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Strongly Nonlinear Parabolic Problems in Musielak-Orlicz-Sobolev Spaces

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ABSTRACT: We prove in this paper the existence of solutions of strongly nonlinear parabolic problems in Musielak-Orlicz-Sobolev spaces. An approximation and a compactness results in inhomogeneous Musielak-Orlicz-Sobolev spaces have also been provided.

Key Words: Inhomogeneous Musielak-Orlicz-Sobolev spaces; parabolic problems; Compactness.

Contents

1. Introduction

Let Ω a bounded open subset of $\mathbb{R}^n$ and let Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$.
We consider the strongly nonlinear parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f \text{ in } Q \\ u(x,t) = 0 \text{ on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \text{ in } \Omega \end{cases} \tag{1}$$

where $A = -\text{ div } (a(x,t,u,\nabla u))$ is an operator of Leray-Lions type, g is a nonlinearity with the sign condition but any restriction on its growth.

This result generalizes analogous ones of Lions [21], Landes [18] when $g \equiv 0$ and of Brezis-Browder [9], Landes.Mustonen [19] for $g \equiv g(x,t,u)$. See also [7,8] for related topics. In these results, the function a is supposed to satisfy a polynomial growth condition with respect to u and ∇u.

In the case where a satisfies a more general growth condition with respect to u and ∇u, it is shown in [12] that the adequate space in which (1) can be studied is the inhomogeneous Orlicz-Sobolev space $W^{1,x}L_M(Q)$ where the N-function M is

2000 *Mathematics Subject Classification:* 46E35, 35K15, 35K20, 35K60 related to the actual growth of a . The solvability of (1) in this setting is proved by Donaldson [12] for $g \equiv 0$ and by Robert [23] for $g \equiv g(x,t,u)$ when A is monotone, $t^2 \ll M(t)$ and $\overline{M}$ satisfies a Δ_2 condition and also by Elmahi [14] for $g = g(x,t,u,\nabla u)$ when M satisfies a Δ' condition and $M(t) \ll t^{N/(N-1)}$ as application of some L_M compactness results in $W^{1,x}L_M(Q)$, see [13].

The solvability of (1) in this setting is proved by Elmahi-Meskine [16] for $g \equiv 0$ and for $g \equiv g(x,t,u,\nabla u)$ in [15], without assuming any restriction on the N-function M.

In a recent work, the authors [2] have established an existence result for problems of the form (1), when $g \equiv 0$, without assuming any restriction on the Musielak function φ.

It is our purpose in this paper to prove the existence of solutions for problem (1) in the setting of Musielak-Orlicz spaces for general Musielak function φ with a nonlinearity $g(x,t,u,\nabla u)$ having natural growth with respect to the gradient. In section 3 some new approximation result in inhomogeneous Musielak-Orlicz-Sobolev spaces (see Theorem 3.2), and, on the other hand, to prove a trace result (see Lemma 4.2). In Section 4, we establish L^1-compactness results in the inhomogeneous Musielak-Orlicz-Sobolev spaces $W^{1,x}L_\varphi(Q)$. Section 5 contains the main result of this paper.

Our result generalizes that of the Elmahi-Meskine in [15] to the case of inhomogeneous Musielak- Orlicz-Sobolev spaces.

Let us point out that our result can be applied in the particular case when $\varphi(x,t) = t^p(x)$, in this case we use the notations $L^{p(x)}(\Omega) = L_\varphi(\Omega)$, and $W^{m,p(x)}(\Omega) = W^m L_\varphi(\Omega)$. These spaces are called Variable exponent Lebesgue and Sobolev spaces.

For some classical and recent results on elliptic and parabolic problems in Orlicz-sobolev spaces and a Musielak-Orlicz-Sobolev spaces, we refer to [1,2,3,6,12,14,15, 16,24].

2. Preliminaries

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. Standard reference is [22]. We also include the definition of inhomogeneous Musielak-Orlicz-Sobolev spaces and some preliminaries Lemmas to be used later.

Musielak-Orlicz-Sobolev spaces : Let Ω be an open subset of $\mathbb{R}^n$.
A Musielak-Orlicz function φ is a real-valued function defined in $\Omega \times \mathbb{R}_+$ such that :

a) $\varphi(x,t)$ is an N-function i.e. convex, nondecreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ for all $t > 0$ and

$$\lim_{t \longrightarrow 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0$$

$$\lim_{t \longrightarrow \infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0.$$

b) $\varphi(.,t)$ is a Lebesgue measurable function

Now, let $\varphi_x(t) = \varphi(x,t)$ and let φ_x^{-1} be the non-negative reciprocal function with respect to t, i.e the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \phi_x^{-1}) = t.$$

For any two Musielak-Orlicz functions φ and γ we introduce the following ordering :

c) if there exists two positives constants c and T such that for almost everywhere $x \in \Omega$:

$$\varphi(x,t) \leq \gamma(x,ct) \text{ for } t \geq T$$

we write $\varphi \prec \gamma$ and we say that γ dominates φ globally if $T = 0$ and near infinity if $T > 0$.

d) if for every positive constant c and almost everywhere $x \in \Omega$ we have

$$\lim_{t\to 0}(\sup_{x\in\Omega} \frac{\varphi(x,ct)}{\gamma(x,t)}) = 0 \text{ or } \lim_{t\to\infty}(\sup_{x\in\varphi} \frac{\varphi(x,ct)}{\gamma(x,t)}) = 0$$

we write $\varphi \prec\prec \gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near infinity respectively.

In the sequel the measurability of a function $u : \Omega \mapsto R$ means the Lebesgue measurability.

We define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx$$

where $u : \Omega \mapsto \mathbb{R}$ is a measurable function.
The set

$$K_\varphi(\Omega) = \left\{u : \Omega \to R \text{ mesurable } / \varrho_{\varphi,\Omega}(u) < +\infty\right\}.$$

is called the Musielak-Orlicz class (the generalized Orlicz class).

The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$.
Equivelently:

$$L_\varphi(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ mesurable } / \varrho_{\varphi,\Omega}(\frac{|u(x)|}{\lambda}) < +\infty, \text{ for some } \lambda > 0\right\}$$

Let

$$\psi(x,s) = \sup_{t\geq 0}\{st - \varphi(x,t)\},$$

ψ is the Musielak-Orlicz function complementary to (or conjugate of) $\varphi(x,t)$ in the sense of Young with respect to the variable s.

On the space $L_\varphi(\Omega)$ we define the Luxemburg norm:

$$||u||_{\varphi,\Omega} = \inf\{\lambda > 0 / \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx, \leq 1\}.$$

and the so-called Orlicz norm :

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_\psi \leq 1} \int_\Omega |u(x)v(x)|dx.$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [22].

The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $E_\psi(\Omega)^* = L_\varphi(\Omega)$ [22].

The following conditions are equivalent:

e) $E_\varphi(\Omega) = K_\varphi(\Omega)$

f) $K_\varphi(\Omega) = L_\varphi(\Omega)$

g) φ has the Δ_2 property.

