Bol. Soc. Paran. Mat. (3s.) **v. 36** 2 (2018): 151–167.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v36i2.31977

Properties of the First Eigenvalue with Sign-changing Weight of the Discrete p-Laplacian and Applications.

Mohammed Chehabi, Omar Chakrone

ABSTRACT: By establishing some results around the first eigenvalue $\lambda_1(m)$ for the following problem:

$$\begin{aligned} -\Delta(\varphi_p(\Delta u(k-1))) &= \lambda m(k)\varphi_p(u(k)), \quad k \in [1,n], \\ u(0) &= 0 = u(n+1), \end{aligned}$$

where $m \in M([1,n]) = \{m : [1,n] \longrightarrow \mathbb{R} \ / \ \exists k_0 \in [1,n], \ m(k_0) > 0\}$, as the constant sign of the first eigenfunction with $\lambda_1(m)$, the simplicity of $\lambda_1(m)$, the strict monotonicity property with respect the weight and sign change of any eigenfunction with λ ($\lambda > \lambda_1(m)$), we prove the existence and non-existence of solutions of the problem (1.1).

Key Words: Difference equations, Discrete p-Laplacian, Variational methods, First eigenvalue, First eigenfunction, Simplicity, Strict monotonicity.

Contents

1. Introduction

In recent years, equations involving the discrete p-Laplacian operator, subject to different boundary conditions, have been widely studied by many authors and several approaches. We recall here the works of Agarwal, Perera and O'Regan [1], D. Jiang, J. Chu, D. O'Regan, and R. P. Agarwal [2], J. Chu, D. Jiang [3], Jong-Ho Kim, Jea-Hyun Park and June-Yub Lee [6]; the variational approach represents an important advance as it allows to prove multiplicity results as well.

Motivations for this interest arose in by different fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics and others.

Consider the boundary value problem:

$$-\Delta(\varphi_p(\Delta u(k-1))) = f(k, u(k)), \quad k \in [1,n],$$

$$u(0) = 0 = u(n+1), \tag{1.1}$$

Submitted May 17, 2016. Published July 17, 2016

where n is an integer greater than or equal to 1, $[1,n]$ is the discrete interval $\{1,...,n\}$, $\Delta u(k)=u(k+1)-u(k)$ is the forward difference operator, and we only assume that $f\in C([1,n]\times\mathbb{R})$.

Define

$$F(k,t)=\int_0^t f(k,s)ds,\ k\in[1,n],\ t\in\mathbb{R}.$$

Let $\lambda_1(m)$ be the first eigenvalue of

$$-\Delta(\varphi_p(\Delta u(k-1)))=\lambda m(k)\varphi_p(u(k)),\quad k\in[1,n], \tag{1.2}$$
$$u(0)=0=u(n+1),$$

where $m\in M([1,n])=\{m:[1,n]\longrightarrow\mathbb{R}\ \ /\ \ \exists k_0\in[1,n],\ m(k_0)>0\}$, $\varphi_p(s)=|s|^{p-2}s$ and $1<p<\infty$.

The class W of functions $u:[0,n+1]\longrightarrow\mathbb{R}$ such that $u(0)=0=u(n+1)$ is an n-dimensional Banach space under the norm

$$\|u\|=\left(\sum_{k=1}^{n+1}|\Delta u(k-1)|^p\right)^{\frac{1}{p}}.$$

We donote by λ_1 and ϕ_1 the first eigenvalue and the positive eigenfunction of (1.2) for $m=1$ such that $\|\phi_1\|=1$, and $S=\{u\in W,\quad \|u\|=1\}\}$.

Define

$$\Phi_f(u)=\sum_{k=1}^{n+1}\left[\frac{1}{p}|\Delta u(k-1)|^p-F(k,u(k))\right],\qquad u\in W,$$

Then the functional Φ_f is C^1 with

$$\begin{aligned}(\Phi_f'(u),v)&=\sum_{k=1}^{n+1}\Big[\varphi_p(\Delta u(k-1))\Delta v(k-1)-f(k,u(k))v(k)\Big]\\&=-\sum_{k=1}^{n}\Big[\Delta\varphi_p(\Delta u(k-1))\Big]v(k)-\sum_{k=1}^{n+1}f(k,u(k))v(k),\end{aligned}$$

$v\in W$, so solutions of (1.1) are precisely the critical points of Φ_f.

Theorem 1.1. *If*

$$\lim_{|t|\longrightarrow+\infty}\sup\frac{pF(k,t)}{|t|^p}\le\lambda_1,\ \ k\in[1,n],$$

$$\textit{and}\ \lim_{|t|\longrightarrow+\infty}\sup\frac{pF(k,t)}{|t|^p}\ne\lambda_1\ \textit{for some}\ k\in[1,n], \tag{1.3}$$

then (1.1) has a solution. More precisely, this solution is a global minimizer of Φ_f.

Remark 1.2. *If*

$$\lim_{|t|\longrightarrow+\infty}\sup\frac{pF(k,t)}{|t|^p}=\lambda_1,\ \ k\in[1,n],$$

then (1.1) does not always have a solution, as is shown in the following proposition (Proposition 1.3).

Proposition 1.3. *If*

$$f(k,t)=\lambda_1(m).m(k)\,|t|^{p-2}\,t+h(k),\ \ k\in[1,n],\ \ t\in\mathbb{R},$$

where $h(k)\geq 0$, $k\in[1,n]$, and $h(k)\neq 0$ for some $k\in[1,n]$, then (1.1) does not have a solution. One can take as particular case $m=1$.

Theorem 1.4. *Suppose that*

$$f(k,t)\geq a_0(k),\quad (k,t)\in[1,n]\times[0,\alpha(k)], \tag{1.4}$$

for some nontrivial function $a_0\geq 0$ and a function α such that $\alpha(k)>0$, $k\in[1,n]$. If

$$\lambda_1\leq\lim_{t\longrightarrow+\infty}\inf\frac{pF(k,t)}{t^p},\ k\in[1,n]\ ,\ \lambda_1\neq\lim_{t\longrightarrow+\infty}\inf\frac{pF(k,t)}{t^p}\ \textit{for some}\ k\in[1,n],$$

$$\textit{and}\ \lambda_1\leq\lim_{t\longrightarrow+\infty}\inf\frac{f(k,t)}{t^{p-1}},\quad k\in[1,n], \tag{1.5}$$

then (1.1) has a solution $u>0$.

