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Infinitely Many Weak Solutions for Fourth-order Equations Depending on Two Parameters

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ABSTRACT: In this paper, by employing Ricceri variational principle, we prove the existence of infinitely many weak solutions for fourth-order problems depending on two real parameters. We also provide some particular cases and a concrete example in order to illustrate the main abstract results of this paper.

Key Words: Ricceri variational principle, infinitely many solutions, fourth-order equations.

Contents

1. Introduction

In this paper, we consider the following fourth-order boundary value problem with two control parameters

$$\begin{cases} u''''h(x,u') - u'' + uh(x,u') \\ \qquad\qquad = [\lambda f(x,u) + \mu g(x,u) + p(u)]h(x,u') \quad \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases} \tag{1.1}$$

where λ is a positive parameter, μ is a non-negative parameter, $f, g : [0,1]\times\mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|p(t_1) - p(t_2)| \le L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, with $p(0) = 0$, and $h : [0,1]\times\mathbb{R} \to [0,+\infty)$ is a bounded and continuous function with $m := \inf_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t) > 0$.

There is an increasing interest in studying fourth-order boundary value problems, because the static form change of a beam can be described by a fourth-order equation, and also a model to study traveling waves in suspension bridges can be described by nonlinear fourth-order equations (for instance, see [13]). More general nonlinear fourth-order elliptic boundary value problems have been studied in recent

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years. Several results are known concerning the existence of multiple solutions for fourth-order boundary value problems, and we refer the reader to [4,5,8,14,18] and the references cited therein.

In particular, authors in [5] proved the existence of at least three weak solutions for following problem:

$$\begin{cases} u''''h(x,u') - u'' = [\lambda f(x,u) + g(u)]h(x,u'), & \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases}$$

where λ is a positive parameter, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, with $g(0) = 0$, and $h : [0,1] \times \mathbb{R} \to [0,+\infty)$ is a bounded and continuous function with $\inf_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t) > 0$.

The main features of this paper are the following:
(I) the presence of a differential operator and the treatment in a suitable Sobolev function space;
(II) the use of the Ricceri variational principle, which is a powerful analytic tool for multiplicity results in nonlinear problems with a variational structure.

Recently in [6], presenting a version of the infinitely many critical points theorem of Ricceri (see [17, Theorem 2.5]), the existence of an unbounded sequence of weak solutions for a Strum-Liouville problem, having discontinuous nonlinearities, has been established. In a such approach, an appropriate oscillating behavior of the nonlinear term either at infinity or at zero is required. This type of methodology has been used in several works in order to obtain existence results for different kinds of problems (see, for instance, [1,2,3,4,7,8,9,10,11,16] and references therein). We refer to [12] for several applications of the Ricceri variational principles.
In [1], the existence of infinitely many classical solutions for the following Dirichlet quasilinear system has been obtained

$$\begin{cases} -(p_i - 1)|u_i'(x)|^{p_i-2}u_i''(x) = \lambda F_{u_i}(x, u_1, \ldots, u_n)h_i(x, u_i'), & x \in (a,b), \\ u_i(0) = u_i(1) = 0 \quad \text{for } 1 \le i \le n, \end{cases}$$

where $p_i > 1$ for $1 \le i \le n$, λ is a positive parameter, $a, b \in \mathbb{R}$ with $a < b$, $h_i : [a,b] \times \mathbb{R} \to [0,+\infty)$ is a bounded and continuous function with $m_i := \inf_{(x,t)\in[a,b]\times\mathbb{R}} h_i(x,t) > 0$ for $1 \le i \le n$, $F : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1, t_2, \ldots, t_n) \to F(x, t_1, \ldots, t_n)$ is in C^1 in $\mathbb{R}^n$ for all $x \in [a,b]$, F_{u_i} is continuous in $[a,b] \times \mathbb{R}^n$ for $1 \le i \le n$, where F_{u_i} denotes the partial derivative of F with respect to u_i, and

$$\sup_{|(t_1,\ldots,t_n)|\le M} |F_{u_i}(x, t_1, \ldots, t_n)| \in L^1([a,b])$$

for all $M > 0$ and all $1 \le i \le n$.
Now, starting from the results obtained in [1] and with the same method, we are interested in looking for a class of perturbations, namely $\mu g + p$, for which (1.1) still preserves multiple solutions. In particular, our goal in this paper is to obtain some sufficient conditions to guarantee that problem (1.1) has infinitely

many weak solutions. To this end, we require that the primitive F of f satisfies a suitable oscillatory behavior either at infinity (for obtaining unbounded solutions) or at zero (for finding arbitrarily small solutions), while G, the primitive of g, has an appropriate growth (see Theorems 3.1 and 3.7). Our approach is fully variational and the main tool is a general critical point theorem (see Lemma 2.1 below) contained in [6]; see also [17].

Here, as an example, we state a special case of our results.