We recall that φ has the Δ_2 property if there exists $k > 0$ independent of $x \in \Omega$ and a nonnegative function h , integrable in Ω such that $\varphi(x, 2t) \leq k\varphi(x, t) + h(x)$ for large values of t, or for all values of t, according to whether Ω has finite measure or not.

Let us define the modular convergence: we say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \varrho_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall|\alpha| \leq m \quad D^\alpha u \in L_\varphi(\Omega)\}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with nonnegative integers α_i; $|\alpha| = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives.
The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space.

Now, the functional

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \varrho_{\varphi,\Omega}(D^\alpha u),$$

for $u \in W^m L_\varphi(\Omega)$ is a convex modular. and

$$||u||^m_{\varphi,\Omega} = \inf\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1\}$$

is a norm on $W^m L_\varphi(\Omega)$.
The pair $\langle W^m L_\varphi(\Omega), ||u||^m_{\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition :

$$\text{there exist a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c,$$

as in [22].

The space $W^m L_\varphi(\Omega)$ will always be identified to a $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \prod L_\varphi$.
Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W^m E_\varphi(\Omega)$ be the space of functions u such that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$, and let $W^m_0 E_\varphi(\Omega)$ be the (norm) closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \{f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega)\}$$

$$W^{-m} E_\psi(\Omega) = \{f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega)\}$$

As we did for $L_\varphi(\Omega)$, we say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

From [22], for two complementary Musielak-Orlicz functions φ and ψ the following inequalities hold:

h) the young inequality :

$$t.s \leq \varphi(x,t) + \psi(x,s) \text{ for } t, s \geq 0, \; x \in \Omega$$

i) the Hölder inequality :

$$\left| \int_\Omega u(x)v(x)\, dx \right| \leq ||u||_{\varphi,\Omega} |||v|||_{\psi,\Omega}.$$

for all $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$.

Inhomogeneous Musielak-Orlicz-Sobolev spaces :

Let Ω an bounded open subset of $\mathbb{R}^n$ and let $Q = \Omega\times]0,T[$ with some given $T > 0$. Let φ be a Musielak function. For each $\alpha \in \mathbb{N}^n$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^n$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows.

$$W^{1,x}L_\varphi(Q) = \{u \in L_\varphi(Q) : \forall|\alpha| \leq 1 \; D_x^\alpha u \in L_\varphi(Q)\}$$

and

$$W^{1,x}E_\varphi(Q) = \{u \in E_\varphi(Q) : \forall|\alpha| \leq 1 \; D_x^\alpha u \in E_\varphi(Q)\}$$

The last space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\leq m} \|D_x^\alpha u\|_{\varphi,Q}.$$

We can easily show that they form a complementary system when Ω is a Lipschitz domain [5]. These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$ which has $(N+1)$ copies. We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$. If $u \in W^{1,x}L_\varphi(Q)$ then the function : $t \longmapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values in $W^1L_\varphi(\Omega)$. If, further, $u \in W^{1,x}E_\varphi(Q)$ then this function is a $W^1E_\varphi(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds: $W^{1,x}E_\varphi(Q) \subset L^1(0,T; W^1E_\varphi(\Omega))$. The space $W^{1,x}L_\varphi(Q)$ is not in general separable, if $u \in W^{1,x}L_\varphi(Q)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto \|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q)$ is defined as the (norm) closure in $W^{1,x}E_\varphi(Q)$ of $\mathcal{D}(Q)$. We can easily show as in [5] that when Ω a Lipschitz domain then each element u of the closure of $\mathcal{D}(Q)$ with respect of the weak * topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is limit, in $W^{1,x}L_\varphi(Q)$, of some subsequence $(u_i) \subset \mathcal{D}(Q)$ for the modular convergence; i.e., there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$,

$$\int_Q \varphi(x, (\frac{D_x^\alpha u_i - D_x^\alpha u}{\lambda}))\, dx\, dt \to 0 \text{ as } i \to \infty,$$

this implies that (u_i) converges to u in $W^{1,x}L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_M, \Pi L_\psi)$. Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi E_\psi)} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi L_\psi)},$$

this space will be denoted by $W_0^{1,x}L_\psi(Q)$. Furthermore, $W_0^{1,x}E_\varphi(Q) = W_0^{1,x}L_\varphi(Q) \cap \Pi E_\varphi$.

We have the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_\varphi(Q) & F \\ W_0^{1,x}E_\varphi(Q) & F_0 \end{pmatrix},$$

F being the dual space of $W_0^{1,x}E_\varphi(Q)$. It is also, except for an isomorphism, the quotient of ΠL_ψ by the polar set $W_0^{1,x}E_\varphi(Q)^\perp$, and will be denoted by $F = W^{-1,x}L_\psi(Q)$ and it is shown that

$$W^{-1,x}L_\psi(Q) = \Big\{ f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in L_\psi(Q) \Big\}.$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\le 1} \|f_\alpha\|_{\psi,Q}$$

where the inf is taken on all possible decompositions

$$f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha, \quad f_\alpha \in L_\psi(Q).$$

The space F_0 is then given by

$$F_0 = \Big\{ f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in E_\psi(Q) \Big\}$$

and is denoted by $F_0 = W^{-1,x}E_\psi(Q)$.

3. Approximation Theorem and Trace Result

In this section, Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ with the segment property and
I is a subinterval of $\mathbb{R}$ (both possibly unbounded) and $Q = \Omega \times I$. It is easy to see that Q also satisfies Lipschitz domain.

Definition 3.1. *We say that $u_n \to u$ in $W^{-1,x}L_\psi(Q) + L^2(Q)$ for the modular convergence if we can write*

$$u_n = \sum_{|\alpha|\le 1} D_x^\alpha u_n^\alpha + u_n^0 \text{ and } u = \sum_{|\alpha|\le 1} D_x^\alpha u^\alpha + u^0$$

with $u_n^\alpha \to u^\alpha$ in $L_\psi(Q)$ for modular convergence for all $|\alpha| \le 1$ and $u_n^\alpha \to u^\alpha$ strongly in $L^2(Q)$.

We shall prove the following approximation theorem, which plays a fundamental role
when the existence of solutions for parabolic problems is proved.

