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Geometric properties of the complex Baskakov-Stancu operators in the unit disk

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abstract: In this article, we determine certain conditions under which the partial sums involving the complex Baskakov-Stancu operators of analytic univalent functions of bounded turning are also of bounded turning. Moreover, we consider some geometric properties such as starlikeness and convexity for these partial sums. The lower bound of the partial sums of univalent functions is computed using the lower bound of the complex Baskakov-Stancu operators of analytic functions.

Key Words: analytic function, bounded turning, univalent, partial sums, unit disk

Contents

1. Introduction

A central concept in Geometric Function Theory is that of univalence. Many sufficient conditions of geometric type that impose univalence are important, like : starlikeness, convexity, close-to-convexity, α-convexity, spirallikeness and bounded turning. All these geometric sufficient conditions for univalence are mainly studied for analytic functions, because in this case they can obviously be expressed by nice (and simple) differential inequalities. Also, because of the Riemann Mapping Theorem, in general it suffices to study these properties on the open unit disk.

Concerning these properties, it is normal to ask how well can be approximated an analytic function having a given property in Geometric Function Theory, by polynomials having the same property. The history of this problem contains three main directions of research, depending on the methods used : approximation preserving geometric properties by the partial sums (for resent work see [1], [2]); approximation preserving geometric properties by Cesàro means (for recent work see [3]);

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approximation of univalent functions by subordinate polynomials, by using the concept of maximal polynomial range (for recent work see [4], [5]).

It was shown that the partial sums of the Libera integral operator of univalent functions is starlike in $|z| < \frac{3}{8}$. The number $\frac{3}{8}$ is sharp ([6]). In [7], it was also shown that the partial sums of the Libera integral operator of functions of bounded turning are also of bounded turning. Moreover, in [1], the authors determined conditions under which the partial sums of some multiplier integral operators of analytic univalent functions of bounded turning are also of bounded turning. In [8], Owa considered the starlikeness and convexity of special classes of partial sums of certain analytic functions in the open unit disk. In [9], Silverman studied the radii properties for the sequence of partial sums of subfamilies of univalent functions. In [10], Latha and Shivarudrappa gave some results concerning partial sums of certain meromorphic functions. In [11], Goyal et. all., concerned on partial sums of certain meromorphic multivalent functions.

In this paper, we use the partial sums method in order to obtain new results concerning the preservation of geometric properties, such as bounded turning, by approximating and interpolating polynomials. These partial sums involve the complex Baskakov-Stancu operators of analytic univalent functions.

2. Preliminaries

2.1. Concepts in geometric function theory

One of the major branches of complex analysis is univalent function theory: the study of one-to-one analytic functions. A domain E of the complex plane is said to be convex if and only if the line segment joining any two points of E lies entirely in E : An analytic, univalent function f in the unit disk U mapping the unit disk onto some convex domain is called a convex function. Moreover, A set $D \subset \mathbb{C}$ is said to be starlike with respect to the point $z_0 \in D$ if the line segment joining z_0 to all points $z \in D$ lies in D. A function $f(z)$ which is analytic and univalent in the unit disk $U, f(0) = 0$ and maps U onto a starlike domain with respect to the origin.

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots .$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \tag{2.1}$$

A function $f \in \mathcal{A}$ is called starlike of order μ if it satisfies the following inequality

$$\Re\{\frac{zf'(z)}{f(z)}\} > \mu, \quad (z \in U)$$

for some $0 \leq \mu < 1$. We denoted this class $\mathcal{S}^*(\mu)$.

A function $f \in \mathcal{A}$ is called convex of order μ if it satisfies the following inequality

$$\Re\{\frac{zf''(z)}{f'(z)} + 1\} > \mu, \quad (z \in U)$$

for some $0 \leq \mu < 1$. We denoted this class $\mathcal{C}(\mu)$. Note that $f \in \mathcal{C}(\mu)$ if and only if $zf' \in \mathcal{S}^*(\mu)$.