Theorem 1.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a non-negative continuous function and $p:\mathbb{R}\to\mathbb{R}$ a Lipschitz continuous function with the Lipschitz constant $1\le L\le \pi^4+1$ and $p(0)=0$. Put $F(\xi):=\int_0^\xi f(t)dt$ for all $\xi\in\mathbb{R}$ and assume*

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}=0,\qquad \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}=+\infty.$$

Then, the problem

$$\begin{cases} u''''-u''+u=f(u)+p(u) & \text{in } (0,1),\\ u(0)=u(1)=0=u''(0)=u''(1) \end{cases}$$

has a sequence of pairwise distinct weak solutions.

2. Preliminaries

The goal of this work is to establish some new criteria for problem (1.1) to have infinitely many weak solutions in X. Our analysis is mainly based on a recent critical point theorem of Bonanno and Molica Bisci [6] (see Lemma (2.1) below) which is a more precise version of Ricceri's variational principle [17, Theorem 2.5].

Lemma 2.1. *Let X be a reflexive real Banach space, let $\Phi,\Psi:X\to\mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r>\inf_X\Phi$, let*

$$\varphi(r):=\inf_{u\in\Phi^{-1}(-\infty,r)}\frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r)}\Psi(v)\right)-\Psi(u)}{r-\Phi(u)},$$

$$\gamma:=\liminf_{r\to+\infty}\varphi(r),\qquad \textit{and}\qquad \delta:=\liminf_{r\to(\inf_X\Phi)^+}\varphi(r).$$

Then the following properties hold:

(a) *For every $r>\inf_X\Phi$ and every $\lambda\in(0,1/\varphi(r))$, the restriction of the functional*

$$I_\lambda:=\Phi-\lambda\Psi$$

to $\Phi^{-1}(-\infty,r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) *If $\gamma<+\infty$, then for each $\lambda\in(0,1/\gamma)$, the following alternative holds: either* (b$_1$) *I_λ possesses a global minimum, or*

(b$_2$) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*

(c$_1$) *there is a global minimum of Φ which is a local minimum of I_λ, or*

(c$_2$) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that converges weakly to a global minimum of Φ.*

Let us introduce some notation which will be used later. Define

$$\begin{aligned} H_0^1([0,1]) &:= \Big\{u \in L^2([0,1]) : u' \in L^2([0,1]),\ u(0) = u(1) = 0\Big\}, \\ H^2([0,1]) &:= \Big\{u \in L^2([0,1]) : u', u'' \in L^2([0,1])\Big\}. \end{aligned}$$

Let $X := H^2([0,1]) \cap H_0^1([0,1])$ be the Sobolev space endowed with the usual norm defined as follows:

$$\|u\| := \left(\int_0^1 |u''(x)|^2\, dx\right)^{1/2}.$$

We recall the following Poincaré type inequalities (see, for instance, [15, Lemma 2.3]):

$$\|u'\|^2_{L^2([0,1])} \le \frac{1}{\pi^2}\|u\|^2, \tag{2.1}$$

$$\|u\|^2_{L^2([0,1])} \le \frac{1}{\pi^4}\|u\|^2 \tag{2.2}$$

for all $u \in X$. For the norm in $C^1([0,1])$,

$$\|u\|_\infty := \max\Big\{\max_{t\in[0,1]} |u(t)|, \max_{t\in[0,1]} |u'(t)|\Big\},$$

we have the following relation.

Proposition 2.1. Let $u \in X$. Then

$$\|u\|_\infty \le \frac{1}{2\pi}\|u\|. \tag{2.3}$$

Proof: Taking (2.1) into account, the conclusion follows from the well-known inequality $\|u\|_\infty \le \frac{1}{2}\|u'\|_{L^2([0,1])}$. □

Let $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ be two L^1-Carathéodory functions. We recall that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if

(a) the mapping $x \longmapsto f(x,\xi)$ is measurable for every $\xi \in \mathbb{R}$;

(b) the mapping $\xi \longmapsto f(x,\xi)$ is continuous for almost every $x \in [0,1]$;

(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,1])$ such that

$$\sup_{|\xi|\le\rho} |f(x,\xi)| \le l_\rho(x)$$

for almost every $x \in [0,1]$.

Corresponding to f, g and p we introduce the functions $F, G : [0,1] \times \mathbb{R} \to \mathbb{R}$, $P : \mathbb{R} \to \mathbb{R}$ and $H : [0,1] \times \mathbb{R} \to [0,+\infty)$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi)\,d\xi, \qquad G(x,t) := \int_0^t g(x,\xi)\,d\xi,$$

$$P(t) := -\int_0^t p(\xi)d\xi,$$

$$H(x,t) := \int_0^t \left(\int_0^\tau \frac{1}{h(x,\delta)}d\delta\right) d\tau$$

for all $x \in [0,1]$ and $t \in \mathbb{R}$.

