{\rho^{\frac{1}{2}} r_e(\rho^{\frac{1}{2}}\,[\![u_h]\!]^\pm)\right\}$
LDG [9]	$\{u_h\} - \beta\cdot[\![u_h]\!]$	$\{\boldsymbol{q}_h\} + \beta\,[\![\boldsymbol{q}_h]\!] + \frac{2\eta_e}{h}\left\{\rho\,[\![u_h]\!]^\pm\right\}$
CDG [20]	$\{u_h\} - \beta\cdot[\![u_h]\!]$	$\{\boldsymbol{q}_h^e\} + \beta\,[\![\boldsymbol{q}_h^e]\!] + \frac{2\eta_e}{h}\left\{\rho\,[\![u_h]\!]^\pm\right\}$

Table 2.1
Numerical fluxes for different DG methods

We note that in the definition of the different DG methods, η_e is a penalty parameter defined on each edge in $\mathcal{E}$, while $r_e(\phi) \in \mathbf{P}^p(\kappa)$ is a local lifting operator defined by:

$$(r_e(\phi), \mathbf{v}_h)_\kappa = \left\langle \phi, \mathbf{v}_h^+ \right\rangle_e \qquad \forall \mathbf{v}_h \in \mathbf{P}^p(\kappa) \tag{2.12}$$

Additionally $\boldsymbol{q}^e$ is given by:

$$\boldsymbol{q}_h^e = -(\rho \nabla u_h - \rho^{\frac{1}{2}} r_e(\rho^{\frac{1}{2}^+} [\![u]\!]^+)) \tag{2.13}$$

For the Local Discontinuous Galerkin (LDG) and Compact Discontinuous Galerkin (CDG) methods, β is a vector which is defined on each edge/face in $\mathcal{E}^i$ as

$$\beta = \frac{1}{2}\left(S_{\kappa^+}^{\kappa^-}\boldsymbol{n}^+ + S_{\kappa^-}^{\kappa^+}\boldsymbol{n}^-\right) \tag{2.14}$$

where $S_{\kappa^+}^{\kappa^-} \in \{0,1\}$ is a switch defined on each face of element κ^+ shared with element κ^-, such that

$$S_{\kappa^+}^{\kappa^-} + S_{\kappa^-}^{\kappa^+} = 1. \tag{2.15}$$

3. The DG discretization from a new perspective. A key component, required for the development and analysis of the algorithms presented, is to express the global bilinear form $a(u_h, w_h)$ as the sum of element-wise contributions $a_\kappa(u_h, w_h)$ such that

$$a(u_h, w_h) = \sum_{\kappa \in \mathcal{T}_h} a_\kappa(u_h, w_h) \tag{3.1}$$

where $a_\kappa(u_h, w_h)$ is a symmetric, positive semi-definite "local bilinear form". In particular, we wish the local bilinear form to have a compact stencil, such that $a_\kappa(u_h, w_h)$ is a function of only u_h, ∇u_h in κ, and u_h^+, ∇u_h^+, ρ^- and $\hat{u}_h$ on $\partial\kappa$. In particular, we note that in (2.5), which is summed over all elements to give $a(u_h, w_h)$, $\hat{\boldsymbol{q}}$ depends in general upon u^+, u^-, ∇u^+, ∇u^-, ρ^+ and ρ^- . We write that local bilinear form as:

$$\begin{aligned} a_\kappa(u_h, w_h) &= (\rho \nabla u_h, \nabla w_h)_\kappa - \left\langle \rho^+(u_h^+ - \hat{u}_h)\boldsymbol{n}^+, \nabla w_h^+ \right\rangle_{\partial\kappa} + \left\langle \hat{\boldsymbol{q}}_h^+, (w_h^+ - \hat{w}_h)\boldsymbol{n}^+ \right\rangle_{\partial\kappa} \\ &= (\rho \nabla u_h, \nabla w_h)_\kappa - \left\langle \rho^+ [\![u]\!]_h^+, \nabla w_h^+ \right\rangle_{\partial\kappa} + \left\langle \hat{\boldsymbol{q}}_h^+, [\![w_h]\!]^+ \right\rangle_{\partial\kappa} \end{aligned} \tag{3.2}$$

where $\hat{\boldsymbol{q}}_h^+ = \hat{\boldsymbol{q}}_h^+(\nabla u_h^+, u_h^+, \hat{u}_h, \rho^+)$ is a "local numerical flux". In particular, in order to recover the original global bilinear form, $\hat{\boldsymbol{q}}_h^\pm$ must satisfy the following relationship on each edge, e:

$$\hat{\boldsymbol{q}}_h [\![w_h]\!] = \hat{\boldsymbol{q}}_h^+ [\![w_h]\!]^+ + \hat{\boldsymbol{q}}_h^- [\![w_h]\!]^- \qquad \forall w_h \in W_h^p \tag{3.3}$$

Table 3.1 lists the numerical traces and local fluxes for the DG methods considered, while Table 3.2 lists the corresponding local bilinear forms. It is simple to verify that (3.3) holds for each of the DG methods considered by using the identities:

$$[\![u_h]\!] = [\![u_h]\!]^+ + [\![u_h]\!]^- \qquad \text{and} \qquad \begin{cases} [\![u_h]\!]^+ = [\![u_h]\!]^- = \frac{1}{2}[\![u_h]\!] & \text{if } \hat{u}_h = \{u_h\} \\ [\![u_h]\!]^+ = [\![u_h]\!], \quad [\![u_h]\!]^- = 0 & \text{if } \hat{u}_h = \{u_h\} - \beta[\![u_h]\!] \text{ and } S_{\kappa^+}^{\kappa^-} = 1 \\ [\![u_h]\!]^+ = 0, \quad [\![u_h]\!]^- = [\![u_h]\!] & \text{if } \hat{u}_h = \{u_h\} - \beta[\![u_h]\!] \text{ and } S_{\kappa^+}^{\kappa^-} = 0 \end{cases} \tag{3.4}$$

We now make an observation on the degrees of freedom involved in the local bilinear form, $a_\kappa(u_h, w_h)$. We consider using a nodal basis on each element κ to define W_h^p. For the Interior Penalty (IP) method and the methods of Bassi and Rebay, and Brezzi et al., the numerical trace $\hat{u}_h$ on an edge/face depends on both u_h^+ and u_h^-. Hence the local bilinear form corresponds to all nodal degrees of freedom defining u_h on κ as well as nodal values on all edge/faces of $\partial\kappa \cap \mathcal{E}^i$ corresponding to the trace of u_h from elements neighbouring κ. On the other hand, for the LDG and CDG methods, the numerical trace $\hat{u}_h$ takes on the value of u_h^+ if $S_{\kappa^+}^{\kappa^-} = 0$ or u_h^- if $S_{\kappa^+}^{\kappa^-} = 1$. Hence the local bilinear form corresponds only to degrees of freedom defining u_h on κ and nodal values corresponding to the trace of u_h on neighbouring elements across edge/faces of $\partial\kappa \cap \mathcal{E}^i$ for which $S_{\kappa^+}^{\kappa^-} = 1$.