For $0 \leq \mu < 1$, let $B(\mu)$ denote the class of functions f of the form (2.1) so that $\Re\{f'\} > \mu \in U$. The functions in $B(\mu)$ are called functions of bounded turning (c.f. [12, Vol. II]). By the Nashiro-Warschowski Theorem (see e.g. [12, Vol. I]) the functions in $B(\mu)$ are univalent and also close-to-convex in U.

We need the following results in the sequel.

Lemma 2.1. *[7] For $z \in U$ we have*

$$\Re\Big\{\sum_{m=1}^{j} \frac{z^m}{m+2}\Big\} > -\frac{1}{3}, \quad (z \in U).$$

Lemma 2.2. *[12, Vol. I] Let $P(z)$ be analytic in U, such that $P(0) = 1$, and $\Re(P(z)) > \frac{1}{2}$ in U. For functions Q analytic in U the convolution function $P * Q$ takes values in the convex hull of the image on U under Q.*

The operator $(*)$ stands for the Hadamard product or convolution of two power series in $\mathcal{A}$,

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in U).$$

2.2. The complex Baskakov-Stancu operators

In the present paper we concern about the following complex Baskakov-Stancu operator:

$$V_n^{\alpha,\beta}(f,z) = \sum_{\nu=0}^{\infty} \frac{n(n+1)...(n+\nu-1)}{(n+\beta)^{\nu}}[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, ..., \frac{\alpha+\nu}{n+\beta}; f]z^{\nu}, \tag{2.2}$$

where $[x_0, ..., x_m; f]$ denotes the divided difference of the function f on the distinct points $x_0, ..., x_m$ and for $\nu = 0$, we put $n(n+1)...(n+\nu-1) = 1$. Note that the operator $V_n^{\alpha,\beta}(f,z)$ is well defined for all $z \in \mathbb{C}$. In [13], Gal et. all. studied the rate of the approximation of analytic functions for the Baskakov-Stancu operator $V_n^{\alpha,\beta}(f,z)$.

Operator (2.2) has the following properties which can be found in [13]:

Lemma 2.3. *For all $n, k \in \mathbb{N} \cup \{0\}$, $0 \le \alpha \le \beta$, $z \in \mathbb{C}$, let us define*

$$V_n^{\alpha,\beta}(e_k, z) = \sum_{\nu=0}^{\infty} \frac{n(n+1)...(n+\nu-1)}{(n+\beta)^\nu} [\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, ..., \frac{\alpha+\nu}{n+\beta}; e_k] z^\nu, \tag{2.3}$$

where $e_k = z^k$. Then

$$\begin{aligned} V_n^{\alpha,\beta}(e_0, z) &= 1 \\ V_n^{\alpha,\beta}(e_1, z) &= \frac{nz+\alpha}{n+\beta} \\ V_n^{\alpha,\beta}(e_2, z) &= \frac{n(n+1)z^2}{(n+\beta)^2} + \frac{nz(1+2\alpha)}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2} \\ &\vdots \\ V_n^{\alpha,\beta}(e_{k+1}, z) &= \frac{z(1+z)}{n+\beta} V_n^{\alpha,\beta}(e_k, z)' + (\frac{nz+\alpha}{n+\beta}) V_n^{\alpha,\beta}(e_k, z). \end{aligned} \tag{2.4}$$

And the recursive relations

$$V_n^{\alpha,\beta}(e_k, z) = \sum_{j=0}^{k} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} V_n(e_j, z), \quad V_n^{0,0}(e_j, z) = V_n(e_j, z). \tag{2.5}$$

Lemma 2.4. *For all $n, k \in \mathbb{N} \cup \{0\}$, $0 \le \alpha \le \beta$, $z \in U$, we have*

$$|V_n^{\alpha,\beta}(e_k, z)| \le (k+1)!.$$

Lemma 2.5. *Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be analytic for all $z \in U$. Then*

$$|V_n^{\alpha,\beta}(f, z) - f(z)| \ge \frac{C}{n},$$

where C is a constant depends on α, β and f.