We say that a function $u \in X$ is a *weak solution* of problem (1.1) if

$$\int_0^1 u''(x)v''(x)\,dx + \int_0^1 \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)}d\tau\right) v'(x)\,dx + \int_0^1 u(x)v(x)\,dx$$
$$-\lambda \int_0^1 f(x,u(x))v(x)\,dx - \mu \int_0^1 g(x,u(x))v(x)\,dx - \int_0^1 p(u(x))v(x)\,dx = 0$$

holds for all $v \in X$.

In the following, let $M := \sup_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t)$ and suppose that the Lipschitz constant $L > 0$ of the function p satisfies the following condition:

(A_0) $1 \le L \le \pi^4 + 1$.

Now, put

$$k_1 := \frac{\pi^2 + m(\pi^4 + L + 1)}{2m\pi^4},$$

$$k_2 := \frac{\pi^4 - L + 1}{\pi^4},$$

$$A := \liminf_{\xi\to+\infty} \frac{\displaystyle\int_0^1 \sup_{|t|\le\xi} F(x,t)\,dx}{\xi^2}$$

and

$$B := \limsup_{\xi\to+\infty} \frac{\displaystyle\int_{3/8}^{5/8} F(x,\xi)\,dx}{\xi^2}.$$

3. Main results

In this section we establish the main abstract result of this paper.

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that* (A_0) *holds and moreover*

(A_1) $F(x,t) \geq 0$ *for all* $(x,t) \in ([0, \frac{3}{8}] \cup [\frac{5}{8}, 1]) \times \mathbb{R}$;

(A_2) $A < \frac{27 k_2 \pi^2}{2048 k_1} B$.

Then, setting

$$\lambda_1 := \frac{4096 k_1}{27 B}, \qquad \lambda_2 := \frac{2 k_2 \pi^2}{A},$$

for each $\lambda \in (\lambda_1, \lambda_2)$ *and for every arbitrary* L^1*-Carathéodory function* $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, *whose potential* $G(x,t) := \int_0^t g(x,\xi)\, d\xi$ *for all* $(x,t) \in [0,1] \times \mathbb{R}$, *is a non-negative function satisfying the condition*

$$g_\infty := \lim_{\xi \to +\infty} \frac{\displaystyle\int_0^1 \sup_{|t| \leq \xi} G(x,t)\, dx}{\xi^2} < +\infty, \tag{3.1}$$

if we put

$$\mu_{g,\lambda} := \frac{2 k_2 \pi^2}{g_\infty} \left(1 - \lambda \frac{A}{2 k_2 \pi^2}\right),$$

where $\mu_{g,\lambda} = +\infty$ *when* $g_\infty = 0$, *problem* (1.1) *has an unbounded sequence of weak solutions for every* $\mu \in [0, \mu_{g,\lambda})$ *in* X.

Proof: Our aim is to apply Lemma 2.1(b) to problem (1.1). To this end, fix $\overline{\lambda} \in (\lambda_1, \lambda_2)$ and g satisfying our assumptions. Since $\overline{\lambda} < \lambda_2$, we have

$$\mu_{g,\overline{\lambda}} := \frac{2 k_2 \pi^2}{g_\infty} \left(1 - \overline{\lambda} \frac{A}{2 k_2 \pi^2}\right) > 0.$$

Now fix $\overline{\mu} \in (0, \mu_{g,\overline{\lambda}})$ and set

$$J(x,\xi) := F(x,\xi) + \frac{\overline{\mu}}{\overline{\lambda}} G(x,\xi)$$

for all $(x,\xi) \in [0,1] \times \mathbb{R}$. For each $u \in X$, let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{2}\|u\|^2 + \int_0^1 H(x, u'(x))\, dx + \frac{1}{2}\int_0^1 |u(x)|^2 dx + \int_0^1 P(u(x))\, dx,$$

$$\Psi(u) := \int_0^1 J(x, u(x))\, dx,$$

and put

$$I_{\overline{\lambda}}(u) := \Phi(u) - \overline{\lambda}\Psi(u), \qquad u \in X.$$

Note that the weak solutions of (1.1) are exactly the critical points of $I_{\overline{\lambda}}$. The functionals Φ, Ψ satisfy the regularity assumptions of Lemma 2.1. Indeed, by standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_0^1 u''(x)v''(x)\,dx + \int_0^1 \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)}d\tau\right) v'(x)\,dx \\ + \int_0^1 u(x)v(x)\,dx - \int_0^1 p(u(x))v(x)\,dx$$

for any $v \in X$. Furthermore, the differential $\Phi' : X \to X^*$ is a Lipschitzian operator. Indeed, taking (2.1) and (2.2) into account, for any $u, v \in X$, there holds