Method	$\hat{u}_h$	$\hat{\boldsymbol{q}}_h^+$
Interior Penalty	$\{u_h\}$	$-\rho^+\nabla u_h^+ + \frac{\eta_e}{h}\rho^+ [\![u_h]\!]^+$
Bassi and Rebay [5]	$\{u_h\}$	$-\rho^+\nabla u_h^+ + \eta_e \rho^{\frac{1}{2}^+} r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+)$
Brezzi et al. [8]	$\{u_h\}$	$\boldsymbol{q}_h^+ + \eta_e \rho^{\frac{1}{2}^+} r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+)$
LDG [9]	$\{u_h\} - \beta \cdot [\![u_h]\!]$	$\boldsymbol{q}_h^+ + \frac{\eta_e}{h}\rho^+ [\![u_h]\!]^+$
CDG [20]	$\{u_h\} - \beta \cdot [\![u_h]\!]$	$\boldsymbol{q}_h^{e+} + \frac{\eta_e}{h}\rho^+ [\![u_h]\!]^+$

TABLE 3.1
Numerical fluxes for different DG methods

Method	$a_\kappa(u_h, w_h)$
Interior Penalty	$g + \sum_{e\in\partial\kappa} \frac{\eta_e}{h_e} \left\langle \rho^+ [\![u_h]\!]^+, [\![w_h]\!]^+ \right\rangle_e$
Bassi and Rebay [5]	$g + \sum_{e\in\partial\kappa} \eta_e \left(r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_e(\rho^{\frac{1}{2}^+} [\![w_h]\!]^+) \right)_\kappa$
Brezzi et al. [8]	$g + \left(r_\kappa(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_\kappa(\rho^{\frac{1}{2}^+} [\![w_h]\!]^+) \right)_\kappa + \sum_{e\in\partial\kappa} \eta_e \left(r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_e(\rho^{\frac{1}{2}^+} [\![w_h]\!]^+) \right)_\kappa$
LDG [9]	$g + \left(r_\kappa(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_\kappa(\rho^{\frac{1}{2}^+} [\![w_h]\!]^+) \right)_\kappa + \sum_{e\in\partial\kappa} \frac{\eta_e}{h_e} \left\langle \rho^+ [\![u_h]\!]^+, [\![w_h]\!]^+ \right\rangle_e$
CDG [20]	$g + \sum_{e\in\partial\kappa} \left(r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_e(\rho^{\frac{1}{2}^+} [\![w_h]\!]^+) \right)_\kappa + \sum_{e\in\partial\kappa} \frac{\eta_e}{h_e} \left\langle \rho^+ [\![u_h]\!]^+, [\![w_h]\!]^+ \right\rangle_e$

Where $g = (\rho\nabla u_h, \nabla w_h)_\kappa - \left\langle \rho^+ [\![u_h]\!]^+, \nabla w_h^+ \right\rangle_{\partial\kappa} - \left\langle \rho^+ \nabla u_h, [\![w_h]\!]^+ \right\rangle_{\partial\kappa}$

TABLE 3.2
Elementwise bilinear form for different DG methods

We denote by ρ_κ the average value of $\rho(\mathbf{x})$ on each element κ and assume that the variation of $\rho(\mathbf{x})$ within an element is uniformly bounded as:

$$c_\rho \rho_\kappa \leq \rho(\mathbf{x}) \leq C_\rho \rho_\kappa \qquad \forall \mathbf{x} \in \kappa, \quad \forall \kappa \tag{3.5}$$

where the constants c_ρ and C_ρ are independent of ρ_κ.

We now give the following lemma regarding the local bilinear form $a_\kappa(u_h, w_h)$.

LEMMA 3.1. *The element-wise bilinear form $a_\kappa(u_h, u_h)$ satisfies*

$$a_\kappa(u_h, u_h) \geq 0 \tag{3.6}$$

with $a_\kappa(u_h, u_h) = 0$ iff $u_h = \hat{u}_h = K$ for some constant K.

Proof. We proceed to show that Lemma 3.1 holds for all of the DG methods considered. The proof of Lemma 3.1 closely follows the proof of boundedness and stability of the different DG methods presented in Arnold et al. [4], though here we consider the contribution of a single element.

For each of the DG methods considered we can show $u_h = \hat{u}_h = K \Rightarrow a_\kappa(u_h, u_h) = 0$ by recognizing $u_h = K \Rightarrow \nabla u_h = 0$ and substituting into the different bilinear forms. It remains to prove $a_\kappa(u_h, u_h) \geq 0$ and $a_\kappa(u_h, u_h) = 0 \Rightarrow u_h = \hat{u}_h = K$.

In order to prove the result for the interior penalty method we employ the following result from Arnold et al [4]:

$$c\left(r_e(\rho^{\frac{1}{2}} w), r_e(\rho^{\frac{1}{2}} w) \right)_\kappa \leq \frac{1}{h_e} \langle \rho w, w \rangle_e \leq C \left(r_e(\rho^{\frac{1}{2}} w), r_e(\rho^{\frac{1}{2}} w) \right)_\kappa \qquad \forall w \in W_h^p \tag{3.7}$$

where c and C are constants which depend only upon the minimum angle of κ the polynomial order p and the constants in (3.5). Hence, choosing η_e sufficiently large for the interior penalty method we have

$$a_{\kappa,\mathrm{IP}}(u_h, u_h) \geq a_{\kappa,\mathrm{BR2}}(u_h, u_h) \tag{3.8}$$

and hence it is sufficient to show that Lemma 3.1 holds for the method of Bassi and Rebay [5]. Specifically, η_e may be chosen for the interior penalty method as described in Shahbazi [21]. For the method of Bassi and Rebay,

$$\begin{aligned}
a_{\kappa,\mathrm{BR2}}(u_h, u_h) &= (\rho \nabla u_h, \nabla u_h)_\kappa - 2\left\langle \rho \nabla u_h, [\![u_h]\!]^+ \right\rangle_{\partial\kappa} + \sum_{e \in \partial\kappa} \eta_e \left(r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) \right)_\kappa \\
&= (\rho \nabla u_h, \nabla u_h)_\kappa - \sum_{e \in \partial\kappa} 2 \left(\rho^{\frac{1}{2}} \nabla u_h, r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) \right)_\kappa + \sum_{e \in \partial\kappa} \eta_e \left(r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) \right)_\kappa \\
&\geq \sum_{e \in \partial\kappa} \frac{1}{N_e} \left(\rho^{\frac{1}{2}} \nabla u_h - r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), \rho^{\frac{1}{2}} \nabla u_h - r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) \right)_\kappa \\
&\qquad + \sum_{e \in \partial\kappa} (\eta_e - N_e) \left(r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) \right)_\kappa \\
&\geq 0
\end{aligned} \tag{3.9}$$