Theorem 1.5. *If (1.4) holds and (1.1) has a supersolution w where $w(k)>\alpha(k)$, $k\in[1,n]$, then (1.1) has a solution $u_1<w$. If, in addition, (1.5) holds, then there is a second solution $u_2>u_1$.*

To prove these theorems we need the following results:

Lemma 1.6. *(see [2]) If*

$$\begin{aligned}-\Delta(\varphi_p(\Delta u(k-1)))\ &\geq\ -\Delta(\varphi_p(\Delta v(k-1))),\quad k\in[1,n],\\ u(0)\ &\geq\ v(0),\ \ u(n+1)\geq v(n+1),\end{aligned}$$

then either $u>v$ in $[1,n]$ or $u\equiv v$. In particular, if

$$\begin{aligned}-\Delta(\varphi_p(\Delta u(k-1)))\ &\geq\ 0,\quad k\in[1,n],\\ u(0)\ &\geq\ 0,\ \ u(n+1)\geq 0,\end{aligned}$$

then either $u>0$ in $[1,n]$ or $u\equiv 0$.

Lemma 1.7. *If $\underline{u}$ is a subsolution of (1.1) and u is a solution of the modified problem:*

$$\begin{aligned} -\Delta(\varphi_p(\Delta u(k-1))) &= f_{\underline{u}}(k,u(k)), \quad k \in [1,n], \\ u(0) &= 0 = u(n+1), \end{aligned}$$

where

$$f_{\underline{u}}(k,t) = \begin{cases} f(k,t), & t \geq \underline{u}(k), \\ f(k,\underline{u}(k)), & t < \underline{u}(k), \end{cases}$$

then $u \geq \underline{u}$.

Recall that $\lambda_1(m)$ is characterized by

$$\lambda_1(m) = \inf\left\{\sum_{k=1}^{n+1} |\Delta u(k-1)|^p\,,\ u \in W \text{ and } \sum_{k=1}^{n} m(k)\,|u(k)|^p = 1\right\}, \tag{1.6}$$

which is also expressed in the form

$$\frac{1}{\lambda_1(m)} = \sup_{u \in W\setminus\{0\}} \left\{\frac{\sum\limits_{k=1}^{n} m(k)\,|u(k)|^p}{\sum\limits_{k=1}^{n+1} |\Delta u(k-1)|^p}\right\},$$

is the first eigenvalue of (1.2) in the sense that $\lambda_1(m) \leq \lambda$ for any other positive eigenvalue λ of (1.2).

Proposition 1.8. *Let u_1 be an eigenfunction with eigenvalue $\lambda_1(m)$, then u_1 does not change sign in $[1,n]$. Moreover u_1 does not vanish in $[1,n]$.*

Proposition 1.9. *Let u be an eigenfunction with eigenvalue λ such that $\lambda_1(m) < \lambda$, then u changes sign in $[1,n]$.*

Proposition 1.10. *The first eigenvalue $\lambda_1(m)$ is simple: let u and v be two eigenfunctions associated with $\lambda_1(m)$, then there exists $c \in \mathbb{R}$ such that $u = cv$.*

Proposition 1.11. *$\lambda_1(m)$ verifies the strict monotonicity property with respect to the weight m: If $m_1, m_2 \in M([1,n])$, such that $m_1 \leq m_2$ and $m_1(k) \neq m_2(k)$ for some $k \in [1,n]$, then $\lambda_1(m_2) < \lambda_1(m_1)$.*

Remarks 1.12. *1- If* $\lim_{t\longrightarrow+\infty}\sup\dfrac{f(k,t)}{t^{p-1}}<\lambda_1,\ k\in[1,n]$, *then* $\lim_{t\longrightarrow+\infty}\sup\dfrac{pF(k,t)}{t^p}$ $<\lambda_1,\ k\in[1,n]$. *The converse is false.*

If $\lambda_1<\lim_{t\longrightarrow+\infty}\inf\dfrac{f(k,t)}{t^{p-1}},\ \ k\in[1,n]$, *then* $\lambda_1<\lim_{t\longrightarrow+\infty}\inf\dfrac{pF(k,t)}{t^p},\ k\in[1,n]$. *The converse is false.*

2- In the Theorem 1.5, one can consider $w(k)=t_1$, *where* $t_1>\alpha(k)$, *and* $f(k,t_1)\leq 0,\ k\in[1,n]$.

In the present paper, we study the problem (1.1) following a variational approach, based on the properties of the first eigenvalue $\lambda_1(m)$. The paper is organized as follows: in Section 2, we prove the Lemma 1.7 and the propositions 1.3, 1.8 1.9, 1.10 and 1.11, in Section 3, we prove the theorems 1.1, 1.4 and 1.5.

2. Proofs of Propositions and Lemma

Proof of Lemma 1.7

By absurd, assume that there exists $h\in[1,n]$ such that $u(h)<\underline{u}(h)$.
Let $k_1=\max\{k\in[0,n+1],\ u(k)<\underline{u}(k)\}$ and $k_0=\max\{k\in[0,k_1],\ u(k)\geq\underline{u}(k)\}$.
We get $k_1\neq n+1$, $u(k_0)\geq\underline{u}(k_0)$, $u(k_1)<\underline{u}(k_1)$, $u(k_1+1)\geq\underline{u}(k_1+1)$ and for $k\in[k_0+1,k_1]$, $u(k)<\underline{u}(k)$. Then for every $k\in[k_0+1,k_1]$,

$$-\Delta(\varphi_p(\Delta u(k-1)))=f_{\underline{u}}(k,u(k))=f(k,\underline{u}(k))\geq-\Delta(\varphi_p(\Delta\underline{u}(k-1))).$$

Let for every $k\in[0,k_1-k_0+1]$, $v(k)=u(k+k_0)$ and $\underline{v}(k)=\underline{u}(k+k_0)$, then

$$\begin{aligned}-\Delta(\varphi_p(\Delta v(k-1)))&\geq-\Delta(\varphi_p(\Delta\underline{v}(k-1))),\quad\forall k\in[1,k_1-k_0],\\ v(0)&=u(k_0)\geq\underline{v}(0)=\underline{u}(k_0),\\ v(k_1-k_0+1)&=u(k_1+1)\geq\underline{v}(k_1-k_0+1)=\underline{u}(k_1+1).\end{aligned}$$

By Lemma 1.6, for every $k\in[1,k_1-k_0],\ \ v(k)\geq\underline{v}(k)$.
In particular $v(k_1-k_0)=u(k_1)\geq\underline{v}(k_1-k_0)=\underline{u}(k_1)$, which is a contradiction.
Thus $u\geq\underline{u}$.