$$\|\Phi'(u) - \Phi'(v)\|_{X^*} = \sup_{\|w\|\le 1} |(\Phi'(u) - \Phi'(v), w)|$$
$$\le \sup_{\|w\|\le 1} \int_0^1 |u''(x) - v''(x)||w''(x)|\,dx$$
$$+ \sup_{\|w\|\le 1} \int_0^1 \left|\int_{u'(x)}^{v'(x)} \frac{1}{h(x,\tau)}d\tau\right| |w'(x)|\,dx$$
$$+ \sup_{\|w\|\le 1} \int_0^1 |u(x) - v(x)||w(x)|\,dx$$
$$+ \sup_{\|w\|\le 1} \int_0^1 |p(u(x)) - p(v(x))||w(x)|\,dx$$
$$\le \|u - v\| + \frac{1}{m} \sup_{\|w\|\le 1} \|u' - v'\|_{L^2(0,1)}\|w'\|_{L^2(0,1)}$$
$$+(1+L) \sup_{\|w\|\le 1} \|u - v\|_{L^2(0,1)}\|w\|_{L^2(0,1)} \le \left(1 + L + \frac{1}{m\pi^2} + \frac{1}{\pi^4}\right) \|u - v\|.$$

Recalling that p is Lipschitz continuous and h is bounded away from zero, the claim is true. In particular, we derive that Φ is continuously differentiable. Also, for any

$u, v \in X$, we have

$$\begin{aligned}
&(\Phi'(u) - \Phi'(v), u - v) = \\
&\|u - v\|^2 + \int_0^1 \left(\int_{u'(x)}^{v'(x)} \frac{1}{h(x,\tau)} d\tau \right) (u'(x) - v'(x))\, dx \\
&+ \int_0^1 |u(x) - v(x)|^2\, dx - \int_0^1 (p(u(x)) - p(v(x)))(u(x) - v(x))\, dx \\
&\geq \|u - v\|^2 + \frac{1}{M} \|u' - v'\|^2_{L^2(0,1)} + (1 - L)\|u - v\|^2_{L^2(0,1)} \\
&\geq \|u - v\|^2 + (1 - L)\frac{1}{\pi^4}\|u - v\|^2 = k_2\|u - v\|^2.
\end{aligned}$$

By the assumption (A_0), it turns out that Φ' is a strongly monotone operator. So, by applying Minty-Browder theorem (Theorem 26.A of [19]), $\Phi' : X \to X^*$ admits a Lipschitz continuous inverse. On the other hand, the fact that X is compactly embedded into $C^0([0,1])$ implies that the functional Ψ is well defined, continuously Gâteaux differentiable and with compact derivative, whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_0^1 f(x, u(x))v(x)\, dx + \frac{\overline{\mu}}{\overline{\lambda}} \int_0^1 g(x, u(x))v(x)\, dx$$

for any $v \in X$. Hence Ψ is sequentially weakly (upper) continuous (see [19, Corollary 41.9]). Since p is Lipschitz continuous and satisfies $p(0) = 0$, while h is bounded away from zero, we have from (2.2) that

$$\Phi(u) \geq \frac{k_2}{2}\|u\|^2 \tag{3.2}$$

for all $u \in X$, and so Φ is coercive. First of all, we will show that $\overline{\lambda} < 1/\gamma$. Hence, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_0^1 \sup_{|t|\leq\xi_n} F(x,t)\, dx}{\xi_n^2} = A.$$

Put

$$r_n := 2k_2\pi^2\xi_n^2$$

for all $n \in \mathbb{N}$. Then, for all $v \in X$ with $\Phi(v) < r_n$, taking (2.3) into account, one has $\|v\|_\infty < \xi_n$. Note that $\Phi(0) = \Psi(0) = 0$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)\right) - \Psi(u)}{r_n - \Phi(u)} \\
&\le \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)}{r_n} \\
&\le \frac{1}{2k_2\pi^2} \frac{\int_0^1 \sup_{|t|\le\xi_n} J(x,t)\,dx}{\xi_n^2} \\
&\le \frac{1}{2k_2\pi^2}\left[\frac{\int_0^1 \sup_{|t|\le\xi_n} F(x,t)\,dx}{\xi_n^2} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_0^1 \sup_{|t|\le\xi_n} G(x,t)\,dx}{\xi_n^2}\right].
\end{aligned}$$

Moreover, from the assumption (A_2) and the condition (3.1), we have $A < +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_0^1 \sup_{|t|\le\xi_n} G(x,t)\,dx}{\xi_n^2} = g_\infty.$$

Therefore,

$$\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le \frac{1}{2k_2\pi^2}\left(A + \frac{\overline{\mu}}{\overline{\lambda}} g_\infty\right) < +\infty. \tag{3.3}$$

The assumption $\overline{\mu} \in (0, \mu_{G,\overline{\lambda}})$ immediately yields

$$\gamma \le \frac{1}{2k_2\pi^2}\left(A + \frac{\overline{\mu}}{\overline{\lambda}} g_\infty\right) < \frac{1}{2k_2\pi^2} A + \frac{1 - \frac{1}{2k_2\pi^2}\overline{\lambda} A}{\overline{\lambda}}.$$