We proceed to construct partial sums involving the complex Baskakov-Stancu operator (2.5). Let $\phi_k = (\frac{n+\beta}{\alpha})^k$, $\alpha \ne 0$; thus, in view of Eq. (2.4) (Lemma 2.3), we define the partial sums

$$\begin{aligned} V_{n,\phi_k}^{\alpha,\beta}(e_k, z) &= \phi_k V_n^{\alpha,\beta}(e_k, z) \\ &= \sum_{j=0}^{k} \binom{k}{j} \frac{n^j \alpha^{k-j} \phi_k}{(n+\beta)^k} V_n(e_j, z) \\ &= 1 + b_1 z + b_2 z^2 + ... + b_k z^k, \end{aligned} \tag{2.6}$$

where $b_0 = 1$,

$$b_1 = \phi_k \Big(\binom{k}{1} [\frac{n\alpha^{k-1}}{(n+\beta)^k}] + \binom{k}{2} [\frac{n^2\alpha^{k-2}}{(n+\beta)^k}](\frac{1}{n}) + \binom{k}{3} [\frac{n^3\alpha^{k-3}}{(n+\beta)^k}](\frac{2}{n^2}) + ... \Big)$$

$$b_2 = \phi_k\Big(\binom{k}{2} [\frac{n^2\alpha^{k-2}}{(n+\beta)^k}](\frac{n+1}{n}) + \binom{k}{3} [\frac{n^3\alpha^{k-3}}{(n+\beta)^k}](\frac{3(n+1)}{n^2}) + ...\Big)$$

$$b_3 = \phi_k\Big(\binom{k}{3} [\frac{n^3\alpha^{k-3}}{(n+\beta)^k}](\frac{(n+1)(n+2)}{n^2}) + ...\Big)$$

$$\vdots.$$

Immediately, in virtue of Lemma 2.2, with $P(z) = 1$ and $Q(z) = V^{\alpha,\beta}_{n,\phi_k}(e_k, z)$, we have $V^{\alpha,\beta}_{n,\phi_k}(e_k, z)$ is a convex function in the unit disk. Moreover, if $\alpha - n \leq \beta$ i.e. $(0 < \phi_k \leq 1)$, then the function $V^{\alpha,\beta}_n(e_k, z)$ is also convex in the unit disk.

Now by employing the Hadamard product of analytic functions $f \in \mathcal{A}$, we may define a linear operator using (2.6) as follows:

$$\begin{aligned} P_k(z) &:= zV^{\alpha,\beta}_{n,\phi_k}(e_k, z) * f(z) \\ &= z + \sum_{j=2}^{k} a_j b_{j-1} z^j, \end{aligned} \tag{2.7}$$

3. Main results

By making use Lemma 2.1 and Lemma 2.2, we illustrate the conditions under which the $k-$th partial sums of the operator (2.6) of analytic univalent functions of bounded turning are also of bounded turning.

Theorem 3.1. *Let $f \in \mathcal{A}$. If $\frac{1}{2} < \mu < 1$ and $f(z) \in B(\mu)$, then*

$$P_k(z) \in B\big(\frac{2+\mu}{3}\big), \quad \alpha \geq 1.$$

Proof: Let f be of the form (2.1) and $f(z) \in B(\mu)$ that is

$$\Re\{f'(z)\} > \mu, \ \ (\frac{1}{2} < \mu < 1, \ \ z \in U).$$

Implies

$$\Re\{1 + \sum_{j=2}^{\infty} j a_j z^{j-1}\} > \frac{1}{2}.$$

Now for $\frac{1}{2} < \mu < 1$ we have

$$\Re\Big\{1 + \sum_{j=2}^{\infty} a_j \frac{j}{1-\mu} z^{j-1}\Big\} > \Re\Big\{1 + \sum_{j=2}^{\infty} j a_j z^{j-1}\Big\}$$

then

$$\Re\Big\{1+\sum_{j=2}^{\infty}\frac{j}{1-\mu}a_jz^{j-1}\Big\}>\frac{1}{2}. \tag{3.1}$$