Proof of Proposition 1.3

Suppose that

$$f(k,t)=\lambda_1(m).m(k)\left|t\right|^{p-2}t+h(k),\ \ k\in[1,n],\ \ t\in\mathbb{R},$$

where $h(k)\geq 0,\ k\in[1,n]$, and $h(k)\neq 0$ for some $k\in[1,n]$, and (1.1) has a solution u, then

$$\begin{cases}-\Delta(\varphi_p(\Delta u(k-1)))=\lambda_1(m).m(k)\left|u(k)\right|^{p-2}u(k)+h(k),\quad k\in[1,n],\\ u(0)=0=u(n+1).\end{cases}\tag{2.1}$$

We prove that $u \geq 0$.

By absurd, suppose that $u^- \neq 0$, where $u^- = \max\{-u, 0\}$ and multiply (2.1) by $-u^-$, we obtain

$$-\sum_{k=1}^{n+1}\varphi_p(\Delta u(k-1))\Delta u^-(k-1) = \lambda_1(m)\sum_{k=1}^{n} m(k)\left|u^-(k)\right|^p - \sum_{k=1}^{n} h(k).u^-(k).$$

Distinguish the cases of signs of $u(k-1)$ and $u(k)$, we prove that

$$\sum_{k=1}^{n+1}\left|\Delta u^-(k-1)\right|^p \leq -\sum_{k=1}^{n+1}\varphi_p(\Delta u(k-1))\Delta u^-(k-1),$$

then

$$\begin{aligned} 0 < &\sum_{k=1}^{n+1}\left|\Delta u^-(k-1)\right|^p \\ &\leq \lambda_1(m)\sum_{k=1}^{n} m(k)\left|u^-(k)\right|^p - \sum_{k=1}^{n} h(k).u^-(k) \\ &\leq \lambda_1(m)\sum_{k=1}^{n} m(k)\left|u^-(k)\right|^p, \end{aligned}$$

so $\dfrac{u^-}{\left(\sum\limits_{k=1}^{n} m(k)\left(u^-(k)\right)^p\right)^{\frac{1}{p}}}$ is also a minimizer of $\lambda_1(m)$ and u^- is an eigenfunction with eigenvalue $\lambda_1(m)$, then $u^- > 0$ (Proposition (1.8)), and hense

$$\sum_{k=1}^{n} h(k).u^-(k) = 0,$$

and $h = 0$, which is a contradiction.

Let ψ a positive eigenfunction with eigenvalue $\lambda_1(m)$ and $c = \min\limits_{k\in[1,n]}\dfrac{u(k)}{\psi(k)}$, there exists $k_1 \in [1, n]$, such that $c = \dfrac{u(k_1)}{\psi(k_1)}$, then $v \leq u$ and $v(k_1) = u(k_1)$, where $v = c\psi$.

We get

$$-\Delta(\varphi_p(\Delta v(k_1-1))) + \lambda_1(m).m^-(k_1)\left(v(k_1)\right)^{p-1} = \lambda_1(m).m^+(k_1)\left(v(k_1)\right)^{p-1},$$

where $m^+ = \max\{m, 0\}$ and $m^- = \max\{-m, 0\}$ so that $m = m^+ - m^-$, thus

$$\begin{aligned} -\Delta(\varphi_p(\Delta u(k_1-1))) + \lambda_1(m).m^-(k_1)\left(u(k_1)\right)^{p-1} &= \lambda_1(m).m^+(k_1)\left(u(k_1)\right)^{p-1} \\ &\quad +h(k_1) \\ &\geq \lambda_1(m).m^+(k_1)\left(v(k_1)\right)^{p-1} \\ &= -(\Delta\varphi_p(\Delta v(k_1-1))) \\ &\quad +\lambda_1(m).m^-(k_1)\left(v(k_1)\right)^{p-1}, \end{aligned}$$

then
$$-\Delta(\varphi_p(\Delta u(k_1-1))) \geq -\Delta(\varphi_p(\Delta v(k_1-1))),$$
and
$$-\varphi_p(\Delta u(k_1)) + \varphi_p(\Delta u(k_1-1)) \geq -\varphi_p(\Delta v(k_1)) + \varphi_p(\Delta v(k_1-1)).$$
We have
$$\Delta u(k_1) = u(k_1+1) - u(k_1) \geq v(k_1+1) - v(k_1) = \Delta v(k_1)$$
and
$$\Delta u(k_1-1) \leq \Delta v(k_1-1),$$
therefore
$$0 \leq \varphi_p(\Delta u(k_1)) - \varphi_p(\Delta v(k_1)) \leq \varphi_p(\Delta u(k_1-1)) - \varphi_p(\Delta v(k_1-1)) \leq 0,$$
and
$$u(k_1-1) = v(k_1-1) \text{ and } u(k_1+1) = v(k_1+1).$$
Step by step we prove that $u = v$, then $h = 0$, which is a contradiction.