given $\eta_e > N_e$, where N_e is the number of edge/faces of κ. In order to show $a_{\kappa,\mathrm{BR2}}(u_h, u_h) = 0 \Rightarrow u_h = \hat{u}_h = K$, we note $a_{\kappa,\mathrm{BR2}}(u_h, u_h) = 0$ implies

$$\sum_{e \in \partial\kappa} \frac{1}{N_e} \left(\rho^{\frac{1}{2}} \nabla u_h - r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), \rho^{\frac{1}{2}} \nabla u_h - r_e(\rho^{\frac{1}{2}} [\![u_h]\!]^+) \right)_\kappa + \sum_{e \in \partial\kappa} (\eta_e - N_e) \left(r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) \right)_\kappa = 0 \tag{3.10}$$

Hence $r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) = 0$ and $\rho^{\frac{1}{2}} \nabla u_h - r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) = 0$ which implies $\hat{u}_h = u_h^+$ on $\partial\kappa$ and $\nabla u_h = 0$ in κ.

Proof of the method of Brezzi et al. [8] follows in a similar manner. Namely:

$$\begin{aligned}
a_{\kappa,\text{Brezzi et al.}}(\rho u_h, u_h) &= (\rho \nabla u_h, \nabla u_h)_\kappa - 2\left\langle \rho^+ \nabla u_h, [\![u_h]\!]^+ \right\rangle_{\partial\kappa} + \left(r_\kappa(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_\kappa(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) \right)_\kappa \\
&\qquad + \sum_{e \in \partial\kappa} \eta_e \left(r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) \right)_\kappa \\
&\geq \left(\rho^{\frac{1}{2}} \nabla u_h - r_\kappa(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), \rho^{\frac{1}{2}} \nabla u_h - r_\kappa(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) \right)_\kappa + \sum_{e \in \partial\kappa} \eta_e \left(r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) \right)_\kappa \\
&\geq 0
\end{aligned} \tag{3.11}$$

provided $\eta_e > 0$. In order to show $a_{\kappa,\text{Brezzi et al}}(u_h, u_h) = 0 \Rightarrow u_h = \hat{u}_h = K$, we note $a_{\kappa,\text{Brezzi et al}}(u_h, u_h) = 0$ implies

$$\left(\rho^{\frac{1}{2}} \nabla u_h - r_\kappa(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+)), \rho^{\frac{1}{2}} \nabla u_h - r_\kappa(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) \right)_\kappa + \sum_{e \in \partial\kappa} \eta_e \left(r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+), r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) \right)_\kappa = 0 \tag{3.12}$$

Hence $r_e(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) = 0$ and $\rho^{\frac{1}{2}} \nabla u_h - r_\kappa(\rho^{\frac{1}{2}^+} [\![u_h]\!]^+) = 0$ which implies $\hat{u}_h = u_h^+$ on $\partial\kappa$ and $\nabla u_h = 0$ in κ.

For the LDG method we have

$$\begin{aligned} a_{\kappa,\mathrm{LDG}}(u_h,u_h) &= (\rho\nabla u_h,\nabla u_h)_\kappa - 2\left\langle \rho^+\nabla u_h, [\![u_h]\!]^+\right\rangle_{\partial\kappa} + \left(r_\kappa(\rho^{\frac{1}{2}^+}[\![u_h]\!]^+), r_\kappa(\rho^{\frac{1}{2}^+}[\![u_h]\!]^+)\right)_\kappa \\ &\quad + \sum_{e\in\partial\kappa}\frac{\eta_e}{h_e}\left\langle \rho^+[\![u_h]\!]^+, [\![u_h]\!]^+\right\rangle_e \\ &= \left(\rho^{\frac{1}{2}}\nabla u_h - r_\kappa(\rho^{\frac{1}{2}^+}[\![u_h]\!]^+), \rho^{\frac{1}{2}}\nabla u_h - r_\kappa(\rho^{\frac{1}{2}^+}[\![u_h]\!]^+)\right)_\kappa + \sum_{e\in\partial\kappa}\frac{\eta_e}{h_e}\left\langle \rho^+[\![u_h]\!]^+, [\![u_h]\!]^+\right\rangle_e \\ &\geq 0 \end{aligned} \tag{3.13}$$

Setting $\eta_e > 0$ ensures $a_{\kappa,\mathrm{LDG}}(u_h,u_h) = 0 \Rightarrow u_h = \hat{u}_h = K$. Namely, $a_{\kappa,\mathrm{LDG}}(u_h,u_h) = 0$ implies

$$\left(\rho^{\frac{1}{2}}\nabla u_h - r_\kappa(\rho^{\frac{1}{2}^+}[\![u_h]\!]^+), \rho^{\frac{1}{2}}\nabla u_h - r_\kappa(\rho^{\frac{1}{2}^+}[\![u_h]\!]^+)\right)_\kappa + \sum_{e\in\partial\kappa}\frac{\eta_e}{h_e}\left\langle \rho^+[\![u_h]\!]^+, [\![u_h]\!]^+\right\rangle_e = 0 \tag{3.14}$$

Hence $[\![u_h]\!]^+ = 0$ and $\rho^{\frac{1}{2}}\nabla u_h + r_\kappa(\rho^{\frac{1}{2}^+}[\![u_h]\!]^+) = 0$, which implies $\nabla u_h = 0$.

Finally for the CDG method, we again use (3.7) and note that if η_e is chosen sufficiently large for the CDG method then we have

$$a_{\kappa,\mathrm{CDG}}(u_h,u_h) \geq a_{\kappa,\mathrm{BR2}}(u_h,u_h) \tag{3.15}$$

Hence, proof of Lemma 3.1 for the CDG method thus follows directly from the proof for the method of Bassi and Rebay. □

We now parameterize the space W_h^p using a standard nodal basis defined at nodes $\mathbf{x}$ on each element κ. The following lemmas show that the bilinear form is equivalent to a quadratic form based on the value of u_h at the nodes $\mathbf{x}$.