Theorem 3.2. *Let φ be an Musielak-Orlicz function satisfying the condition (1.7) of [5]. If $u \in W^{1,x}L_\varphi(Q) \cap L^2(Q)$ (respectively $W_0^{1,x}L_\varphi(Q) \cap L^2(Q)$) and $\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(Q) + L^2(Q)$, then there exists a sequence (v_j) in $\mathcal{D}(\overline{Q})$ (respectively $\mathcal{D}((\overline{I}), \mathcal{D}(\Omega))$) such that $v_j \to u$ in $W^{1,x}L_\varphi(Q) \cap L^2(Q)$ and $\frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t}$ in $W^{-1,x}L_\psi(Q) + L^2(Q)$ for the modular convergence.*

Proof: Let $u \in W^{1,x}L_\varphi(Q) \cap L^2(Q)$ such that $\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(Q) + L^2(Q)$ and let $\varepsilon > 0$ be given. Writing $\frac{\partial u}{\partial t} = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$, where $u^\alpha \in L_\psi(Q)$ for all $|\alpha| \leq 1$ and $u^0 \in L^2(Q)$, we will show that there exists $\lambda > 0$(depending only on u and N)
and there exists $v \in \mathcal{D}(\overline{Q})$ for which we can write $\frac{\partial v}{\partial t} = \sum_{|\alpha|\leq 1} D_x^\alpha v^\alpha + v^0$ with $v^\alpha, v^0 \in \mathcal{D}(\overline{Q})$ such that

$$\int_Q \varphi(x, \frac{D_x^\alpha v - D_x^\alpha u}{\lambda})dxdt \leq \varepsilon, \forall |\alpha| \leq 1, \tag{2}$$

$$||v - u||_{L^2(Q)} \leq \varepsilon, \tag{3}$$

$$||v^0 - u^0||_{L^2(Q)} \leq \varepsilon, \tag{4}$$

$$\int_Q \psi(x, \frac{v^\alpha - u^\alpha}{\lambda})dxdt \leq \varepsilon, \forall |\alpha| \leq 1, \tag{5}$$

The equation (3) flows from a slight adaptation of the arguments of [5],
(4) and (5) flow also from classical approximation results.
Regrading the equation (6) it is enough to prove that $\mathcal{D}(\overline{Q})$ is dense in $L_\psi(Q)$ for this end.
We use the fact that the log-HÖlder continuity(commutes with the complementarity) i.e : if φ is log-HÖlder the its complementary ψ also it is, and proceed as in [5] (with φ and ψ interchanged) and using of course $\mathbb{R}^{N+1}$ instead of $\mathbb{R}^N$ and $Q = \Omega \times (0, T)$ instead of Ω.
These facts lead us to prove that

$$||K_\varepsilon f||_{\psi,Q} \leq C||f||_{\psi,Q}, \forall f \in L_\psi(Q)$$

(with $K_\varepsilon f(x,t) = k_\varepsilon^{-1} \int_Q K_\varepsilon(x-y)f(k_\varepsilon y, t)dy$,$K_\varepsilon(x) = \frac{1}{\varepsilon^N}K(\frac{x}{\varepsilon})$ and $K(x)$ is a measurable function with support in the ball $B_R = B(0, R)$ see [5]).
And then we deduce that $\mathcal{D}(\overline{Q})$ is dense in $L_\psi(Q)$ for the modular convergence which gives the desired conclusion. □

The case of $W_0^{1,x}L_\varphi(Q) \cap L^2(Q)$ is similar to the above arguments as in [5].

Remark 3.3. *If, in the statement of Theorem 3.2, one consider $\Omega \times \mathbb{R}$ instead of Q,*
we have $\mathcal{D}(\Omega \times \mathbb{R})$ is dense in $u \in W_0^{1,x}L_\varphi(\Omega \times \mathbb{R}) \cap L^2(\Omega \times \mathbb{R}) : \frac{\partial u}{\partial t} \in W_0^{1,x}L_\psi(\Omega \times \mathbb{R}) + L^2(\Omega \times \mathbb{R})$ for the modular convergence. This follows trivially from the fact that $\mathcal{D}(\mathbb{R}, \mathcal{D}(\Omega)) \equiv \mathcal{D}(\Omega \times \mathbb{R})$.
A first application of Theorem 3.2 is the following trace result generalizing a classical result which states that if u belong to $L^2(a, b; H_0^1(\Omega))$ and $\frac{\partial u}{\partial t}$ belongs to $L^2(a, b; H^{-1}(\Omega))$, then u is in $C([a, b], L^2(\Omega))$.

Lemma 3.4. . *Let $a < b \in \mathbb{R}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then $\{u \in W_0^{1,x}L_\varphi(\Omega\times(a,b))\cap L^2(\Omega\times(a,b)) : \frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(\Omega\times(a,b)) + L^2(\Omega\times(a,b))\}$ is a subset of $C([a,b], L^2(\Omega))$.*

Proof: Let $u \in W_0^{1,x}L_\varphi(\Omega\times(a,b))\cap L^2(\Omega\times(a,b))$ such that $W^{-1,x}L_\psi(\Omega\times(a,b))+ L^2(\Omega\times(a,b))$. After two consecutive reflection first with respect to $t = b$ and then with respect to $t = b$,
$\hat{u}(x,t) = u(x,t)\chi_{(a,b)} + u(x, 2b-t)\chi_{(b,2b-a)}$ on $\Omega\times(a, 2b-a)$
$\tilde{u}(x,t) = \hat{u}(x,t)\chi_{(a,2b-a)} + \hat{u}(x, 2a-t)\chi_{(3a-2b,a)}$ on $\Omega\times(3a-2b, 2b-a)$,
we get a function $\tilde{u} \in W_0^{1,x}L_\varphi(\Omega\times(3a-2b, 2b-a))\cap L^2(\Omega\times(3a-2b, 2b-a))$ such that $\frac{\partial\tilde{u}}{\partial t} \in W^{-1,x}L_\psi(\Omega\times(3a-2b, 2b-a)) + L^2(\Omega\times(3a-2b, 2b-a))$. Now, by letting a function
$\eta \in \mathcal{D}(\mathbb{R})$ with $\eta = 1$ on $[a,b]$ and supp$\eta \subset (3a-2b, 2b-a)$, setting $\overline{u} = \eta\tilde{u}$, and using standard arguments (see [[9], Lemme IV, Remarque 10, p. 158]), we have $\overline{u} = u$ on $\Omega\times(a,b)$ $\tilde{u} \in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R})\cap L^2(\Omega\times\mathbb{R})$ $\frac{\partial\tilde{u}}{\partial t} \in W^{-1,x}L_\psi(\Omega\times\mathbb{R})+ L^2(\Omega\times\mathbb{R})$.
Now let $v_j \in \mathcal{D}(\Omega\times\mathbb{R})$ be the sequence given by Theorem 3.2 corresponding to $\overline{u}$, that is,

$$v_j \to \overline{u} \in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R})\cap L^2(\Omega\times\mathbb{R}) \text{ and } \frac{\partial v_j}{\partial t} \to \frac{\partial\overline{u}}{\partial t} \in W^{-1,x}L_\psi(\Omega\times\mathbb{R})+L^2(\Omega\times\mathbb{R})$$

for the modular convergence.
We have

$$\int_\Omega (v_i(\tau) - v_j(\tau))^2 dx = 2\int_\Omega\int_{-\infty}^{\tau}(v_i - v_j)(\frac{\partial v_i}{\partial t} - \frac{\partial v_j}{\partial t})dxdt \to 0, \text{ as } i,j\to\infty$$

from which one deduces that v_j is a Cauchy sequence in $C(\mathbb{R}, L^2(\Omega))$, and since the limit of v_j in $L^2(\Omega\times\mathbb{R})$ is $\overline{u}$, we have $v_j \to \overline{u}$ in$C(\mathbb{R}, L^2(\Omega))$. Consequently, $u \in C([a,b], L^2(\Omega))$.
In order to deal with the time derivative, we introduce a time mollification of a function $u \in L_\varphi(Q)$.
Thus we define, for all $\mu > 0$ and all $(x,t) \in Q$

$$u_\mu(x,t) = \mu\int_{-\infty}^{t}\tilde{u}(x,s)\exp(\mu(s-t))ds, \tag{6}$$

where $\tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s)$ is the zero extension of u. □

Throughout the paper the index μ always indicates this mollification.