Proof of Proposition 1.8

Let u be an eigenfunction with the first eigenvalue $\lambda_1(m)$ of the problem (1.2), then

$$\begin{cases} -(\Delta\varphi_p(\Delta u(k-1))) = \lambda_1(m).m(k)\,|u(k)|^{p-2}\,u(k), \quad k \in [1,n], \\ u(0) = 0 = u(n+1). \end{cases} \tag{2.2}$$

If $u^- \neq 0$, multiply (2.2) by $-u^-$, we obtain
$$-\sum_{k=1}^{n+1}\varphi_p(\Delta u(k-1))\Delta u^-(k-1) = \lambda_1(m)\sum_{k=1}^{n} m(k)\left(u^-(k)\right)^p.$$
As
$$\sum_{k=1}^{n+1}\left|\Delta u^-(k-1)\right|^p \leq -\sum_{k=1}^{n+1}\varphi_p(\Delta u(k-1))\Delta u^-(k-1),$$
we get
$$0 < \sum_{k=1}^{n+1}\left|\Delta u^-(k-1)\right|^p \leq \lambda_1(m)\sum_{k=1}^{n} m(k)\left(u^-(k)\right)^p,$$
and u^- is an eigenfunction with eigenvalue $\lambda_1(m)$.

Let us prove that $u^- > 0$. By absurd, suppose that there exists $k_0 \in [1,n]$ such that $u^-(k_0) = 0$, then
$$-\Delta(\varphi_p(\Delta u^-(k_0-1))) = 0,$$

and

$$\varphi_p(\Delta u^-(k_0)) = \varphi_p(\Delta u^-(k_0-1)),$$

so $u^-(k_0+1)+u^-(k_0-1)=0$ and $u^-(k_0+1)=u^-(k_0-1)=0$.
Step by step we prove that $u^-=0$, which is a contradiction.
Thus $u^->0$ and $u=-u^-<0$.
If $u^-=0$, then $u\geq 0$ and $u>0$ (by the same techniques as the above).

Proof of Proposition 1.9
Let (u,λ) a solution of (1.2) with $\lambda>\lambda_1(m)$. Suppose that u keeps a constant sign, we can suppose that $u>0$, we have

$$-\Delta(\varphi_p(\Delta u(k-1))) + \lambda.m^-(k)\,(u(k))^{p-1} = \lambda.m^+(k)\,(u(k))^{p-1}\,.$$

Claim

$$\lambda = \inf\left\{\sum_{k=1}^{n+1}|\Delta v(k-1)|^p + \lambda.\sum_{k=1}^{n} m^-(k)\,|v(k)|^p\,,\ v\in W \text{ and } \sum_{k=1}^{n} m^+(k)\,|v(k)|^p = 1\right\}$$

Proof of Claim
Let

$$\mu = \inf\left\{\sum_{k=1}^{n+1}|\Delta v(k-1)|^p + \lambda.\sum_{k=1}^{n} m^-(k)\,|v(k)|^p\,,\ v\in W \text{ and } \sum_{k=1}^{n} m^+(k)\,|v(k)|^p = 1\right\},$$

and prove that $\mu=\lambda$.
It's clear that $u_0 = \dfrac{u}{\left(\sum\limits_{k=1}^{n} m^+(k)\,|u(k)|^p\right)^{\frac{1}{p}}} \in W$ and $\sum\limits_{k=1}^{n} m^+(k)\,|u_0(k)|^p = 1$, so

$$\mu \leq \sum_{k=1}^{n+1}|\Delta u_0(k-1)|^p + \lambda.\sum_{k=1}^{n} m^-(k)\,|u_0(k)|^p = \lambda.$$

There exists $w\in W$ such that $\sum\limits_{k=1}^{n} m^+(k)\,|w(k)|^p = 1$ and

$$\mu = \sum_{k=1}^{n+1}|\Delta w(k-1)|^p + \lambda.\sum_{k=1}^{n} m^-(k)\,|w(k)|^p\,.$$

As

$$\sum_{k=1}^{n+1}|\Delta\,|w|\,(k-1)|^p \leq \sum_{k=1}^{n+1}|\Delta w(k-1)|^p\,,$$

then, $|w|$ minimizes also μ, so we can suppose that $w\geq 0$ and $w\neq 0$.

We have

$$- (\Delta\varphi_p(\Delta w(k-1))) + \lambda.m^-(k)\,(w(k))^{p-1} = \mu.m^+(k)\,(w(k))^{p-1}\,. \tag{2.3}$$

Let $c = \max\limits_{k\in[1,n]} \dfrac{w(k)}{u(k)}$, there exists $k_0 \in [1,n]$, such that $c = \dfrac{w(k_0)}{u(k_0)}$.
We have $w \leq v$ and $w(k_0) = v(k_0)$, where $v = cu$.
We get

$$\begin{aligned} -\Delta(\varphi_p(\Delta w(k-1))) + \lambda.m^-(k)\,(w(k))^{p-1} &= \mu.m^+(k)\,(w(k))^{p-1} \\ &\leq \lambda.m^+(k)\,(v(k))^{p-1} \\ &= -\Delta(\varphi_p(\Delta v(k-1))) \\ &\quad +\lambda.m^-(k)\,(v(k))^{p-1}\,, \end{aligned}$$

in particular for $k = k_0$, we obtain

$$-\Delta(\varphi_p(\Delta w(k_0-1))) \leq -\Delta(\varphi_p(\Delta v(k_0-1))),$$

and

$$0 \leq \varphi_p(\Delta w(k_0-1)) - \varphi_p(\Delta v(k_0-1)) \leq \varphi_p(\Delta w(k_0)) - \varphi_p(\Delta v(k_0)) \leq 0,$$

so

$$w(k_0+1) = v(k_0+1) \text{ and } w(k_0-1) = v(k_0-1).$$

Step by step we prove that $w = v$.
Replacing w by v in (2.3),

$$\begin{aligned} \mu.m^+(k)\,(v(k))^{p-1} &= -(\Delta\varphi_p(\Delta v(k-1))) + \lambda.m^-(k)\,(v(k))^{p-1} \\ &= \lambda.m^+(k)\,(v(k))^{p-1}\,,\ k \in [1,n], \end{aligned}$$

then $\mu = \lambda$.