Hence,

$$\overline{\lambda} = \frac{1}{\frac{1}{2k_2\pi^2} A + (1 - \frac{1}{2k_2\pi^2}\overline{\lambda} A)/\overline{\lambda}} < \frac{1}{\gamma}.$$

Let $\overline{\lambda}$ be fixed. We claim that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}} < \frac{27}{4096 k_1} B,$$

there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n\to+\infty} \eta_n = +\infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{27}{4096 k_1} \frac{\int_{3/8}^{5/8} F(x,\eta_n)\,dx}{\eta_n^2} \tag{3.4}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$ define

$$w_n(x) = \begin{cases} -\frac{64}{9}\eta_n(x^2 - \frac{3}{4}x), & x \in [0, \frac{3}{8}[, \\ \eta_n, & x \in [\frac{3}{8}, \frac{5}{8}], \\ -\frac{64}{9}\eta_n(x^2 - \frac{5}{4}x + \frac{1}{4}), & x \in]\frac{5}{8}, 1]. \end{cases} \tag{3.5}$$

For any fixed $n \in \mathbb{N}$, it is easy to see that $w_n \in X$ and, in particular, one has

$$\|w_n\|^2 = \frac{4096}{27}\eta_n^2. \tag{3.6}$$

Taking (2.1), (2.2) and (3.6) into account, we have

$$\Phi(w_n) \le k_1\|w_n\|^2 = \frac{4096k_1}{27}\eta_n^2. \tag{3.7}$$

On the other hand, bearing (A_1) in mind and since G is non-negative, from the definition of Ψ, we infer

$$\Psi(w_n) = \int_0^1 \left[F(x, w_n(x)) + \frac{\overline{\mu}}{\overline{\lambda}}G(x, w_n(x))\right] dx \ge \int_{3/8}^{5/8} F(x, \eta_n)\, dx. \tag{3.8}$$

By (3.4), (3.7) and (3.8), we observe that

$$I_{\overline{\lambda}}(w_n) \le \frac{4096k_1}{27}\eta_n^2 - \overline{\lambda}\int_{3/8}^{5/8} F(x, \eta_n)\, dx < \frac{4096k_1}{27}\eta_n^2(1 - \overline{\lambda}\tau) \tag{3.9}$$

for every $n \in \mathbb{N}$ large enough. Since $\overline{\lambda}\tau > 1$ and $\lim_{n\to+\infty} \eta_n = +\infty$, we have

$$\lim_{n\to+\infty} I_{\overline{\lambda}}(w_n) = -\infty.$$

Then, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, by Lemma 2.1(b), there exists a sequence $\{u_n\}$ of critical points of $I_{\overline{\lambda}}$ such that

$$\lim_{n\to+\infty} \|u_n\| = +\infty,$$

and the conclusion is achieved. □

Remark 3.2. Under the conditions $A = 0$ and $B = +\infty$, from Theorem 3.1 we see that for every $\lambda > 0$ and for each $\mu \in \big[0, \frac{2k_2\pi^2}{g_\infty}\big)$, problem (1.1) admits a sequence of weak solutions which is unbounded in X. Moreover, if $g_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu \ge 0$.

The following result is a special case of Theorem 3.1 with $\mu = 0$.

Theorem 3.3. *Assume that all the assumptions in the Theorem 3.1 hold. Then, for each*

$$\lambda \in \left(\frac{4096k_1}{27B}, \frac{2k_2\pi^2}{A} \right),$$

the problem

$$\begin{cases} u''''h(x,u') - u'' + uh(x,u') = [\lambda f(x,u) + p(u)]h(x,u'), & \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1) \end{cases}$$

has an unbounded sequence of weak solutions in X.

Remark 3.4. Theorem 1.1 in the Introduction immediately follows from Theorem 3.3, setting $h(x,t) \equiv 1$ for all $(x,t) \in [0,1] \times \mathbb{R}$.

Here, we point out the following consequence of Theorem 3.3.

Corollary 3.5. *Assume that the assumption* (A_1) *in the Theorem 3.1 holds. Suppose that*

$$A < 2k_2\pi^2, \qquad B > \frac{4096k_1}{27}.$$

Then, the problem

$$\begin{cases} u''''h(x,u') - u'' + uh(x,u') = [f(x,u) + p(u)]h(x,u') & \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1) \end{cases}$$

has an unbounded sequence of weak solutions in X.

Corollary 3.6. *Let $g_1 : [0,1] \to \mathbb{R}$ be a non-negative continuous function. Put $G_1(\xi) := \int_0^\xi g_1(t)\,dt$ for all $\xi \in \mathbb{R}$ and assume that*

(A_3) $\liminf_{\xi \to +\infty} \frac{G_1(\xi)}{\xi^2} < +\infty$;

(A_4) $\limsup_{\xi \to +\infty} \frac{G_1(\xi)}{\xi^2} = +\infty$.