Proposition 3.5. *If $u \in L_\varphi(Q)$ then u_μ is measurable in Q and $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and if $u \in \mathcal{L}_\varphi(Q)$ then*

$$\int_Q \varphi(x, u_\mu)dxdt \le \int_Q \varphi(x,u)dxdt.$$

Proof: Since $(x,t,s) \mapsto u(x,s)exp(\mu(s-t))$ is measurable in $\Omega\times[0,T]\times[0,T]$, we deduce that u_μ is measurable by Fubini's theorem. By Jensen's integral inequality [see [4]] we have, since $\int_{-\infty}^0 \mu exp(\mu s)ds = 1$,

$$\varphi(x, \int_{-\infty}^t \mu\tilde{u}(x,s)exp(\mu(s-t))ds) = \varphi(x, \int_{-\infty}^0 \mu exp(\mu s)\tilde{u}(x,s+t)ds)$$
$$\leq \int_{-\infty}^0 \mu exp(\mu s)\varphi(x,\tilde{u}(x,s+t))ds$$

which implies

$$\int_Q \varphi(x,u_\mu(x,t))dxdt \leq \int_{\Omega\times\mathbb{R}}(\int_{-\infty}^0 \mu exp(\mu s)\varphi(x,\tilde{u}(x,s+t)ds))dxdt$$
$$\leq \int_{-\infty}^0 \mu exp(\mu s)(\int_{\Omega\times\mathbb{R}} \varphi(x,\tilde{u}(x,s+t))dxdt)ds$$
$$\leq \int_{-\infty}^0 \mu exp(\mu s)(\int_Q \varphi(x,u(x,t))dxdt)ds$$
$$= \int_Q \varphi(x,u)dxdt.$$

Furthermore
$\frac{\partial u_\mu}{\partial t} = \lim_{\delta\to 0}\frac{1}{\delta}(exp(-\mu\delta)-1)u_\mu(x,t)+\lim_{\delta\to 0}\frac{1}{\delta}\int_t^{t+\delta} u(x,s)exp(\mu(s-(t+\delta))ds =$
$-\mu u_\mu + \mu u.$ □

Proposition 3.6. *(1) If $u \in L_\varphi(Q)$ then $u_\mu \to u$ as $\mu\to\infty$ in $L_\varphi(Q)$ for the modular convergence.*
(2) If $u \in W^{1,x}L_\varphi(Q)$ then $u_\mu \to u$ as $\mu\to\infty$ in $W^{1,x}L_\varphi(Q)$ for the modular convergence.

Proof: (1) Let $(\phi_k) \subset \mathcal{D}(Q)$ such that $\phi_k \to u$ in $L_\varphi(Q)$ for the modular convergence.
Let $\lambda > 0$ large enough such that

$$\frac{u}{\lambda} \in \mathcal{L}_\varphi(Q) \text{ and } \int_Q \varphi(x, \frac{\phi_k - u}{\lambda})dxdt \to 0 \text{ as } k\to\infty.$$

For a.e. $(x,t)\in Q$ we have

$$|(\phi_k)_\mu(x,t) - (\phi_k)(x,t)| = \frac{1}{\mu}|\frac{\partial\phi_k}{\partial t}(x,t)| \leq \frac{1}{\mu}||\frac{\partial\phi_k}{\partial t}||_\infty.$$

On the other hand

$$\begin{aligned}\int_Q \varphi(x, \frac{u_\mu - u}{3\lambda})dxdt &\leq \frac{1}{3}\int_Q \varphi(x, \frac{u_\mu - (\phi_k)_\mu}{\lambda})dxdt \\ &+\frac{1}{3}\int_Q \varphi(x, \frac{(\phi_k)_\mu - \phi_k}{\lambda})dxdt \\ &+\frac{1}{3}\int_Q \varphi(x, \frac{\phi_k - u}{\lambda})dxdt \\ &\leq \frac{1}{3}\int_Q \varphi(x, \frac{(\phi_k - u)_\mu}{\lambda})dxdt \\ &+\frac{1}{3}\int_Q \varphi(x, \frac{(\phi_k)_\mu - \phi_k}{\lambda})dxdt \\ &+\frac{1}{3}\int_Q \varphi(x, \frac{\phi_k - u}{\lambda})dxdt.\end{aligned}$$

This implies that

$$\int_Q \varphi(x, \frac{u_\mu - u}{3\lambda})dxdt \leq \frac{2}{3}\int_Q \varphi(x, \frac{\phi_k - u}{\lambda})dxdt + \frac{1}{3}\varphi(x, \frac{1}{\mu\lambda}||\frac{\partial \phi_k}{\partial t}||_\infty)meas(Q).$$

Let $\varepsilon > 0$. There exists k such that

$$\int_Q \varphi(x, \frac{\phi_k - u}{\lambda})dxdt \leq \varepsilon,$$

and there exists μ_0 such that

$$\varphi(x, \frac{1}{\mu\lambda}||\frac{\partial \phi_k}{\partial t}||_\infty)meas(Q) \leq \varepsilon \text{ for all } \mu \geq \mu_0.$$

Hence

$$\int_Q \varphi(x, \frac{u_\mu - u}{3\lambda})dxdt \leq \varepsilon \text{ for all } \mu \geq \mu_0.$$

(2) Since $\forall\alpha, |\alpha| \leq 1$, we have $D_x^\alpha(u_\mu) = (D_x^\alpha u)_\mu$, consequently, the first part above applied on each $D_x^\alpha u$, gives the result. □

Remark 3.7. *If $u \in E_\varphi(Q)$, we can choose λ arbitrary small since $\mathcal{D}(Q)$ is (norm) dense in $E_\varphi(Q)$.*
Thus, for all $\lambda > 0$

$$\int_Q \varphi(x, \frac{u_\mu - u}{\lambda})dxdt \to 0 \text{ as } \mu \to \infty$$

and $u_\mu \to u$ strongly in $E_\varphi(Q)$.Idem for $W^{1,x}E_\varphi(Q)$.

Proposition 3.8. *If $u_n \to u$ in $W^{1,x}L_\varphi(Q)$ strongly (resp., for the modular convergence)*
then $(u_n)_\mu \to u_\mu$ in $W^{1,x}L_\varphi(Q)$ strongly (resp., for the modular convergence).

Proof: . For all $\lambda > 0$ (resp., for some $\lambda > 0$),

$$\int_Q \varphi(x, \frac{D_x^\alpha((u_n)\mu) - D_x^\alpha(u)_\mu}{\lambda})dxdt \leq \int_Q \varphi(x, \frac{D_x^\alpha(u_n) - D_x^\alpha u}{\lambda})dxdt \to 0 \text{ as } n \to \infty,$$

then $(u_n)_\mu \to u_\mu$ in $W^{1,x}L_\varphi(Q)$ strongly (resp., for the modular convergence). □

4. Compactness Results

In this section, we shall prove some compactness theorems in inhomogeneous Musielak-Orlicz- Sobolev spaces which will be applied to get existence theorem for parabolic problems.