Returning to the proof of Proposition **1.9**
Let ψ a positive eigenfunction with eigenvalue $\lambda_1(m)$ such that
$\sum\limits_{k=1}^{n} m^+(k)\,(\psi(k))^p = 1$. We have

$$\begin{aligned} \sum_{k=1}^{n+1} |\Delta\psi(k-1)|^p + \lambda_1(m).\sum_{k=1}^{n} m^-(k)\,(\psi(k))^p &= \lambda_1(m).\sum_{k=1}^{n} m^+(k)\,(\psi(k))^p \\ &= \lambda_1(m), \end{aligned}$$

so

$$0 < \sum_{k=1}^{n+1} |\Delta\psi(k-1)|^p = \lambda_1(m)\left(1 - \sum_{k=1}^{n} m^-(k)\,(\psi(k))^p\right). \tag{2.4}$$

By the Claim

$$\lambda \leq \sum_{k=1}^{n+1} |\Delta\psi(k-1)|^p + \lambda.\sum_{k=1}^{n} m^-(k)\,(\psi(k))^p\,,$$

then

$$\lambda \leq \lambda_1(m)\left(1-\sum_{k=1}^{n} m^-(k)\,(\psi(k))^p\right) + \lambda.\sum_{k=1}^{n} m^-(k)\,(\psi(k))^p\,,$$

and

$$\lambda\left(1-\sum_{k=1}^{n} m^-(k)\,(\psi(k))^p\right) \leq \lambda_1(m)\left(1-\sum_{k=1}^{n} m^-(k)\,(\psi(k))^p\right),$$

by (2.4), we get

$$\lambda \leq \lambda_1(m).$$

Which is a contadiction.

Proof of Proposition 1.10

Let u and v be two eigenfunctions associated with $\lambda_1(m)$ of the problem (1.2), then u and v does not change sign in $[1,n]$. We can suppose that $u>0$ and $v>0$. Let $c = \min_{k\in[1,n]} \dfrac{v(k)}{u(k)}$, there exists $k_0 \in [1,n]$, such that $c = \dfrac{v(k_0)}{u(k_0)}$. Then

$$\begin{aligned}
\varphi_p(\Delta v(k_0-1)) - \varphi_p(\Delta v(k_0)) &= \lambda_1(m)m(k_0)\varphi_p(v(k_0)) \\
&= \lambda_1(m)m(k_0)\varphi_p(cu(k_0)) \\
&= \varphi_p(\Delta(cu)(k_0-1)) - \varphi_p(\Delta(cu)(k_0)),
\end{aligned}$$

and

$$0 \leq \varphi_p(\Delta v(k_0)) - \varphi_p(\Delta(cu)(k_0)) = \varphi_p(\Delta v(k_0-1)) - \varphi_p(\Delta(cu)(k_0-1)) \leq 0,$$

so

$$v(k_0+1) = (cu)(k_0+1) \text{ and } v(k_0-1) = (cu)(k_0-1),$$

Step by step we prove that $v = cu$.

Proof of Proposition 1.11

Let $m_1, m_2 \in M([1,n])$ such that $m_1 \leq m_2$, $m_1(k) \neq m_2(k)$ for some $k \in [1,n]$ and u_1 the positive eigenfunction with $\lambda_1(m_1)$ such that $\|u_1\| = 1$. We get

$$\begin{aligned}
\frac{1}{\lambda_1(m_1)} &= \sup_{u\in S}\left\{\sum_{k=1}^{n} m_1(k)\,|u(k)|^p\right\}\\
&= \sum_{k=1}^{n} m_1(k)\,|u_1(k)|^p\\
&< \sum_{k=1}^{n} m_2(k)\,|u_1(k)|^p\\
&\leq \sup_{u\in S}\left\{\sum_{k=1}^{n} m_2(k)\,|u(k)|^p\right\} = \frac{1}{\lambda_1(m_2)},
\end{aligned}$$

thus $\lambda_1(m_2) < \lambda_1(m_1)$.

3. Proofs of Theorems

Proof of Theorem 1.1. By (1.3), for $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F(k,t) \leq (\delta(k)+\varepsilon)\frac{|t|^p}{p} + C_\varepsilon,$$

where $\delta(k) = \lim\limits_{|t|\longrightarrow\infty} \sup \dfrac{pF(k,t)}{|t|^p}$. For $u \in W$,

$$\begin{aligned}
\Phi_f(u) &= \sum_{k=1}^{n+1}\left[\frac{1}{p}\,|\Delta u(k-1)|^p - F(k,u(k))\right]\\
&\geq \sum_{k=1}^{n+1}\left[\frac{1}{p}\,|\Delta u(k-1)|^p - \left((\delta(k)+\varepsilon)\frac{|u(k)|^p}{p} + C_\varepsilon\right)\right]\\
&\geq \frac{1}{p}\left[\|u\|^p - \sum_{k=1}^{n+1}\delta(k)\,|u(k)|^p - \varepsilon\sum_{k=1}^{n+1}|u(k)|^p\right] + (n+1)C_\varepsilon,
\end{aligned}$$

then, by (1.6)

$$\Phi_f(u) \geq \frac{1}{p}\left[1 - \frac{1}{\lambda_1(\delta)} - \frac{\varepsilon}{\lambda_1}\right]\|u\|^p + (n+1)C_\varepsilon.$$

On the other hand, $\delta(k) \leq \lambda_1$, $k \in [1,n]$, and $\delta(k) \neq \lambda_1$ for some $k \in [1,n]$, by the strict monotonicity property with respect to the weight (Proposition 1.11) and (1.6), we get $\lambda_1(\lambda_1) < \lambda_1(\delta)$ and $\lambda_1(\lambda_1) = 1$, then $1 - \dfrac{1}{\lambda_1(\delta)} > 0$. We choose $\varepsilon > 0$ such that $1 - \dfrac{1}{\lambda_1(\delta)} - \dfrac{\varepsilon}{\lambda_1} > 0$, thus Φ_f is bounded from below and coercive, then the problem (1.1) has a solution.