Applying the convolution properties of power series to $P_k'(z)$ we may write

$$\begin{aligned}P_k'(z)&=1+\sum_{j=2}^{k}ja_jb_{j-1}z^{j-1}\\&=\Big[1+\sum_{j=2}^{k}\frac{j}{(1-\mu)}a_jz^{j-1}\Big]*\Big[1+\sum_{j=2}^{k}b_{j-1}(1-\mu)z^{j-1}\Big]\\&:=P(z)*Q(z).\end{aligned} \tag{3.2}$$

Now in virtue of Lemma 2.1, we receive

$$\Re\Big\{\sum_{j=2}^{k}\frac{z^{j-1}}{j+1}\Big\}\geq-\frac{1}{3}. \tag{3.3}$$

Furthermore, for sufficient large n, b_j satisfies the inequality

$$b_j=\psi_{k,n}\sum_{j=1}^{k}\frac{\alpha^{k-j}}{j!(k-j)!}\geq\sum_{j=1}^{k}\frac{\alpha^{k-j}}{j!(k-j)!}\geq\sum_{j=1}^{k}\frac{1}{j+2}, \tag{3.4}$$

where $\psi_{k,n}:=\frac{e(n)k!}{\alpha^k}$ and $\alpha\geq 1$. Thus (3.3) and (3.4) yield

$$\Re\Big\{\sum_{j=2}^{k}b_{j-1}z^{j-1}\Big\}\geq\Re\Big\{\sum_{j=2}^{k}\frac{z^{j-1}}{j+1}\Big\}. \tag{3.5}$$

Hence

$$\Re\Big\{\sum_{j=2}^{k}b_{j-1}z^{j-1}\Big\}\geq-\frac{1}{3}. \tag{3.6}$$

A computation gives

$$\Re\Big\{Q(z)\Big\}=\Re\Big\{1+\sum_{j=2}^{k}b_{j-1}(1-\mu)z^{j-1}\Big\}>\frac{2+\mu}{3}.$$

On the other hand, the power series

$$P(z)=\Big[1+\sum_{j=2}^{k}\frac{n}{(1-\mu)}a_jz^{j-1}\Big],\ \ (z\in U)$$

satisfies: $P(0) = 1$ and

$$\Re\Big\{P(z)\Big\} = \Re\Big\{1 + \sum_{j=2}^{k} \frac{n}{(1-\mu)} a_j z^{j-1}\Big\} > \frac{1}{2}, \ \ (z \in U).$$

Therefore, by Lemma 2.2, we have

$$\Re\Big\{P_k'(z)\Big\} > \frac{2+\mu}{3}, \ \ (z \in U).$$

This completes the proof of Theorem 3.1. □

We define the function S_k which is a partial sum of $f \in \mathcal{A}$ by

$$S_k(z) = z + a_k b_{k-1} z^k := z + A_k z^k, \quad k \geq 2, \ a_k \neq 0. \tag{3.7}$$

Now we proceed to compute the radii of starlikness and convexity of $S_k(z)$.

Theorem 3.2. *The function $S_k(z)$ satisfies*

$$\frac{1 - k!\phi_{k-1}|a_k|r^{k-1}}{1 - |A_k|r^{k-1}} \leq \Re\Big(\frac{zS_k'(z)}{S_k(z)}\Big) \leq \frac{1 + k!\phi_{k-1}|a_k|r^{k-1}}{1 + |A_k|r^{k-1}} \tag{3.8}$$

for

$$0 \leq r < \sqrt[k-1]{\frac{1}{|A_k|}} \leq 1, \, |A_k| \neq 0.$$

Furthermore, $S_k(z) \in S^*(\gamma)$, $0 \leq \gamma < 1$ for

$$0 \leq r < \sqrt[k-1]{\frac{1-\gamma}{k!\phi_{k-1}|a_k| - \gamma|A_k|}} \leq 1, \quad |a_k| \neq 0.$$