Then, for every $\alpha_i \in L^1([0,1])$ for $1 \le i \le n$, with $\min_{x\in[0,1]}\{\alpha_i(x) : 1 \le i \le n\} \ge 0$ and with $\alpha_1 \neq 0$, and for every non-negative continuous $g_i : \mathbb{R} \to \mathbb{R}$ for $2 \le i \le n$, satisfying

$$\max\left\{ \sup_{\xi\in\mathbb{R}} G_i(\xi) : 2 \le i \le n \right\} \le 0$$

and

$$\min\left\{ \liminf_{\xi\to+\infty} \frac{G_i(\xi)}{\xi^2} : 2 \le i \le n \right\} > -\infty,$$

where $G_i(\xi) := \int_0^{\xi} g_i(t)\,dt$ *for all* $\xi \in \mathbb{R}$ *for* $2 \le i \le n$, *for each*

$$\lambda \in \left]0, \frac{2k_2\pi^2}{\liminf_{\xi\to+\infty} \frac{G_1(\xi)}{\xi^2} \int_0^1 \alpha_1(x)\,dx}\right[,$$

the problem

$$\begin{cases} u''''h(x,u') - u'' + uh(x,u') = [\lambda \sum_{i=1}^n \alpha_i(x) g_i(u) + p(u)]h(x,u') & \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1) \end{cases}$$

has an unbounded sequence of weak solutions in X.

Proof: Set $f(x,t) = \sum_{i=1}^n \alpha_i(x) g_i(t)$ for all $(x,t) \in [0,1]\times\mathbb{R}$. From the assumption (A_4) and the condition

$$\min\left\{\liminf_{\xi\to+\infty} \frac{G_i(\xi)}{\xi^2} : 2 \le i \le n\right\} > -\infty$$

we have

$$\limsup_{\xi\to+\infty} \frac{\int_{3/8}^{5/8} F(x,\xi)\,dx}{\xi^2} = \limsup_{\xi\to+\infty} \frac{\sum_{i=1}^n \left(G_i(\xi)\int_{3/8}^{5/8} \alpha_i(x)\,dx\right)}{\xi^2} = +\infty.$$

Moreover, from the assumption (A_3) and the condition

$$\max\left\{\sup_{\xi\in\mathbb{R}} G_i(\xi) : 2 \le i \le n\right\} \le 0$$

we have

$$\liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|t|\le\xi} F(x,t)\,dx}{\xi^2} \le \left(\int_0^1 \alpha_1(x)\,dx\right) \liminf_{\xi\to+\infty} \frac{G_1(\xi)}{\xi^2} < +\infty.$$

Hence, applying Theorem 3.3 the desired conclusion follows. □

Now, put

$$\begin{aligned} A' &:= \liminf_{\xi\to0^+} \frac{\int_0^1 \sup_{|t|\le\xi} F(x,t)\,dx}{\xi^2}, \\ B' &:= \limsup_{\xi\to0^+} \frac{\int_{3/8}^{5/8} F(x,\xi)\,dx}{\xi^2}. \end{aligned}$$

Using Lemma 2.1(c) and arguing as in the proof of Theorem 3.1, we can obtain the following result.

Theorem 3.7. *Assume that the assumption* (A_1) *in the Theorem 3.1 holds and*

(A_5) $A' < \frac{27k_2\pi^2}{2048k_1}B'$.

Then, setting

$$\lambda_3 := \frac{4096k_1}{27B'}, \qquad \lambda_4 := \frac{2k_2\pi^2}{A'},$$

for every $\lambda \in (\lambda_3, \lambda_4)$ *and for every arbitrary* L^1*-Carathéodory function* $g : [0,1] \times \mathbb{R} \to \mathbb{R}$*, whose potential* $G(x,t) := \int_0^t g(x,\xi)\,d\xi$ *for all* $(x,t) \in [0,1] \times \mathbb{R}$*, is a non-negative function satisfying the condition*

$$g_0 := \limsup_{\xi \to 0^+} \frac{\int_0^1 \sup_{|t|\le \xi} G(x,t)\,dx}{\xi^2} < +\infty, \tag{3.10}$$

if we put

$$\mu'_{g,\lambda} := \frac{2k_2\pi^2}{g_0}\left(1 - \lambda\frac{A'}{2k_2\pi^2}\right),$$

where $\mu'_{g,\lambda} = +\infty$ *when* $g_0 = 0$*, for every* $\mu \in [0, \mu'_{g,\lambda})$ *problem* (1.1) *has a sequence of weak solutions, which strongly converges to zero in* X*.*

Proof: Fix $\overline{\lambda} \in (\lambda_3, \lambda_4)$ and let g be a function that satisfies the condition (3.10). Since $\overline{\lambda} < \lambda_4$, we obtain

$$\mu'_{g,\overline{\lambda}} := \frac{2k_2\pi^2}{g_0}\left(1 - \overline{\lambda}\frac{A'}{2k_2\pi^2}\right) > 0.$$