Lemma 3.1. *If*

$$\lambda_1 \leq \lim_{t\longrightarrow+\infty}\inf \frac{pF(k,t)}{t^p}\ k\in[1,n],\ \lambda_1 \neq \lim_{t\longrightarrow+\infty}\inf \frac{pF(k,t)}{t^p}\ \textit{for some } k\in[1,n],$$

$$\lambda_1 \leq \lim_{t\longrightarrow+\infty}\inf \frac{f(k,t)}{t^{p-1}}\ \textit{and}\ \lim_{t\longrightarrow-\infty}\frac{f(k,t)}{|t|^{p-1}}=0,\quad k\in[1,n], \tag{3.1}$$

then Φ_f satisfies the Palais-Smale compactness condition (PS): every sequence (u_j) in W such that $\Phi_f(u_j)$ is bounded and $\Phi'_f(u_j)\longrightarrow 0$ has a convergent subsequence.

Proof of lemma 3.1

It suffices to show that (u_j) is bounded since W is finite dimentional , so suppose that $\rho_j := \|u_j\| \longrightarrow \infty$ for some subsequence. We have

$$o(1)\left\|u_j^-\right\| = (\Phi'_g(u_j), u_j^-),$$

where $u_j^- = \max\{-u_j, 0\}$ is the negative part of u_j.

$$\begin{aligned}(\Phi'_g(u_j), u_j^-) &= \sum_{k=1}^{n+1}\left[\varphi_p(\Delta u_j(k-1))\Delta u_j^-(k-1)\right] - \sum_{k=1}^{n+1} f(k,u_j(k))u_j^-(k)\\ &= \sum_{k=1}^{n+1}\left[\varphi_p(\Delta u_j(k-1))\Delta u_j^-(k-1)\right] - \sum_{k=1}^{n+1} f(k,-u_j^-(k))u_j^-(k).\end{aligned}$$

Distinguishing the case of signs of $u_j(k-1)$ and $u_j(k)$, we prove that

$$\sum_{k=1}^{n+1}\left[\varphi_p(\Delta u_j(k-1))\Delta u_j^-(k-1)\right] \leq -\left\|u_j^-\right\|^p,$$

so

$$(\Phi'_g(u_j), u_j^-) \leq -\left\|u_j^-\right\|^p - \sum_{k=1}^{n+1} f(k,-u_j^-(k))u_j^-(k),$$

it follows from (3.1) that $(u_j^-)_j$ is bounded. So for a further subsequence, $\widetilde{u}_j := \dfrac{u_j}{\rho_j}$ converges to some $\widetilde{u}\geq 0$ in W with $\left\|\widetilde{u}\right\| = 1$. We may assume that for each k, either $(u_j(k))_j$ is bounded or $u_j(k)\to+\infty$.

If $(u_j(k))_j$ is bounded, then $\widetilde{u}_j(k)\longrightarrow 0$, $\widetilde{u}(k)=0$ and $\dfrac{f(k,u_j(k))}{\rho_j^{p-1}}\longrightarrow 0$.

If $u_j(k)\to+\infty$, since $\displaystyle\lim_{t\longrightarrow+\infty}\inf\frac{f(k,t)}{t^{p-1}}\geq\lambda_1$, we get $f(k,u_j(k))\geq 0$ for large j, by (3.1). Since $\displaystyle\sum_{k=1}^{n+1}\frac{f(k,u_j(k))}{\rho_j^{p-1}}v(k)\longrightarrow 0$, it follows from

$$o(1) = \frac{(\Phi_g'(u_j), v)}{\rho_j^{p-1}} = \sum_{k=1}^{n+1} \left[\varphi_p(\Delta \widetilde{u}_j(k-1))\Delta v(k-1) - \frac{f(k, u_j(k))}{\rho_j^{p-1}} v(k) \right], \quad (3.2)$$

that

$$\sum_{k=1}^{n+1} \varphi_p(\Delta \widetilde{u}(k-1))\Delta v(k-1) \geq 0, \qquad \forall v \geq 0, \quad (3.3)$$

and hense, $\widetilde{u} > 0$ in $[1, n]$ by Lemma 1.6. Then $u_j(k) \to \infty$ for each k, and hence, (3.2) can be written as

$$\sum_{k=1}^{n+1} \left[\varphi_p(\Delta \widetilde{u}_j(k-1))\Delta v(k-1) - \alpha_j(k) \widetilde{u}_j(k)^{p-1} v(k) \right] = o(1), \quad (3.4)$$

where $\alpha_j(k) = \dfrac{f(k, u_j(k))}{u_j(k)^{p-1}}$.

Now choosing v appropriately and passing to the limit, we get $\alpha_j(k)$ converges to some $\alpha(k) \geq \lambda_1$ and

$$-\Delta(\varphi_p(\Delta \widetilde{u}(k-1))) = \alpha(k) \left(\widetilde{u}(k) \right)^{p-1}, \quad k \in [1, n], \quad (3.5)$$

$$\widetilde{u}(0) = 0 = \widetilde{u}(n+1).$$

Suppose that for each $k \in [1, n]$, $\alpha(k) = \lambda_1$. Since $\lambda_1 \leq \lim_{t \longrightarrow +\infty} \inf \dfrac{pF(k,t)}{|t|^p}$, $k \in [1, n]$, and $\lim_{t \longrightarrow +\infty} \inf \dfrac{pF(k,t)}{|t|^p} \neq \lambda_1$ for some $k \in [1, n]$, for $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F(k, t) \geq (\beta(k) - \varepsilon) \frac{|t|^p}{p} + C_\varepsilon,$$

where $\beta(k) = \lim_{t \longrightarrow +\infty} \inf \dfrac{pF(k,t)}{|t|^p}$. As

$$\Phi_f(u_j) = \frac{1}{p} \|u_j\|^p - \sum_{k=1}^{n+1} F(k, u_j(k)),$$

then

$$\Phi_f(u_j) \leq \frac{1}{p} \|u_j\|^p - \sum_{k=1}^{n+1} \left((\beta(k) - \varepsilon) \frac{|u_j(k)|^p}{p} + C_\varepsilon \right),$$

and

$$\frac{\Phi_f(u_j)}{\|u_j\|^p} \leq \frac{1}{p} \left(1 - \sum_{k=1}^{n+1} \left((\beta(k) - \varepsilon) \frac{|u_j(k)|^p}{\|u_j\|^p} \right) \right) - (n+1) \frac{C_\varepsilon}{\|u_j\|^p}.$$