For each $h > 0$, define the usual translated $\tau_h f$ of the function f by$\tau_h f(t) = f(t+h)$.
If f is defined on $[0,T]$ then $\tau_h f$ is defined on $[-h, T-h]$.
First of all, recall the following compactness result proved by Simon [25].

Lemma 4.1. *Let φ be a Musielak function. Let Y be a Banach space such that the following continuous imbedding holds $L^1(\Omega) \subset Y$. Then for all $\varepsilon > 0$ and all $\lambda > 0$, there is $C_\varepsilon > 0$ such that for all $u \in W_0^{1,x}L_\varphi(Q)$, with $\frac{|\nabla u|}{\lambda} \in \mathcal{L}_\varphi(Q)$,*

$$||u||_{L^1(Q)} \leq \varepsilon\lambda(\int_Q \varphi(x, \frac{|\nabla u|}{\lambda})dxdt + T) + C_\varepsilon ||u||_{L^1(0,T;Y)}.$$

Proof: . Since $W_0^1 L_\varphi(\Omega) \subset L^1(\Omega)$ with compact imbedding, then for all $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that for all $v \in W_0^1 L_\varphi(\Omega)$:

$$||v||_{L^1(\Omega)} \leq \varepsilon ||\nabla v||_{L_\varphi(\Omega)} + C_\varepsilon ||v||_Y. \tag{7}$$

Indeed, if the above assertion holds false, there is $\varepsilon_0 > 0$ and $v_n \in W_0^1 L_\varphi(\Omega)$ such that

$$||v_n||_{L^1(\Omega)} \geq \varepsilon_0 ||\nabla v_n||_{L_\varphi(\Omega)} + n||v_n||_Y.$$

This gives, by setting $w_n = \frac{v_n}{||\nabla v_n||_{L_\varphi(\Omega)}}$:

$$||w_n||_{L^1(\Omega)} \geq \varepsilon_0 + n||w_n||_Y, ||\nabla w_n||_{L_\varphi(\Omega)} = 1.$$

Since (w_n) is bounded in $W_0^1 L_\varphi(\Omega)$ then for a subsequence

$$w_n \rightharpoonup w \text{ in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \text{ and strongly in } L^1(\Omega).$$

Thus $||w_n||_{L^1(\Omega)}$ is bounded and $||w_n||_Y \to 0$ as $n \to \infty$.We deduce $w_n \to 0$ in Y and that $w = 0$ implying that $\varepsilon_0 \leq ||w_n||_{L^1(\Omega)} \to 0$, a contradiction.
Using $v = u(t)$ in (7) for all $u \in W_0^{1,x} L_\varphi(Q)$ with $\frac{|\nabla u|}{\lambda} \in \mathcal{L}_\varphi(Q)$ and a.e. t in $(0,T)$, we have

$$||u(t)||_{L^1(\Omega)} \leq \varepsilon ||\nabla u(t)||_{L_\varphi(\Omega)} + C_\varepsilon ||u(t)||_Y.$$

Since $\int_Q \varphi(x, \frac{|\nabla u(x,t)|}{\lambda}) dxdt < \infty$ we have thanks to Fubini's theorem $\int_\Omega \varphi(x, \frac{|\nabla u(x,t)|}{\lambda}) dx < \infty$ for a.e t in $(0,T)$, and then

$$||\nabla u(t)||_{L_\varphi(\Omega)} \leq \lambda (\int_\Omega \varphi(x, \frac{|\nabla u(x,t)|}{\lambda}) dx + 1),$$

which implies that

$$||u(t)||_{L^1(\Omega)} \leq \varepsilon \lambda (\int_\Omega \varphi(x, \frac{|\nabla u(x,t)|}{\lambda}) dx + 1) + C_\varepsilon ||u(t)||_{Y)}.$$

Integrating this over $(0,T)$ yields

$$||u||_{L^1(Q)} \leq \varepsilon \lambda (\int_Q \varphi(x, \frac{|\nabla u(x,t)|}{\lambda}) dxdt + T) + C_\varepsilon \int_0^T ||u(t)||_{Y)} dt$$

and finally

$$||u||_{L^1(Q)} \leq \varepsilon \lambda (\int_Q \varphi(x, \frac{|\nabla u|}{\lambda}) dxdt + T) + C_\varepsilon ||u||_{L^1(,0,T;Y)}.$$

□

We also prove the following lemma which allows us to enlarge the space Y whenever necessary.

Lemma 4.2. *If F is bounded in $W_0^{1,x} L_\varphi(Q)$ and is relatively compact in $L^1(0,T;Y)$ then F is relatively compact in $L^1(Q)$ (and also in $E_\gamma(Q)$ for all Musielak function $\gamma \ll \varphi$).*

Proof: Let $\varepsilon > 0$ be given. Let $C > 0$ be such that $\int_Q \varphi(x, \frac{|\nabla f|}{C}) dxdt \leq 1$ for all $f \in F$.
By the previous lemma, there exists $C_\varepsilon > 0$ such that for all $u \in W_0^{1,x} L_\varphi(Q)$ with $\frac{|\nabla u|}{C} \in \mathcal{L}_\varphi(Q)$,
$||u(t)||_{L^1(Q)} \leq \frac{2\varepsilon C}{4C(1+T)} (\int_Q \varphi(x, \frac{|\nabla u|}{2C}) dxdt + T) + C_\varepsilon ||u||_{L^1(0,T;Y)}$.
Moreover, there exists a finite sequence (fi) in F satisfying

$$\forall f \in F, \exists f_i \text{ such that } ||f - f_i||_{L^1(0,T;Y)} \leq \frac{\varepsilon}{2C_\varepsilon}$$

so that

$$||f-f_i||_{L^1(Q)} \leq \frac{\varepsilon}{2(1+T)}\left(\int_Q \varphi(x, \frac{|\nabla f-\nabla f_i|}{2C})dxdt + T\right) + C_\varepsilon ||f-f_i||_{L^1(0,T;Y)} \leq \varepsilon$$

and hence F is relatively compact in $L^1(Q)$.
Since $\gamma \ll \varphi$ then by using Vitali's theorem, it is easy to see that F is relatively compact in $E_\gamma(Q)$. □

Remark 4.3. *(see [14]). If $F \subset L^1(0,T;B)$ is such that $\{\frac{\partial f}{\partial t} : f \in F\}$ is bounded in $F \subset L^1(0,T;B)$ then*
$||\tau_h f - f||_{L^1(0,T;B)} \to 0$ as $h \to 0$ uniformly with respect to $f \in F$.