Lemma 3.2. *There exist constants c and C independent of h and ρ_κ such that for all $u_h \in W_h^p$*

$$c a_\kappa(u_h,u_h) \leq \rho_\kappa h_\kappa^{n-2}\sum_{\mathbf{x}_i,\mathbf{x}_j\in\kappa\cup\kappa'}(u_h(\mathbf{x}_i)-u_h(\mathbf{x}_j))^2 \leq C a_\kappa(u_h,u_h) \tag{3.16}$$

where $\mathbf{x}_i,\mathbf{x}_j$ are the nodes on κ defining the basis for u_h and nodes on $\partial\kappa'$ defining a basis for the trace u_h^- from neighbours κ' of κ. (We note that for the LDG and CDG methods nodes $\mathbf{x}_i,\mathbf{x}_j$ include nodes defining a basis for u_h^- only on faces for which $S_{\kappa^+}^{\kappa^-} = 1$.)

Proof. Lemma 3.2 is a direct consequence of Lemma 3.1 and a scaling argument. See [10] Lemma 4.3 for the equivalent proof for a mixed finite element discretization. □

We note that constants c and C in Lemma 3.2 depend, in general, on the polynomial order p. Throughout this paper all generic constants will, unless explicitly stated otherwise, depend on the polynomical order p.

Lemma 3.3. *Consider a region $\omega \subset \Omega$ composed of elements in $\mathcal{T}_h$. Denote by ρ_ω the average value of ρ on ω and suppose that ρ is uniformly bounded on ω such that exists constants c_ρ and C_ρ independent of ρ_ω*

$$c_\rho\rho_\omega \leq \rho \leq C_\rho\rho_\omega. \tag{3.17}$$

Then there exist different constants c and C independent of h, $|\omega|$ and ρ_ω such that for all $u_h \in W_h^p$

$$c a_\omega(u_h,u_h) \leq \rho_\omega h^{n-2}\sum_{\kappa\in\omega}\sum_{\mathbf{x}_i,\mathbf{x}_j\in\kappa\cup\kappa'}(u_h(\mathbf{x}_i)-u_h(\mathbf{x}_j))^2 \leq C a_\omega(u_h,u_h) \tag{3.18}$$

Proof. Lemma 3.3 follows directly from Lemma 3.2 and a summation over all element $\kappa \in \omega$. Note, we have used the assumption of a quasi-uniform family of triangulations (namely, $h_\kappa \leq C_h h$ for C_h independent of h) to replace h_κ with h while ensuring that the constants in Lemma 3.3 are independent of h. Similarly, the bound in (3.17), allows us to replace ρ_κ with ρ_ω while ensuring the constants are independent of ρ_ω. Clearly, the constant in Lemma 3.3 will depend in general upon C_h, c_ρ and C_ρ.
□

4. Domain Decomposition. In this section we present a domain decomposition of the discrete form of the DG discretization and derive a Schur complement problem for the interfaces between subdomains. The presentation of the BDDC algorithm follows that presented in [16] for the case of continuous finite elements. We consider a partition of the domain Ω into substructures Ω_i such that $\bar{\Omega} = \cup_{i=1}^N \bar{\Omega}_i$. The substructures Ω_i are disjoint shape regular polygonal regions of diameter $O(H)$, consisting of a union of elements in $\mathcal{T}_h$.

We denote by ρ_i the average value of $\rho(\mathbf{x})$ on Ω_i. We assume that large jumps in $\rho(\mathbf{x})$ are aligned with the subdomain interfaces such that $\rho(\mathbf{x})$ and ρ_κ may be uniformly bounded as:

$$c_\rho \rho_i \leq \rho(\mathbf{x}) \leq C_\rho \rho_i \qquad \forall \mathbf{x} \in \Omega_i, \quad \forall \Omega_i \tag{4.1}$$

$$c_\rho \rho_i \leq \rho_\kappa \leq C_\rho \rho_i \qquad \forall \kappa \in \Omega_i, \quad \forall \Omega_i \tag{4.2}$$

with constants c_ρ and C_ρ independent of ρ_i. We also make the following assumption:

ASSUMPTION 4.1. *Each element κ in Ω_i with an edge/face e on $\partial\Omega_i \cap \partial\Omega_j$ has neighbours only in $\Omega_i \cup \Omega_j$.* We note that while this assumption may appear limiting, in practice it is always possible to locallly split elements on corners/edges in 2D/3D respectively in order to satisfy this requirement.

We next define the local interface $\Gamma_i = \partial\Omega_i \backslash \partial\Omega$ and global interface Γ by $\Gamma = \cup_{i=1}^N \Gamma_i$. We denote by $W_\Gamma^{(i)}$ the space of discrete nodal values on Γ_i which correspond to degrees of freedom shared between Ω_i and neighbouring subdomains Ω_j, while $W_I^{(i)}$ denotes the space of discrete unknowns local to a single substructure Ω_i. In particular, we note that for the Interior penalty method, and the methods of Bassi and Rebay, and Brezzi et al. $W_\Gamma^{(i)}$ includes for each edge/face $e \in \Gamma_i$ degrees of freedom defining two sets of trace values u^+ from $\kappa^+ \in \Omega_i$ and u^- for $\kappa^- \in \Omega_j$. Thus, $W_I^{(i)}$ corresponds to nodal values strictly interior to Ω_i or on $\partial\Omega_i \backslash \Gamma_i$. On the other hand, for the CDG and LDG methods $W_\Gamma^{(i)}$ includes for each edge/face $e \in \Gamma_i$ degrees of freedom defining a single trace value corresponding to either u^+ from $\kappa^+ \in \Omega_i$ if $S_{\kappa^+}^{\kappa^-} = 0$ or u^- from $\kappa^- \in \Omega_j$ if $S_{\kappa^+}^{\kappa^-} = 1$. Hence, $W_I^{(i)}$ corresponds to nodal values interior to Ω_i and on $\partial\Omega_i \backslash \Gamma_i$ as well as nodal values defining u^+ on $e \in \Gamma_i$ for which $S_{\kappa^+}^{\kappa^-} = 1$.

Similarly, we define the spaces $\hat{W}_\Gamma$ and W_I which correspond to the space of discrete unknowns associated with coupled degrees of freedom on Γ and local degrees of freedom on substructures Ω_i respectively. We note that W_I is equal to the product of spaces $W_I^{(i)}$ (i.e. $W_I := \Pi_{i=1}^N W_I^{(i)}$), while in general $\hat{W}_\Gamma \subset W_\Gamma := \Pi_{i=1}^N W_\Gamma^{(i)}$. We define local operators $R_\Gamma^{(i)} : \hat{W}_\Gamma \to W_\Gamma^{(i)}$ which extract the local degrees of freedom on Γ_i from those on Γ. Additionally we define a global operator $R_\Gamma : \hat{W}_\Gamma \to W_\Gamma$ which is formed by a direct assembly of $R_\Gamma^{(i)}$.