Since $(\Phi_f(u_j))_j$ is bounded and $\|u_j\| \longrightarrow +\infty$, we get

$$0 \leq \frac{1}{p}\left(1 - \sum_{k=1}^{n+1}\left((\beta(k)-\varepsilon)\left|\widetilde{u}(k)\right|^p\right)\right),$$

we tends $\varepsilon \longrightarrow 0$,

$$0 \leq \frac{1}{p}\left(1 - \sum_{k=1}^{n+1}\beta(k)\left|\widetilde{u}(k)\right|^p\right).$$

Since $1 = \left\|\widetilde{u}\right\|^p = \sum_{k=1}^{n+1}\alpha(k)\left|\widetilde{u}(k)\right|^p = \lambda_1\sum_{k=1}^{n+1}\left|\widetilde{u}(k)\right|^p$, we get

$$0 \leq \frac{1}{p}\left(\sum_{k=1}^{n+1}(\lambda_1 - \beta(k))\left|\widetilde{u}(k)\right|^p\right),$$

as $\lambda_1 \leq \beta(k)$, $k \in [1,n]$, and $\lambda_1 \neq \beta(k)$ for some $k \in [1,n]$, we get

$$\frac{1}{p}\left(\sum_{k=1}^{n+1}(\lambda_1 - \beta(k))\left|\widetilde{u}(k)\right|^p\right) < 0,$$

hence the contradiction. So $\alpha(k) \geq \lambda_1$, $k \in [1,n]$, and $\lambda_1 \neq \alpha(k)$ for some $k \in [1,n]$, and from the strict monotonicity property with respect to the weight (Proposition 1.11),we get $1 = \lambda_1(\lambda_1) > \lambda_1(\alpha)$.

Since (3.5) and $\widetilde{u} > 0$, then by Proposition 1.8 and Proposition 1.9**,** $\widetilde{u}$ is the first eigenfunction and $\lambda_1(\alpha) = 1$. Hence the contradiction.

Proof of Theorem 1.4

The problem

$$\begin{aligned} -\Delta(\varphi_p(\Delta u(k-1))) &= a_0(k), \quad k \in [1,n], \\ u(0) &= 0 = u(n+1), \end{aligned}$$

has a unique solution $u_0 > 0$ by Lemma 1.6. Let $\varepsilon \in]0,1[$ such that $\underline{u} := \varepsilon u_0 < \alpha(k)$, $k \in [1,n]$. Then by (1.4)

$$-\Delta(\varphi_p(\Delta \underline{u}(k-1))) - f(k,\underline{u}(k)) \leq -(1-\varepsilon^{p-1})a_0(k) \leq 0, \quad k \in [1,n],$$

and $\underline{u}$ is a subsolution of (1.1). Let

$$f_{\underline{u}}(k,t) = \begin{cases} f(k,t), & t \geq \underline{u}(k), \\ f(k,\underline{u}(k)), & t < \underline{u}(k). \end{cases}$$

Consider $F_{\underline{u}}(k,t) = \int_0^t f_{\underline{u}}(k,s)ds$, $k \in [1,n]$, $t \in \mathbb{R}$.

It's clear that, $\lim\limits_{t\to-\infty}\sup \dfrac{pF_{\underline{u}}(k,t)}{|t|^p} < \lambda_1$ and $\lim\limits_{t\to+\infty}\sup \dfrac{pF_{\underline{u}}(k,t)}{t^p} =$ $= \lim\limits_{t\to+\infty}\sup \dfrac{pF(k,t)}{t^p} \leq \lambda_1$, $k \in [1,n]$ and $\lim\limits_{t\to+\infty}\sup \dfrac{pF_{\underline{u}}(k,t)}{t^p} \neq \lambda_1$ for some $k \in [1,n]$.

Thus $\lim\limits_{|t|\to\infty}\sup \dfrac{pF_{\underline{u}}(k,t)}{|t|^p} \leq \lambda_1$, $k \in [1,n]$ and $\lim\limits_{|t|\to\infty}\sup \dfrac{pF_{\underline{u}}(k,t)}{|t|^p} \neq \lambda_1$ for some $k \in [1,n]$, so the modified problem

$$\begin{aligned} -\Delta(\varphi_p(\Delta u(k-1))) &= f_{\underline{u}}(k,u(k)), \quad k \in [1,n], \\ u(0) &= 0 = u(n+1), \end{aligned} \tag{3.6}$$

has a solution u which is a global minimizer of $\Phi_{f_{\underline{u}}}$ by Theorem 1.1. By lemma 1.7, $u \geq \underline{u}$, and hence, also a solution of (1.1).

Proof of Theorem 1.5

As in the proof of theorem 1.4, $\underline{u}$ is a subsolution of (1.1).

We have w is a supersolution of (3.6), let

$$f^+_{\underline{u}}(k,t) = \begin{cases} f_{\underline{u}}(k,w(k)), & t > w(k), \\ f_{\underline{u}}(k,t), & t \leq w(k). \end{cases}$$

Consider $F^+_{\underline{u}}(k,t) = \int_0^t f^+_{\underline{u}}(k,s)ds$, $k \in [1,n]$, $t \in \mathbb{R}$.

It's clear that $\lim\limits_{|t|\longrightarrow\infty}\sup \dfrac{pF^+_{\underline{u}}(k,t)}{|t^p|} \leq \lambda_1$, $k \in [1,n]$ and $\lim\limits_{|t|\longrightarrow\infty}\sup \dfrac{pF^+_{\underline{u}}(k,t)}{|t^p|} \neq \lambda_1$ for some $k \in [1,n]$, so, the modified problem

$$\begin{aligned} -\Delta(\varphi_p(\Delta u(k-1))) &= f^+_{\underline{u}}(k,u(k)), \quad k \in [1,n], \\ u(0) &= 0 = u(n+1), \end{aligned} \tag{3.7}$$

has a solution u_1 which is a local minimizer of $\Phi_{f_{\underline{u}}}$. Indeed, u_1 is a global minimizer of $\Phi_{f^+_{\underline{u}}}$ by Theorem 1.1. By Lemma 1.7 and Lemma 1.6, $\underline{u} \leq u_1 \leq w$ and $u_1 < w$, so $f^+_{\underline{u}}(k,u_1(k)) = f(k,u_1(k))$ and u_1 is a solution of (1.1). Thus $\Phi_{f^+_{\underline{u}}} = \Phi_{f_{\underline{u}}}$ near u_1. Let

$$f_{u_1}(k,t) = \begin{cases} f(k,t), & t \geq u_1(k), \\ f(k,u_1(k)), & t < u_1(k). \end{cases}$$

As u_1 is a subsolution of (1.1), by the same argument with u_1 in place of $\underline{u}$, we see that $\Phi_{f_{u_1}}$ has a local minimizer strict u_1^* also noted by u_1.