Now fix $\overline{\mu} \in (0, \mu'_{g,\overline{\lambda}})$ and set

$$J(x,t) := F(x,\xi) + \frac{\overline{\mu}}{\overline{\lambda}}G(x.\xi)$$

for all $(x,t) \in [0,1] \times \mathbb{R}$. We take Φ, Ψ and $I_{\overline{\lambda}}$ as in the proof of Theorem 3.1. Now, as it has been pointed out before, the functionals Φ and Ψ satisfy the regularity assumptions required in Lemma 2.1. As first step, we will prove that $\overline{\lambda} < 1/\delta$. Then, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty}\xi_n = 0$ and

$$\lim_{n\to+\infty} \frac{\int_0^1 \sup_{|t|\le \xi_n} F(x,t)\,dx}{\xi_n^2} = A'.$$

By the fact that $\inf_X \Phi = 0$ and the definition of δ, we have $\delta = \liminf_{r\to 0^+} \varphi(r)$. Then, as in showing (3.3) in the proof of Theorem 3.1, we can prove that $\delta < +\infty$. From $\overline{\mu} \in (0, \mu'_{g,\overline{\lambda}})$, the following inequalities hold

$$\delta \leq \frac{1}{2k_2\pi^2}\Big(A' + \frac{\overline{\mu}}{\overline{\lambda}} g_0\Big) < \frac{1}{2k_2\pi^2} A' + \frac{1 - \frac{1}{2k_2\pi^2}\overline{\lambda}A'}{\overline{\lambda}}.$$

Therefore

$$\overline{\lambda} = \frac{1}{\frac{1}{2k_2\pi^2}A' + \big(1 - \frac{1}{2k_2\pi^2}\overline{\lambda}A'\big)/\overline{\lambda}} < \frac{1}{\delta}.$$

Let $\overline{\lambda}$ be fixed. We claim that the functional $I_{\overline{\lambda}}$ has not a local minimum at zero. Since

$$\frac{1}{\overline{\overline{\lambda}}} < \frac{27}{4096k_1} B',$$

there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n\to+\infty} \eta_n = 0^+$ and

$$\frac{1}{\overline{\overline{\lambda}}} < \tau < \frac{27}{4096k_1} \frac{\displaystyle\int_{3/8}^{5/8} F(x, \eta_n)\, dx}{\eta_n^2}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$, let $w_n(x)$ defined by (3.5) with the above η_n. Note that $\overline{\lambda}\tau > 1$. Then, as in showing (3.9), we can obtain that

$$\begin{aligned}
I_{\overline{\lambda}}(w_n) &= \Phi(w_n) - \overline{\lambda}\Psi(w_n) \\
&\leq \frac{4096k_1}{27}\eta_n^2 - \overline{\lambda}\int_{3/8}^{5/8} F(x, \eta_n)\, dx \\
&< \frac{4096k_1}{27}\eta_n^2(1 - \overline{\lambda}\tau) < 0
\end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Then, since

$$\lim_{n\to+\infty} I_{\overline{\lambda}}(w_n) = I_{\overline{\lambda}}(0) = 0,$$

we see that zero is not a local minimum of $I_{\overline{\lambda}}$. This, together with the fact that zero is the only global minimum of Φ, we deduce that the energy functional $I_{\overline{\lambda}}$ has not a local minimum at the unique global minimum of Φ. Therefore, by Lemma 2.1(c), there exists a sequence $\{u_n\}$ of critical points of $I_{\overline{\lambda}}$ which converges weakly to zero. In view of the fact that the embedding $X \hookrightarrow C^0([0, 1])$ is compact, we know that the critical points converge strongly to zero, and the proof is complete. □

Remark 3.8. Under the conditions $A' = 0$ and $B' = +\infty$, Theorem 3.7 ensures that for every $\lambda > 0$ and for each $\mu \in \big[0, \frac{2k_2\pi^2}{g_0}\big)$, problem (1.1) admits a sequence of weak solutions which strongly converges to 0 in X. Moreover, if $g_0 = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

Remark 3.9. Applying Theorem 3.7, results similar to Theorems 1.1 and 3.3 Corollaries 3.5 and 3.6 can be obtained. We omit the discussions here.

We conclude this paper with the following example to illustrate our results.

Example 3.10. *Put*

$$a_n := \frac{2n!(n+2)!-1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)!+1}{4(n+1)!}$$

for every $n \in \mathbb{N}$, *and define the non-negative continuous function* $f : \mathbb{R} \to \mathbb{R}$ *by*

$$f(\xi) = \begin{cases} \frac{32(n+1)!^2[(n+1)!^2-n!^2]}{\pi}\sqrt{\frac{1}{16(n+1)!^2} - \left(\xi - \frac{n!(n+2)}{2}\right)^2} & \text{if } \xi \in \bigcup_{n\in\mathbb{N}}[a_n, b_n], \\ 0, & \text{otherwise.} \end{cases}$$