Theorem 4.4. *Let φ be a Musielak function. If F is bounded in $W^{1,x}L_\varphi(Q)$ and $\{\frac{\partial f}{\partial t} : f \in F\}$ is bounded in $W^{-1,x}L_\psi(Q)$, then F is relatively compact in $L^1(Q)$.*

Proof: Let γ and θ be Musielak functions such that $\gamma \ll \varphi$ and $\theta \ll \psi$ near infinity.
For all $0 < t_1 < t_2 < T$ and all $f \in F$, we have

$$\begin{aligned} ||\int_{t_1}^{t_2} f(t)dt||_{W_0^1 E_\gamma(\Omega)} &\leq \int_0^T ||f(t)||_{W_0^1 E_\gamma(\Omega)}dt \\ &\leq C_1 ||f||_{W_0^{1,x}E_\gamma(Q)} \leq C_2 ||f||_{W_0^{1,x}E_\varphi(Q)} \leq C, \end{aligned}$$

where we have used the following continuous imbedding:

$$W_0^{1,x}L_\varphi(Q) \subset W_0^{1,x}E_\gamma(Q) \subset L^1(0,T;W_0^1 E_\gamma(\Omega)).$$

Since the imbedding $W_0^1 L_\gamma(\Omega) \subset L^1(\Omega)$ is compact we deduce that $(\int_{t_1}^{t_2} f(t)dt)_{f\in F}$ is relatively compact in $L^1(\Omega)$ and in $W^{-1,1}(\Omega)$ as well.
On the other hand $\{\frac{\partial f}{\partial t} : f \in F\}$ is bounded in $W^{-1,x}L_\psi(Q)$ and $L^1(0,T;W^{-1,1}(\Omega)$ as well, since

$$W^{-1,x}L_\psi(Q) \subset W^{-1,x}E_\theta(Q) \subset L^1(0,T;W^{-1}E_\theta(\Omega)) \subset L^1(0,T;W^{-1,1}(\Omega))$$

with continuous imbedding.
By Remark 3 of [14], we deduce that $||\tau_h f - f||_{L^1(0,T;W^{-1,1}(\Omega))} \to 0$ uniformly in $f \in F$ when $h \to 0$ and by using Theorem 2 of [14],F is relatively compact in $L^1(0,T;W^{-1,1}(\Omega))$.
Since $L^1(\Omega) \subset W^{-1,1}(\Omega))$ with continuous imbedding we can apply Lemma 4.2 to conclude that F is relatively compact in $L^1(Q)$. □

Corollary 4.5. *Let φ be a Musielak function.*
Let (u_n) be a sequence of $W^{1,x}L_\varphi(Q)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi L_\psi)$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \text{ in } \mathcal{D}'(Q)$$

with h_n bounded in $W^{-1,x}L_\psi(Q)$ and (k_n) bounded in the space $\mathcal{M}(Q)$ of measures on Q.
then $u_n \to u$ strongly in $L^1_{loc}(Q)$.
If further $u_n \in W_0^{1,x}L_\varphi(Q)$ then $u_n \to u$ strongly in $L^1(Q)$.

Proof: . It is easily adapted from that given in [8] by using Theorem 4.4 and Remark 4.3 instead of Lemma 8 of [25]. □

5. Existence Result

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N(N \geq 2)$, $T > 0$ and set $Q = \Omega \times (0,T)$.
Throughout this section, we denote $Q_\tau = \Omega \times (0,\tau)$ for every $\tau \in [0,T]$.
Let φ and γ two Musielak-Orlicz functions such that $\gamma \ll \varphi$.
Consider a second-order operator $A : D(A) \subset W^{1,x}L_\varphi(Q) \to W^{-1,x}L\psi(Q)$ of the form

$$A(u) = -diva(x,t,u,\nabla u),$$

where $a : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, for almost every$(x,t) \in \Omega \times [0,T]$ and all $s \in \mathbb{R}, \xi \neq \xi^* \in \mathbb{R}^N$,

$$|a(x,t,s,\xi)| \leq \beta(c_1(x,t) + \psi_x^{-1}\gamma(x,\vartheta|s|) + \psi_x^{-1}\varphi(x,\vartheta|\xi|)) \tag{8}$$

$$(a(x,t,s,\xi) - a(x,t,s,\xi^*))(\xi - \xi^*) > 0 \tag{9}$$

$$a(x,t,s,\xi)\xi \geq \alpha\varphi(x,\frac{|\xi|}{\lambda}) - d(x,t) \tag{10}$$

with $c_1(x,t) \in E_\psi(Q), c_1 \geq 0, d(x,t) \in L^1(Q), \alpha,\beta,\vartheta > 0$.
Assume that $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, for almost every$(x,t) \in \Omega \times [0,T]$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$:

$$|g(x,t,s,\xi)| \leq b(|s|)(c_2(x,t) + \varphi(x,|\xi|)) \tag{11}$$

$$g(x,t,s,\xi)s \geq 0 \tag{12}$$

with $c_2(x,t) \in L^1(Q)$ and $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function.
Furtheremore let

$$f \in W^{-1,x}E_\psi(Q) \tag{13}$$

Consider then the following parabolic initial-boundary value problem.

$$\begin{cases} \frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f \text{ in } Q \\ u(x,t) = 0 \text{ on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \text{ in } \Omega \end{cases} \tag{14}$$

where u_0 is a given function in $L^2(\Omega)$.
We shall prove the following existence theorem.

Theorem 5.1. *Assume that (8)-(13) hold true. Then the problem (14) admits at least one weak solution $u \in D(A) \cap W_0^{1,x}L_\varphi(Q) \cap \mathcal{C}(([0,T], L^2(\Omega))$ such that $g(x,t,u,\nabla u) \in L^1(Q), g(x,t,u,\nabla u)u \in L^1(Q)$. Furthermore $u(x,0) = u_0(x)$ for almost every $x \in \Omega$, and for all $v \in W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q)$ with $\frac{\partial v}{\partial t} \in W^{-1,x}L\psi(Q) + L^2(Q)$ and for all $\tau \in [0,T]$, we have*

$$\langle \frac{\partial v}{\partial t}, u\rangle_{Q_\tau} + [\int_\Omega u(t)v(t)dx]_0^\tau + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla v dxdt + \int_{Q_\tau} g(x,t,u,\nabla u)v dxdt = \langle f, v\rangle_{Q_\tau} \quad (15)$$

and for $v = u$, which gives the energy equality

$$\frac{1}{2}\int_\Omega u^2(\tau)dx - \frac{1}{2}\int_\Omega u_0^2 dx + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla u dxdt + \int_{Q_\tau} g(x,t,u,\nabla u)v dxdt = \langle f, u\rangle_{Q_\tau}$$