We write the discrete form of (2.6) as:

$$\begin{bmatrix} A_{II} & A_{\Gamma I}^T \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_I \\ u_\Gamma \end{bmatrix} = \begin{bmatrix} b_I \\ b_\Gamma \end{bmatrix}. \tag{4.3}$$

where u_I and u_Γ corresponds to degrees of freedom associated with W_I and $\hat{W}_\Gamma$ respectively. Since the degrees of freedom associated with W_I are local to a particular substructure we may locally eliminate them to obtain a system

$$\hat{S}_\Gamma u_\Gamma = g_\Gamma \tag{4.4}$$

where

$$\hat{S}_\Gamma = A_{\Gamma\Gamma} - A_{\Gamma I} A_{II}^{-1} A_{\Gamma I}^T \qquad\qquad g_\Gamma = b_{\Gamma\Gamma} - A_{\Gamma I} A_{II}^{-1} b_{\Gamma I} \tag{4.5}$$

Additionally we note that $\hat{S}_\Gamma$ and g_Γ may be formed by a direct assembly:

$$\hat{S}_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} S_\Gamma^{(i)} R_\Gamma^{(i)} \qquad\qquad g_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} g_\Gamma^{(i)} \tag{4.6}$$

where

$$S_\Gamma^{(i)} = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} A_{\Gamma I}^{(i)^T} \qquad\qquad g_\Gamma^{(i)} = b_\Gamma^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} b_I^{(i)}. \tag{4.7}$$

Here $\begin{bmatrix} A_{II}^{(i)} & A_{\Gamma I}^{(i)^T} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix}$ and $\begin{bmatrix} b_I^{(i)} \\ b_\Gamma^{(i)} \end{bmatrix}$ correspond to the contributions of a single substructure to the global system (4.3). We may also write $\hat{S}_\Gamma$ as

$$\hat{S}_\Gamma = R_\Gamma^T S_\Gamma R_\Gamma \tag{4.8}$$

where

$$S_\Gamma = \begin{bmatrix} S_\Gamma^{(1)} & & \\ & \ddots & \\ & & S_\Gamma^{(N)} \end{bmatrix} \tag{4.9}$$

5. BDDC method. In this section we introduce the BDDC preconditioner for the Schur complement problem given in (4.4). In order to define the BDDC preconditioner we reparameterize $W_\Gamma^{(i)}$ into two orthogonal spaces $W_\Pi^{(i)}$ and $W_\Delta^{(i)}$. The primal space $W_\Pi^{(i)}$ is the space of discrete unknowns corresponding to functions with a constant value of $\hat{u}$ on each edge (face if $n = 3$) $\mathcal{F}^{ij}$ of substructure Ω_i. The dual space, $W_\Delta^{(i)}$ is the space of discrete unknowns corresponding to functions which have zero mean value of $\hat{u}$ on Γ_i. We note that the reparameterization to obtain $W_\Pi^{(i)}$ and $W_\Delta^{(i)}$ may be performed locally on each subdomain as described in[16]. We next define the partially assembled space

$$\tilde{W}_\Gamma = \hat{W}_\Pi \oplus \left(\Pi_{i=1}^N W_\Delta^{(i)} \right) \tag{5.1}$$

where $\hat{W}_\Pi$ is the assembled global primal space, single valued on Γ, which is formed by assembling the local primal spaces, $W_\Pi^{(i)}$. We define additional local operators $\bar{R}_\Gamma^{(i)} : \tilde{W}_\Gamma \to W_\Gamma^{(i)}$ which extract the degrees of freedom in $\tilde{W}_\Gamma$ corresponding to Γ_i. The global operator $\bar{R}_\Gamma : \tilde{W}_\Gamma \to W_\Gamma$ is formed by a direct assembly of $\bar{R}_\Gamma^{(i)}$. We also define the global operator $\tilde{R}_\Gamma : \hat{W}_\Gamma \to \tilde{W}_\Gamma$. We now define the partially assembled Schur complement matrix $\tilde{S}$, given by:

$$\tilde{S}_\Gamma = \sum_{i=1}^N \bar{R}_\Gamma^{(i)^T} S_\Gamma^{(i)} \bar{R}_\Gamma^{(i)} \tag{5.2}$$

We note that we may also write $\tilde{S}_\Gamma$ as $\tilde{S}_\Gamma = \tilde{R}_\Gamma^T S_\Gamma \tilde{R}_\Gamma$ where S_Γ is given in (4.9). In order to complete the definition of the BDDC preconditioner we define a positive scaling factor $\delta_i^\dagger$ defined for each nodal degree of freedom on $\partial\Omega_i \cap \partial\Omega_j$, corresponding to $W_\Gamma^{(i)}$ by

$$\delta_i^\dagger = \frac{\rho_i^\gamma}{\rho_i^\gamma + \rho_j^\gamma} \qquad\qquad \gamma \in [1/2, \infty) \tag{5.3}$$

where $\mathcal{N}_x$ is the set of indices of subdomains which share that particular degree of freedom. We define the scaled operator $\tilde{R}_{D,\Gamma} : \hat{W}_\Gamma \to \tilde{W}_\Gamma$ which is obtained by multiplying the entries of $\tilde{R}_\Gamma$ corresponding to $W_\Delta^{(i)}$ by $\delta_i^\dagger(x)$. Using $\tilde{R}_\Gamma$ and $\tilde{R}_{D,\Gamma}$ we define the interface averaging operator $E_D : \tilde{W}_\Gamma \to \tilde{W}_\Gamma$ as

$$E_D = \tilde{R}_\Gamma \tilde{R}_{D,\Gamma}^T \tag{5.4}$$

The BDDC preconditioner $M_{\text{BDDC}}^{-1} : \hat{W}_\Gamma \to \hat{W}_\Gamma$ is given by:

$$M_{\text{BDDC}}^{-1} = \bar{R}_{D,\Gamma}^T \tilde{S}_\Gamma^{-1} \bar{R}_{D,\Gamma} \tag{5.5}$$

We note that this preconditioner can be efficiently implemented in parallel, as the only globally coupled degrees of freedom of $\tilde{S}$ are those associated with the primal space W_Π. Additionally, in the following section we will show that this preconditioner is quasi-optimal in that the condition number of the preconditioned system, $M_{\text{BDDC}}^{-1}\hat{S}$, is independent of the number of subdomains and depends only weakly upon the number of degrees of freedom on each subdomain.

6. Analysis. In the following section we present the technical tools required to obtain the condition number bound. The analysis presented in the section closely follows that presented in [23] for mixed finite element methods, which in turn builds upon [10]. In particular, we note that all of the results presented in this section are simply the DG equivalents of similar results presented in [23] or [10]. The innovation which allows us to extend these results to DG discretizations is the new perspective presented in Section 3.