For $t > T = \max\limits_{1\leq k\leq n}(u_1(k))$, we get $f_{u_1}(k,t) = f(k,t)$ and

$$\begin{aligned} F_{u_1}(k,t) &= \int_0^t f_{u_1}(k,s)ds \\ &= \int_0^T (f_{u_1}(k,s) - f(k,s))ds + F(k,t), \end{aligned}$$

so $\lim_{t\longrightarrow+\infty}\inf \dfrac{pF_{u_1}(k,t)}{t^p} = \lim_{t\longrightarrow+\infty}\inf \dfrac{pF(k,t)}{t^p}$. For $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$F_{u_1}(k,t) \geq (\beta(k)-\varepsilon)\frac{|t|^p}{p} + C_\varepsilon,$$

where $\beta(k) = \lim_{t\longrightarrow+\infty}\inf \dfrac{pF(k,t)}{|t|^p}$. As

$$\Phi_{f_{u_1}}(u) = \sum_{k=1}^{n+1}\left[\frac{1}{p}\left|\Delta u(k-1)\right|^p - F_{u_1}(k,u(k))\right],$$

we get

$$\begin{aligned}
\Phi_{f_{u_1}}(t\phi_1) &= \sum_{k=1}^{n+1}\left[\frac{1}{p}\left|\Delta\left(t\phi_1\right)(k-1)\right|^p - F_{u_1}(k,t\phi_1(k))\right]\\
&\leq \frac{1}{p}\left\|t\phi_1\right\|^p - \sum_{k=1}^{n+1}\beta(k)\frac{|t\phi_1(k)|^p}{p} + \varepsilon\sum_{k=1}^{n+1}\frac{|t\phi_1(k)|^p}{p} - (n+1)C_\varepsilon,
\end{aligned}$$

so,

$$\begin{aligned}
\Phi_{f_{u_1}}(t\phi_1) &\leq \frac{1}{p}\left\|t\phi_1\right\|^p - \sum_{k=1}^{n+1}\beta(k)\frac{|t\phi_1(k)|^p}{p} + \varepsilon\sum_{k=1}^{n+1}\frac{|t\phi_1(k)|^p}{p} - (n+1)C_\varepsilon\\
&\leq -\frac{t^p}{p}\left(-1+\frac{1}{\lambda_1(\beta)}+\frac{\varepsilon}{\lambda_1}\right) - (n+1)C_\varepsilon.
\end{aligned}$$

We have $\beta(k)\geq\lambda_1$, $k\in[1,n]$ and $\beta(k)\neq\lambda_1$ for some $k\in[1,n]$, so, from the strict monotonicity property with respect to the weight (Proposition 1.11), we get

$$\lambda_1(\beta) < \lambda_1\left(\lambda_1\right) = 1 \text{ and } -1+\frac{1}{\lambda_1(\beta)} > 0.$$

We choose $\varepsilon>0$, such that $-1+\dfrac{1}{\lambda_1(\beta)}+\dfrac{\varepsilon}{\lambda_1}>0$, thus

$$\lim_{t\longrightarrow\infty} -\frac{t^p}{p}\left(-1+\frac{1}{\lambda_1(\beta)}+\frac{\varepsilon}{\lambda_1}\right) - (n+1)C_\varepsilon = -\infty.$$

So, for t large, we get

$$\Phi_{f_{u_1}}(t\phi_1) \leq -\frac{t^p}{p}\left(-1+\frac{1}{\lambda_1(\beta)}+\frac{\varepsilon}{\lambda_1}\right) - (n+1)C_\varepsilon < \Phi_{f_{u_1}}(u_1).$$

Since $\Phi_{f_{u_1}}$ satisfies (PS) by Lemma 3.1, the mountain- pass lemma [7] now gives a second critical point u_2, which is greater than u_1 by Lemma 1.6.

References

1. R.P. Agarwal, K. Perera, and D. O'Regan, Multiple positive solutions of singular discrete p-Laplacian problems via variational methods, Adv. Diff. Equ, 2 (2005) 93–99.

2. D. Jiang, J. Chu, D. O'Regan, and R. P. Agarwal, Positive solutions for continuous and discrete boundary value problems to the one-dimension p-Laplacian, Math. Inequal. Appl, 7 (4) (2004) 523–534.

3. J. Chu, D. Jiang, Eigenvalues and discrete boundary value problems for the one-dimensional p-Laplacian, J. Math. Anal. Appl, 305 (2005) 452–465.

4. G. Bonanno, P. Candito, Nonlinear difference equations investigated via critical points methods, Nonl. Anal. TMA, 70 (2009) 3180–3186.

5. P. Candito, N. Gionvannelli, Multiple solutions for a discrete boundary value problem, Comput. Math. Appl, 56 (2008) 959–964.

6. Jong-Ho Kim, Jea-Hyun Park and June-Yub Lee, Multiple positive solutions for discrete p-Laplacian equations with potential term, Appl. Anal. Discrete Math, 7 (2013), 27-342.

7. Dajun G. Nonlinear Functional Analysis. Shandong Science and Technology Press, 1985.

M. Chehabi, O. Chakrone, Department of Mathematics and Computer,
Laboratory Nonlinear Analysis, Faculty of Science,
University Mohammed 1st, Oujda, Morocco.
E-mail address: chehb_md@yahoo.fr
E-mail address: chakrone@yahoo.fr