Remark 5.2. *As in the elliptic case (see, [6]), γ is introduced instead of φ in (8) is done only to guarantee the boundedness in $L_\psi(Q)$ of $\psi_x^{-1}\gamma(x,\vartheta|u_n|)$ and $\psi_x^{-1}\gamma(x,\vartheta|\nabla u_n|)$ whenever u_n is bounded in $W^{1,x}L_\varphi(Q)$.*
In the elliptic case,one usually takes $\gamma = \varphi$ in the term $\psi_x^{-1}\gamma(x,\vartheta|u_n|)$ since u_n is bounded in a smaller space $L_\theta(\Omega)$ with $\varphi \ll \theta$; see [6].
However, in the parabolic case, we cannot conclude that there is the boundedness. Nevertheless,we can take $\gamma = \varphi$ if one of the following assertions holds true.
(1) φ satisfies a $\triangle_2$ condition near infinity.
(2) A is monotone, that is $\langle A(u)-A(v), u-v\rangle \geq 0$ for all $u, v \in D(A)\cap W_0^{1,x}L_\varphi(Q)$.
Indeed, suppose first that φ satisfies a $\triangle_2$ condition. Therefore (8) with now $\gamma = \varphi$, imply that, for all $\varepsilon > 0$,

$$|a(x,t,s,\xi)| \leq \beta_\varepsilon(c_\varepsilon(x,t) + \psi_x^{-1}\varphi(x,\varepsilon|s|) + \psi_x^{-1}\varphi(x,\varepsilon|\xi|)),$$

which allows us to deduce the boundedness in $L_\psi(Q)$ of $a(x,t,u_n,\nabla u_n)$ and $a(x,t, u_n,\nabla u_n)$.
Assume now that A is monotone. We have, for all $\phi \in W_0^{1,x}E_\varphi(Q), \langle A(u_n) - A(\phi), u_n - \phi\rangle \geq 0$. This gives $\langle A(u_n), \phi\rangle \leq \langle A(u_n), u_n\rangle - \langle A(\phi), u_n - \phi\rangle$, which implies that, since u_n is bounded in $W_0^{1,x}L_\varphi(Q)$ and $\langle A(u_n), u_n\rangle$ is bounded from above, thanks to the a priori estimates,

$$\langle A(u_n), \phi\rangle \leq C_\phi \text{ for all } \phi \in W_0^{1,x}E_\varphi(Q),$$

where C_ϕ is a constant depending on ϕ but not n. Therefore, the Banach-Steinhauss theorem applies so that we can obtain the boundedness of $A(u_n)$ in $W^{-1,x}L_\psi(Q)$.

Proof of Theorem 5.1. We divide the proof in four steps.
Step 1. A priori estimates.
Consider the sequence of approximate problems:

$$\begin{cases} u_n \in D(A)\cap W_0^{1,x}L_\varphi(Q)\cap \mathcal{C}(([0,T],L^2(\Omega)), u_n(x,0)=u_0(x) a.e. \in \Omega, \\ \langle \frac{\partial u_n}{\partial t}, v\rangle + \langle A(u_n), v\rangle + \int_Q g_n(x,t,u_n,\nabla u_n) v dxdt = \langle f, v\rangle \\ \text{for all } v\in W_0^{1,x}L_\varphi(Q) \end{cases} \tag{16}$$

where

$$g_n(x,t,s,\xi) = T_n(g(x,t,s,\xi))$$

and where for $k>0$, T_k means for the usual truncation operator at k defined on $\mathbb{R}$ by

$$T_k(s) = \max(-k, \min(k,s))$$

Note that $g_n(x,t,s,\xi)s \geq 0, |g_n(x,t,s,\xi)| \leq |g(x,t,s,\xi)|$ and $|g_n(x,t,s,\xi)| \leq n$.
Since g_n is bounded for any fixed $n>0$, there exists at last one solution u_n of(16),(the existence of u_n can be obtained from Galerkin solutions corresponding to the Equation (16) as in [19], see Theorem 1 of [2] for more details).
Note also that $\langle u_n', v\rangle$ is defined in the sense of distributions(where $u_n' = \frac{\partial u_n}{\partial t}$ means for the time derivative of u_n). Since $u_n' = f - A(u_n) - g_n$ is in $W^{-1,x}L_\psi(Q)$ we can extend $\langle u_n', v\rangle$ to all $v\in W_0^{1,x}L_\varphi(Q)$.
Using in (16) the test function u_n, we get

$$\begin{aligned} \frac{1}{2}\int_\Omega u_n^2(T)dx - \frac{1}{2}\int_\Omega u_0^2(x)dx \quad &+ \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \\ &+ \int_Q g_n(x,t,u_n,\nabla u_n)u_n dxdt = \langle f, u_n\rangle \end{aligned}$$

which implies that

$$\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \leq \langle f, u_n\rangle + C$$

Where here and below C is a positive constant not depending on n.
By theorem 1 and theorem 5 of [3] we can say that:

$$\begin{cases} (u_n) \text{ is bounded in } W_0^{1,x}L_\varphi(Q), \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \leq C \\ \text{and } \int_Q g_n(x,t,u_n,\nabla u_n)u_n dxdt \leq C \end{cases} \tag{17}$$

To prove that $a(x,t,u_n,\nabla u_n)$ is a bounded sequence in $(L_\psi(Q))^N$. Let $\phi \in (E_\varphi(Q))^N$ with $||\phi||_{\varphi,Q} = 1$.
In view of (9), we have

$$\int_Q [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\phi)][\nabla u_n - \phi]dxdt \geq 0,$$

which gives

$$\begin{aligned}\int_Q a(x,t,u_n,\nabla u_n)\phi dxdt &\leq \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \\ &\quad + \int_Q a(x,t,u_n,\phi)[\nabla u_n - \phi]dxdt.\end{aligned}$$

Using (8) and (17), we easily see that

$$\int_Q a(x,t,u_n,\nabla u_n)\phi dxdt \leq C$$

And so $a(x,t,u_n,\nabla u_n)$ is a bounded sequence in $(L_\psi(Q))^N$. Splitting Q into $|u_n| \leq 1$ and $|u_n| > 1$ and using (11), we can write

$$\begin{aligned}\int_Q |g_n(x,t,u_n,\nabla u_n)|dxdt &\leq b(1)\int_{\{|u_n|\leq 1\}} (c_2(x,t) + \varphi(|\nabla T_1(u_n)|))dxdt \\ &\quad + \int_{\{|u_n|>1\}} g_n(x,t,u_n,\nabla u_n)u_n dxdt \leq C.\end{aligned}$$

And then $g_n(x,t,u_n,\nabla u_n)$ is a bounded sequence in $L^1(Q)$ implying that $\frac{\partial u_n}{\partial t}$ is a bounded sequence in $W^{-1,x}L_\psi(Q) + L^1(Q)$, therefore Corollary 4.5 allows us to deduce that $u_n \to u$ strongly in $L^1(Q)$. Thus, for some subsequence still denoted by u_n and for some $h \in (L_\psi(Q))^N$:

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi), \text{ strongly in } L^1(Q) \\ \text{and a.e. in } Q \text{ and } a(x,t,u_n,\nabla u_n) \rightharpoonup h \text{ in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi, \Pi E_\varphi). \end{cases} \tag{18}$$

Step 2. Almost everywhere convergence of gradients.
Fix $k > 0$ and let $\phi(s) = s\exp(\delta s^2), \delta > 0$.It is well known that when $\delta \geq (\frac{b(k)}{2\alpha})^2$ one has

$$\phi'(s) - \frac{b(k)}{\alpha}|\phi(s)| \geq \frac{1}{2} \text{ for all } s \in \mathbb{R} \tag{19}$$