The main tools developed in this section connect the DG discretization to a related continuous finite element discretization on a subtriangulation of $\mathcal{T}_h$. Using these tools we are able to leverage the theory for continuous finite element to obtain the desired condition number bound. In order to define the related continuous finite element discretization we consider a special reparameterization of the space W_h^p on each subdomain Ω_i. Specifically, a nodal basis is employed on each element using a special set of nodal locations on each element κ. Specifically, on elements, κ, which do not touch $\partial\Omega_i$ nodal locations are chosen strictly interior to κ. On elements κ which touch $\partial\Omega_i$ nodal location are chosen on $\partial\kappa \cap \partial\Omega_i$ such that $\hat{u}|_{\partial\kappa\cap\partial\Omega_i}$ is uniquely defined by nodal values on $\partial\kappa$, while remaining nodal location are chosen interior to κ. We use this reparameterization so that each node defining the basis corresponds to a unique coordinate $\tilde{\mathbf{x}}$, and $\hat{u}|_{\partial\Omega_i}$ is determined by nodal values on $\partial\Omega_i$. The following Lemma connects the two different parameterizations of the space W_h^p.

Lemma 6.1. *There exist constants c and C independent of h such that for each element κ.*

$$c \sum_{\mathbf{x}_i \in \kappa} \phi(\mathbf{x}_i)^2 \leq \sum_{\tilde{\mathbf{x}}_i \in \kappa} \phi(\tilde{\mathbf{x}}_i)^2 \leq C \sum_{\mathbf{x}_i \in \kappa} \phi(\mathbf{x}_j)^2 \qquad \forall \phi \in P^p(\kappa) \tag{6.1}$$

and

$$c \sum_{\mathbf{x}_i, \mathbf{x}_j \in \kappa} (\phi(\mathbf{x}_i) - \phi(\mathbf{x}_j))^2 \leq \sum_{\tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}_j \in \kappa} (\phi(\tilde{\mathbf{x}}_i) - \phi(\tilde{\mathbf{x}}_j))^2 \leq C \sum_{\mathbf{x}_i, \mathbf{x}_j \in \kappa} (\phi(\mathbf{x}_i) - \phi(\mathbf{x}_j))^2 \qquad \forall \phi \in P^p(\kappa) \tag{6.2}$$

Proof. Proof of Lemma 6.1 follows directly from the fact that using either nodes $\mathbf{x}$ or $\tilde{\mathbf{x}}$ we can form a Lagrange basis for $\phi \in P^p(\kappa)$, with basis function bounded as in [22] Lemma B.5. ∎

We now define the subtriangulation $\hat{\mathcal{T}}_h$ of $\mathcal{T}_h$ by considering each element $\kappa \in \mathcal{T}_h$. The subtriangulation on each element κ consists of the primary vertices used to define W_h^p, and secondary vertices corresponding to nodes on $\partial\kappa \backslash \partial\Omega_i$ required to form a quasi-uniform triangulation of κ. We note that such a subtriangulation may be obtained on the reference element $\hat{\kappa}$ then mapped to $\mathcal{T}_h$. As an example, Figure 6.1 shows the nodes defining the reparameterization as well as the subtriangulation for a $p = 1$ triangular element.

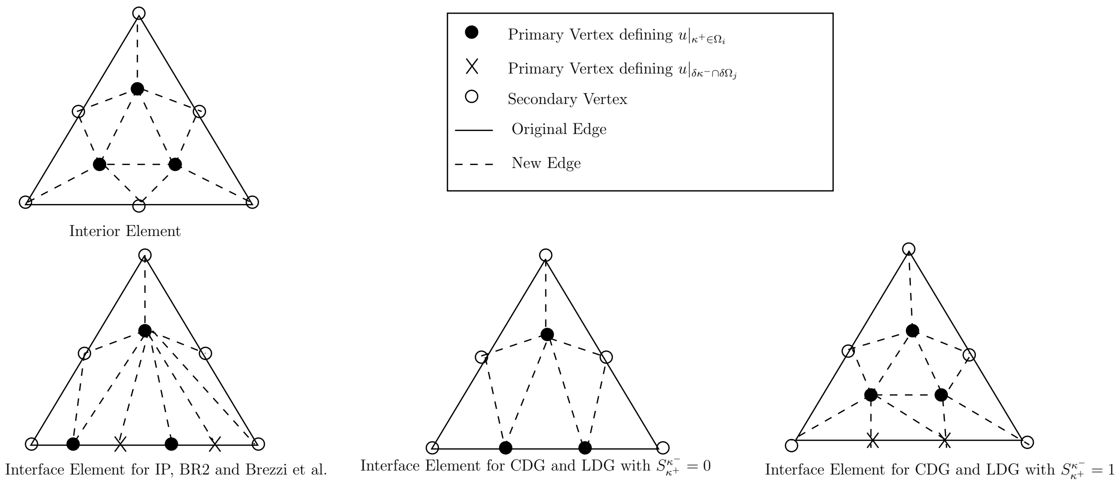

Fig. 6.1. *Examples of subtriangulations of $p = 1$ triangular elements*

Define $U_h(\Omega)$ to be the continuous linear finite element space defined on the triangulation $\hat{\mathcal{T}}_h$. Additionally we define $U_h(\Omega_i)$ and $U_h(\partial\Omega_i)$, as the restriction of $U_h(\Omega)$ to Ω_i and $\partial\Omega_i$ respectively. We now define a mapping $I_h^{\Omega_i}$ from any function ϕ defined at the primary vertices in Ω_i to $U_h(\Omega_i)$ as

$$I_h^{\Omega_i}\phi(\mathbf{x}) = \begin{cases} \phi(\mathbf{x}), \text{ if } \mathbf{x} \text{ is a primary vertex;} \\ \\ \text{the average of all adjacent primary vertices on } \partial\Omega_i, \\ \text{if } \mathbf{x} \text{ is a secondary vertex on } \partial\Omega_i; \\ \\ \text{the average of all adjacent primary vertices on } \Omega_i, \\ \text{if } \mathbf{x} \text{ is a secondary vertex in the interior of } \Omega_i; \\ \\ \text{the linear interpolation of the vertex values,} \\ \text{if } \mathbf{x} \text{ is not a vertex of } \mathcal{T}_h. \end{cases} \tag{6.3}$$

Since $(I_h^{\Omega_i}\phi)|_{\partial\Omega_i}$ is uniquely defined by $\phi|_{\partial\Omega_i}$, we may define the map $I_h^{\partial\Omega_i}$ from a function defined on the primary vertices on $\partial\Omega_i$ to $U_h(\partial\Omega_i)$ such that $I_h^{\partial\Omega_i}\phi|_{\partial\Omega_i} = (I_h^{\Omega_i}\phi)|_{\partial\Omega_i}$. We define $\tilde{U}_h(\Omega_i) \subset U_h(\Omega_i)$ and $\tilde{U}_h(\partial\Omega_i) \subset U_h(\partial\Omega_i)$ as the range of $I_h^{\Omega_i}$ and $I_h^{\partial\Omega_i}$ respectively.