One has

$$\int_{n!}^{(n+1)!} f(t)dt = \int_{a_n}^{b_n} f(t)dt = (n+1)!^2 - n!^2$$

for every $n \in \mathbb{N}$. *Then, one has*

$$\lim_{n\to+\infty} \frac{F(a_n)}{a_n^2} = 0 \quad \text{and} \quad \lim_{n\to+\infty} \frac{F(b_n)}{b_n^2} = 4.$$

Note that there is no sequence $\{c_n\}$ *such that* $\lim_{n\to+\infty} F(c_n)/c_n^2 > 4$. *Therefore,*

$$\lim_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0 \quad \text{and} \quad \lim_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 4.$$

Hence, by choosing $p(t) = -t$ *for all* $t \in \mathbb{R}$ *and* $h(x,t) \equiv 1$ *for all* $(x,t) \in [0,1] \times \mathbb{R}$, *we have*

$$0 = \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} < \frac{27\pi^6}{4096(\pi^4+\pi^2+2)} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = \frac{27\pi^6}{1024(\pi^4+\pi^2+2)}.$$

So, using Theorem 3.1, problem

$$\begin{cases} u'''' - u'' + 2u = \lambda f(u) \quad \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1) \end{cases}$$

has a sequence of weak solutions which is unbounded in X.

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References

1. Afrouzi, G. A., Hadjian, A., *Infinitely many solutions for a class of Dirichlet quasilinear elliptic systems*, J. Math. Anal. Appl., **393**, 265-272, (2012).

2. Afrouzi, G. A., Hadjian A., Heidarkhani, S., *Infinitely many solutions for a mixed doubly eigenvalue boundary value problem*, Mediterr. J. Math., **10**, 1317-1331, (2013).

3. Afrouzi, G. A., Heidarkhani, S., Shokooh, S., *Infinitely many solutions for Steklov problems associated to non-homogeneous differential operators through Orlicz-Sobolev spaces*, Complex Var. Elliptic Equ., **60**(11), 1505-1521, (2015).

4. Afrouzi, G. A., Shokooh, S., *Existence of infinitely many solutions for quasilinear problems with a $p(x)$-biharmonic operator*, Electron. J. Diff. Eqs., **317**, 1-14, (2015).

5. Afrouzi, G. A., Shokooh, S., *Three solutions for a fourth-order boundary-value problem*, Electron. J. Diff. Eqs., **45**, 1-11, (2015).

6. Bonanno, G., Molica Bisci, G., *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities*, Bound. Value Probl., **2009**, 1-20, (2009).

7. Candito, P., *Infinitely many solutions to the Neumann problem for elliptic equations involving the p- Laplacian and with discontinuous nonlinearities*, Proc. Edin. Math. Soc., **45**, 397-409, (2002).

8. Candito, P., Li, L., Livrea, R., *Infinitely many solutions for a perturbed nonlinear Navier boundary value problem involving the p-biharmonic*, Nonlinear Anal., **75**, 6360-6369, (2012).

9. Graef, J.R., Heidarkhani, S., Kong, L., *Infinitely many solutions for systems of multi-point boundary value problems using variational methods*, Topol. Methods Nonlinear Anal., to appear.

10. Heidarkhani, S., *Infinitely many solutions for systems of two-point Kirchhoff-type boundary value problems*, Ann. Polon. Math., **107**, 133-152, (2013).

11. Heidarkhani, S., Henderson, J., *Infinitely many solutions for nonlocal elliptic systems of $(p_1, ..., p_n)$-Kirchhoff type*, Ann. Polon. Math., **69** (2012), 1-15.

12. Kristály, A., Rădulescu, V., Varga, C., *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, No. 136, Cambridge University Press, Cambridge, (2010).

13. Lazer, A.C., McKenna, P.J., *Large amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis*, SIAM Rev., **32**, 537–578, (1990).

14. Liu, X.L., Li, W.T., *Existence and multiplicity of solutions for fourth-order boundary value problems with parameters*, J. Math. Anal. Appl., **327**, 362–375, (2007).

15. Peletier, L. A., Troy, W. C., Van der Vorst, R. C. A. M., *Stationary solutions of a fourth order nonlinear diffusion equation*, (Russian) Translated from the English by V. V. Kurt. *Differentsialnye Uravneniya* **31**, 327-337, (1995). English translation in *Differential Equations* **31**, 301-314, (1995).

16. Rădulescu, V., Repovš, D., *Partial Differential Equations with Variable Exponents. Variational Methods and Qualitative Analysis, Monographs and Research Notes in Mathematics*, CRC Press, Boca Raton, FL, 2015.

17. Ricceri, B., *A general variational principle and some of its applications*, J. Comput. Appl. Math., **113**, 401-410, (2000).

18. Yin, H., Xu, M., *Existence of three solutions for a Navier boundary value problem involving the $p(x)$-biharmonic operator*, Ann. Polon. Math., **109**, 47-54, (2013).

19. Zeidler, E., *Nonlinear Functional Analysis and its Applications*, vol. II/B and III, Berlin-Heidelberg-New York, (1990).

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