Proof: Noting that

$$\frac{zS_k'(z)}{S_k(z)} = k - \frac{(k-1)}{1 + A_k z^{k-1}}.$$

It follows from Lemma 2.4

$$b_{k-1} \leq |V_n^{\alpha,\beta}(e_{k-1}, z)| \leq \phi_{k-1} k!$$

and that for $\cos\theta \to 1$, we obtain

$$\begin{aligned}\Re\Big(\frac{zS_k'(z)}{S_k(z)}\Big) &= k - (k-1)\frac{1 + |A_k|\cos\theta r^{k-1}}{1 + 2|A_k|r^{k-1}\cos\theta + (A_k)^2 r^{2(k-1)}} \\ &\leq \frac{1 + k!\phi_{k-1}|a_k|r^{k-1}}{1 + |A_k|r^{k-1}}.\end{aligned}$$

Moreover, we also observe that

$$\Re\Big(\frac{zS_k'(z)}{S_k(z)}\Big) \geq \frac{1-k!\phi_{k-1}|a_k|r^{k-1}}{1-|A_k|r^{k-1}}.$$

Now assume that

$$\frac{1-k!\phi_{k-1}|a_k|r^{k-1}}{1-|A_k|r^{k-1}} > \gamma$$

for

$$0 \leq r < \sqrt[k-1]{\frac{1-\gamma}{k!\phi_{k-1}|a_k|-\gamma|A_k|}} \leq 1.$$

This completes the proof. □

Next we drive the radiuses of convexity.

Theorem 3.3. $S_k(z) \in \mathcal{C}(\mu)$, $0 \leq \mu < 1$, *for*

$$0 \leq r < \sqrt[k-1]{\frac{1-\mu}{(k-\mu)|A_k|}} \leq 1,\ |A_k| \neq 0.$$

Proof: A computation gives

$$1+\frac{zS_k''(z)}{S_k'(z)} = k - \frac{(k-1)}{1+kA_kz^{k-1}}.$$

Therefor, for $\cos\theta \to 1$, we obtain

$$\begin{aligned}\Re\Big(1+\frac{zS_k''(z)}{S_k'(z)}\Big) &= k-(k-1)\frac{1+|A_k|\cos\theta r^{k-1}}{1+2k|A_k|r^{k-1}\cos\theta+k^2|A_k|^2r^{2(k-1)}}\\ &\leq \frac{1+k|A_k|r^{k-1}}{1+|A_k|r^{k-1}}.\end{aligned}$$

Moreover, we impose

$$\Re\Big(1+\frac{zS_k''(z)}{S_k'(z)}\Big) \geq \frac{1-k|A_k|r^{k-1}}{1-|A_k|r^{k-1}}.$$

Consider that

$$\frac{1-k|A_k|r^{k-1}}{1-|A_k|r^{k-1}} > \mu$$

for

$$0 \leq r < \sqrt[k-1]{\frac{1-\mu}{(k-\mu)|A_k|}} \leq 1,\ |A_k| \neq 0.$$

This completes the proof. □

Finally we compute the lower bound of the partial sums

$$f_m(z) = z + \sum_{j=2}^{m} z^j, \quad z \in U. \tag{3.9}$$

Theorem 3.4. *Let f_m be the partial sums defined in (3.9). Then*

$$|f_m| \geq \frac{C}{n} - \sum_{k=1}^{m} (k+1)! \geq 0,$$

where C is a sufficient large positive constant.

Proof: By applying Lemma 2.5. □

4. Conclusion

We defined new partial sums in the unit disk using the modified complex Baskakov-Stancu operator. We studied these partial sums geometrically by employing some concepts of the geometric functions theory. Partial sums play important roles in the univalent function theory. It has been shown that the proposal partial sums preserved most of the geometric properties such as bounded turning, starlikeness and convexity.

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