We now connect the original DG discretization to the continuous finite element discretization on $\tilde{\mathcal{T}}_h$ by showing that both discretizations are equivalent to a quadratic form in terms of the nodal values on $\tilde{\mathcal{T}}_h$. The following lemmas and theorems are the equivalent of similar theorems for mixed finite element discretizations presented in [10] and [23]. These results are a direct consequence of Lemma 3.1, which is the DG equivalent of Lemma 4.2 of [10]. We list the relevant results from [10] and [23] and refer to these papers for the proofs.

Lemma 6.2. *For Ω_i composed of elements κ in $\mathcal{T}_h$, there exist constants c and C independent of h, H and ρ_κ such that for all $u_h \in W_h^p$*

$$ca_i(u_h, u_h) \le \sum_{\kappa\in\Omega_i} \rho_\kappa h^{n-2} \sum_{\tilde{\mathbf{x}}_i,\tilde{\mathbf{x}}_j\in\kappa\cup\kappa'} (u_h(\tilde{\mathbf{x}}_i) - u_h(\tilde{\mathbf{x}}_j))^2 \le Ca_i(u_h, u_h) \tag{6.4}$$

Proof. Lemma 6.2 follows directly from Lemmas 3.3 and 6.1. $\square$

Lemma 6.3. *There exists a constant $C > 0$ independent of h and H such that*

$$\left|I_h^{\partial\Omega_i}\phi\right|_{H^1(\Omega_i)} \leq C\,|\phi|_{H^1(\Omega_i)} \qquad \forall \phi \in U_h(\Omega_i), \tag{6.5}$$

$$\| I_h^{\partial\Omega_i}\phi \|_{L^2(\Omega_i)} \leq C \| \phi \|_{L^2(\Omega_i)} \qquad \forall \phi \in U_h(\Omega_i), \tag{6.6}$$

Proof. See [10] Lemma 6.1. □

We define the following scaled norms:

$$\| \phi \|^2_{H^1(\Omega_i)} = |\phi|^2_{H^1(\Omega_i)} + \frac{1}{H_i^2} \| \phi \|^2_{L^2(\Omega_i)} \tag{6.7}$$

$$\| \phi \|^2_{H^{1/2}(\partial\Omega_i)} = |\phi|^2_{H^{1/2}(\partial\Omega_i)} + \frac{1}{H_i} \| \phi \|^2_{L^2(\partial\Omega_i)} \tag{6.8}$$

Lemma 6.4. *There exist constants $c, C > 0$ independent of h and H such that for any $\hat{\phi} \in \tilde{U}_h(\partial\Omega_i)$.*

$$c \| \hat{\phi} \|_{H^{1/2}(\partial\Omega_i)} \leq \inf_{\substack{\phi \in \tilde{U}_h(\Omega_i) \\ \phi|_{\partial\Omega_i} = \hat{\phi}}} \| \phi \|_{H^1(\Omega_i)} \leq C \| \hat{\phi} \|_{H^{1/2}(\partial\Omega_i)} \tag{6.9}$$

$$c \left|\hat{\phi}\right|_{H^{1/2}(\partial\Omega_i)} \leq \inf_{\substack{\phi \in \tilde{U}_h(\Omega_i) \\ \phi|_{\partial\Omega_i} = \hat{\phi}}} |\phi|_{H^1(\Omega_i)} \leq C \left|\hat{\phi}\right|_{H^{1/2}(\partial\Omega_i)} \tag{6.10}$$

Proof. See [10] Lemma 6.2. □

Lemma 6.5. *There exists a constant $C > 0$ independent of h and H such that*

$$\| I_h^{\partial\Omega_i}\hat{\phi} \|_{H^{1/2}(\partial\Omega_i)} \leq C \| \hat{\phi} \|_{H^{1/2}(\partial\Omega_i)} \qquad \forall \hat{\phi} \in U_h(\partial\Omega_i) \tag{6.11}$$

Proof. See [10] Lemma 6.3. □

Lemma 6.6. *There exist constants c and C independent of h, H and ρ_i such that for all $u_\Gamma^{(i)} \in W_\Gamma^{(i)}$,*

$$c\rho_i \left|I_h^{\partial\Omega_i} u_i\right|^2_{H^{1/2}(\partial\Omega_i)} \leq |u_i|_{S_\Gamma^{(i)}} \leq C\rho_i \left|I_h^{\partial\Omega_i} u_i\right|^2_{H^{1/2}(\partial\Omega_i)} \tag{6.12}$$

Proof. See [10] Theorem 6.5. □

Lemma 6.7. *There exist constants c and C independent of h and H such that for all $u_\Gamma \in \tilde{W}_\Gamma$*

$$|E_D u_\Gamma|^2_{\tilde{S}_\Gamma} \leq C(1 + \log(H/h))^2 \, |u_\Gamma|^2_{\tilde{S}_\Gamma} \tag{6.13}$$

Proof. The proof of Lemma 6.7 closely follows that of [23] Lemma 5.5. We note Assumption 4.1 is essential for this result. In particular, if Assumption 4.1 were not valid then $(E_D u_\Gamma)_j$ the restriction of $E_D u_\Gamma$ to degrees of freedom on Ω_j would necessarily depend on degrees of freedom u_k corresponding to a subdomain Ω_k which does not neighbour Ω_j however are connected through the element κ in Ω_i which has edges/faces on both $\partial\Omega_i \cap \partial\Omega_j$ and $\partial\Omega_i \cap \partial\Omega_k$. □

(a) $p = 1$

	$\frac{1}{H}$				
$\frac{H}{h}$	2	4	8	16	32
2	4	16	24	29	29
4	13	20	30	31	31
8	15	21	32	34	34
16	15	24	34	35	35

(b) $p = 3$

	$\frac{1}{H}$				
$\frac{H}{h}$	2	4	8	16	32
2	8	17	25	30	30
4	13	21	29	31	31
8	13	22	31	33	32
16	12	23	33	34	33

(c) $p = 5$

	$\frac{1}{H}$				
$\frac{H}{h}$	2	4	8	16	32
2	8	20	25	31	31
4	12	21	30	33	31
8	11	23	32	35	33
16	11	26	34	36	35

Table 7.1
Iteration count for BDDC preconditioner using Interior Penalty Method

(a) $p = 1$

	$\frac{1}{H}$				
$\frac{H}{h}$	2	4	8	16	32
2	4	19	28	33	33
4	14	24	34	36	36
8	18	26	36	38	38
16	18	28	37	40	40

(b) $p = 3$

	$\frac{1}{H}$				
$\frac{H}{h}$	2	4	8	16	32
2	8	21	29	34	33
4	16	25	33	35	35
8	16	27	34	36	36
16	15	28	36	37	37

(c) $p = 5$

	$\frac{1}{H}$				
$\frac{H}{h}$	2	4	8	16	32
2	9	23	30	35	34
4	16	26	34	36	36
8	15	28	35	36	36
16	14	29	37	38	38

Table 7.2
Iteration count for BDDC preconditioner using the method of Bassi and Rebay

We now give the main theoretical result of the paper.

Theorem 6.8. *The condition number of the preconditioner operator $M_{BDDC}^{-1}\hat{S}$ is bounded by $C(1+\log(H/h))^2$ where C is a constant independent of h, H or ρ_i, though in general dependent upon the polynomial order p.*

Proof. Theorem 6.8 follows directly from Lemma 6.7. (See for example [23] Theorem 6.1). ∎

7. Numerical Results. In this section we present numerical results for the BDDC preconditioner introduced in Section 5. For each numerical experiment we solve the linear system resulting from the DG discretization using a Preconditioned Conjugate Gradient (PCG) method. The PCG algorithm is run until the initial l_2 norm of the residual is decreased by a factor of 10^{10}. We consider a domain $\Omega \in \mathbb{R}^2$ with $\Omega = (0,1)^2$. We divide Ω into $N \times N$ square subdomains Ω_i with side lengths H such that $N = \frac{1}{H}$. Each subdomain is the union of triangular elements obtained by bisecting squares of side length h, ensuring that Assumption 4.1 is satisfied. Thus each subdomain has n_i elements, where $n_i = 2\left(\frac{H}{h}\right)^2$.

In the first set of numerical experiments we solve (2.1) on Ω with f chosen such that the exact solution is given by $u = \sin(\pi x)\sin(\pi y)$. We discretize using each of the DG methods discussed in Section 2 for polynomial orders $p = 1$, 3, and 5. Tables 7.1- 7.5 show the corresponding number of PCG iteration required to converge for the considered DG methods. As predicted by the analysis the number of iterations is independent of the number of subdomains and only weakly dependent upon the number of elements per subdomain. In addition we note that the number of iterations also appears to be only weakly dependent on the solution order p. Finally, we note that the number of iterations required for the solution of the LDG and CDG discretizations is smaller than those of the other DG methods. For the LDG and CDG methods the Schur complement problem has approximately half the number of degrees of freedom as for the other DG methods, hence it is not surprising that a smaller number of iterations is required to converge.

In the second numerical experiment we examine the behaviour of the preconditioner for large jumps in the coefficient ρ. We partition the domain in a checkerboard pattern and set $\rho = 1$ on half of the subdomains and set $\rho = 1000$ in

(a) $p = 1$

$\frac{H}{h}$ \ $\frac{1}{H}$	2	4	8	16	32
2	4	16	23	25	24
4	13	19	27	28	28
8	15	20	28	30	29
16	16	22	30	32	32

(b) $p = 3$

$\frac{H}{h}$ \ $\frac{1}{H}$	2	4	8	16	32
2	7	17	25	26	26
4	14	20	28	29	28
8	15	22	29	31	30
16	14	24	31	33	33

(c) $p = 5$

$\frac{H}{h}$ \ $\frac{1}{H}$	2	4	8	16	32
2	8	18	26	27	27
4	14	22	28	29	29
8	15	23	30	32	32
16	14	25	33	34	33

Table 7.3
Iteration count for BDDC preconditioner using the method of Brezzi et al.

(a) $p = 1$

$\frac{H}{h}$ \ $\frac{1}{H}$	2	4	8	16	32
2	12	18	20	20	20
4	13	20	23	23	23
8	14	23	26	26	26
16	14	25	28	29	28

(b) $p = 3$

$\frac{H}{h}$ \ $\frac{1}{H}$	2	4	8	16	32
2	11	20	22	22	22
4	12	22	25	25	25
8	12	24	28	28	27
16	12	25	29	30	30

(c) $p = 5$

$\frac{H}{h}$ \ $\frac{1}{H}$	2	4	8	16	32
2	12	21	24	24	23
4	12	23	27	28	27
8	11	25	29	30	30
16	11	26	31	32	31

Table 7.4
Iteration count for BDDC preconditioner using the LDG method

the remaining subdomains. We solve (2.1) with $f = 1$. We discretize this problem using the CDG method. Initially we set $\delta_i^\dagger = \frac{1}{|\mathcal{N}_x|}$, where $|\mathcal{N}_x|$ is the number of elements in the set $\mathcal{N}_x$. We note that this choice of $\delta_i^\dagger$ corresponds to setting $\gamma = 0$ in (5.3), which does not satisfy the assumption $\gamma \in [1/2, \infty)$. Hence, we obtain poor convergence of the BDDC algorithm as shown in Table 7.5(a). Next we set $\delta_i^\dagger$ as in (5.3) with $\gamma = 1$. With this choice of $\delta_i^\dagger$ the good convergence properties of the BDDC algorithm are recovered as shown in Table 7.5(b).

8. Conclusions. We have extended the BDDC preconditioner to a large class of DG discretizations for second-order elliptic problems. The analysis shows that the condition number of the preconditioned system is bounded by $C(1+\log(H/h))^2$, with constant C independent of h, H or large jumps in the coefficient ρ. Numerical results confirm the theory.

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(a) $p = 1$

$\frac{H}{h}$ \ $\frac{1}{H}$	2	4	8	16	32
2	12	19	20	20	19
4	12	20	23	23	22
8	13	23	25	25	25
16	13	24	28	28	27

(b) $p = 3$

$\frac{H}{h}$ \ $\frac{1}{H}$	2	4	8	16	32
2	11	20	22	22	22
4	12	21	24	25	24
8	12	23	27	27	27
16	12	25	28	29	29

(c) $p = 5$

$\frac{H}{h}$ \ $\frac{1}{H}$	2	4	8	16	32
2	11	22	25	24	24
4	12	24	27	27	26
8	12	24	29	29	29
16	11	26	31	31	31

Table 7.5
Iteration count for BDDC preconditioner using the CDG method

(a) $\delta_i^\dagger = \frac{1}{2}$

p \ $\frac{1}{H}$	2	4	8	16	32
0	17	69	118	138	147
1	51	119	179	215	232
2	52	129	192	241	252
3	55	133	207	267	316
4	58	144	226	285	304
5	59	153	242	306	361

(b) $\delta_i^\dagger = \frac{\rho_i}{\sum_j \rho_j}$

p \ $\frac{1}{H}$	2	4	8	16	32
0	4	6	13	15	16
1	4	7	14	18	19
2	4	7	13	17	18
3	4	7	15	18	19
4	4	7	14	19	20
5	4	7	14	19	20

Table 7.6
Iteration count for BDDC preconditioner using the CDG method with $\rho = 1$ or 1000

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