Let $v_j \in \mathcal{D}(Q)$ be a sequence such that

$$v_j \to u \text{ in } W_0^{1,x}L_\varphi(Q) \text{ for the modular convergence} \tag{20}$$

and let $w_i \in \mathcal{D}(\Omega)$ be a sequence which converges strongly to u_0 in $L^2(\Omega)$.
Set $\omega^i_{\mu,j} = T_k(v_j)_\mu + \exp(-\mu t)T_k(w_i)$ where $T_k(v_j)_\mu$ is the mollification with respect to time of $T_k(v_j)$,
see (6).
Note that $\omega^i_{\mu,j}$ is a smooth function having the following properties:

$$\begin{cases} \frac{\partial}{\partial t}(\omega^i_{\mu,j}) = \mu(T_k(v_j) - \omega^i_{\mu,j}), \omega^i_{\mu,j}(0) = T_k(v_j), |\omega^i_{\mu,j}| \leq k, \\ \omega^i_{\mu,j} \to T_k(u)_\mu + \exp(-\mu t)T_k(w_i) \text{ in } W_0^{1,x}L_\varphi(Q) \\ \qquad \text{for the modular convergence as } j \to \infty, \\ T_k(u)_\mu + \exp(-\mu t)T_k(w_i) \to T_k(u) \text{ in } W_0^{1,x}L_\varphi(Q) \\ \qquad \text{for the modular convergence as } \mu \to \infty. \end{cases}$$

Using in (16) the test function $Z_{n,j}^{\mu,i} = \phi(T_k(u_n) - \omega_{\mu,j}^i)$ which belongs to $W_0^{1,x}L_\varphi(Q)$, we get

$$\langle u_n', Z_{n,j}^{\mu,i}\rangle + \int_Q a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla\omega_{\mu,j}^i]\phi'(T_k(u_n) - \omega_{\mu,j}^i)dxdt$$
$$+ \int_Q g_n(x,t,u_n,\nabla u_n)\phi(T_k(u_n) - \omega_{\mu,j}^i)dxdt = \langle f, \phi(T_k(u_n) - \omega_{\mu,j}^i)\rangle,$$

which implies since $g_n(x,t,u_n,\nabla u_n)\phi(T_k(u_n) - \omega_{\mu,j}^i) \geq 0$ on $|u_n| > k$:

$$\langle u_n', Z_{n,j}^{\mu,i}\rangle + \int_Q a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla\omega_{\mu,j}^i]\phi'(T_k(u_n) - \omega_{\mu,j}^i)dxdt$$
$$+ \int_Q g_n(x,t,u_n,\nabla u_n)\phi(T_k(u_n) - \omega_{\mu,j}^i)dxdt \leq \langle f, \phi(T_k(u_n) - \omega_{\mu,j}^i)\rangle. \quad (21)$$

In the sequel and throughout the paper, we will omit for simplicity the dependence on x and t in the function $a(x,t,s,\xi)$ and denote $\varepsilon(n,j,\mu,i,s)$ all quantities (possibly different) such that

$$\lim_{s\to\infty}\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\varepsilon(n,j,\mu,i,s) = 0$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then j, μ, i and finally s. Similarly,we will write only $\varepsilon(n)$, or $\varepsilon(n,j)$,... to mean that the limits are made only on the specified parameters.
We will deal with each term of (21). First of all, observe that

$$\langle f, \phi(T_k(u_n) - \omega_{\mu,j}^i)\rangle = \varepsilon(n,j,\mu) \quad (22)$$

since $T_k(u_n) - \omega_{\mu,j}^i \rightharpoonup T_k(u) - \omega_{\mu,j}^i$ weakly in $W_0^{1,x}L_\varphi(Q)$ as $n\to\infty$,
and $T_k(u) - \omega_{\mu,j}^i \to T_k(u) - T_k(u)_\mu + \exp(-\mu t)T_k(w_i)$ in $W_0^{1,x}L_\varphi(Q)$ for the modular convergence
and so for the topology $\sigma(\Pi L_\varphi, \Pi L\psi)$ as $j\to\infty$,
and finally $T_k(u) - T_k(u)_\mu + \exp(-\mu t)T_k(w_i) \to 0$ in $W_0^{1,x}L_\varphi(Q)$ for the modular convergence as $\mu\to\infty$.
From (16) one deduces that $u_n \in W_0^{1,x}L_\varphi(Q)\cap L^2(Q)$ and $\frac{\partial u_n}{\partial t} \in W^{-1,x}L_\psi(Q)$ and then, by Theorem 3.2, there exists a smooth function $u_{n\sigma}$ such that,
as $\sigma\to\infty$,$u_{n\sigma}\to u_n$ in $W_0^{1,x}L_\varphi(Q)\cap L^2(Q)$ and $\frac{\partial u_{n\sigma}}{\partial t} \to \frac{\partial u_n}{\partial t}$ in $W^{-1,x}L_\psi(Q) + L^2(Q)$ for modular convergence. Consequently

$$\langle u_n', Z_{n,j}^{\mu,i}\rangle = \lim_{\sigma\to\infty}\int_Q u_{n\sigma}'\phi(T_k(u_{n\sigma}) - \omega_{\mu,j}^i)dxdt$$
$$= \lim_{\sigma\to\infty}\int_Q [(T_k(u_{n\sigma}))' + (G_k(u_{n\sigma}))']\phi(T_k(u_{n\sigma}) - \omega_{\mu,j}^i)dxdt,$$

where $G_k(s) = s - T_k(s)$. Hence

$$\begin{aligned}\langle u'_n, Z^{\mu,i}_{n,j}\rangle &= \lim_{\sigma\to\infty}\int_Q (T_k(u_{n\sigma}) - \omega^i_{\mu,j})'\phi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})dxdt \\ &+ \int_Q (\omega^i_{\mu,j})'\phi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})dxdt \\ &+ \int_Q (G_k(u_{n\sigma}))\phi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})dxdt \\ &= \lim_{\sigma\to\infty}(I_1(\sigma) + I_2(\sigma) + I_3(\sigma)).\end{aligned}$$

Setting $\Phi(s) = \int_0^s \phi(r)dr$, it is easy to see that $\Phi(s) \geq 0$,

$$\begin{aligned}I_1(\sigma) &= [\int_\Omega \Phi(T_k(u_{n\sigma})(t) - \omega^i_{\mu,j}(t))dx]_0^T \\ &\geq -\int_\Omega \Phi(T_k(u_{n\sigma})(0) - T_k(w_i))dx.\end{aligned}$$

Since, as $\sigma \to \infty$, the last side goes to $-\int_\Omega \Phi(T_k(u_0) - T_k(w_i))dx$ which is of the form $\varepsilon(i)$, we get

$$\limsup_{\sigma\to\infty} I_1(\sigma) \geq \varepsilon(i).$$

About $I_2(\sigma)$, we have, since $(\omega^i_{\mu,j})' = \mu(T_k(v_j) - \omega^i_{\mu,j})$ and $\phi(s)s \geq 0$,

$$\begin{aligned}I_2(\sigma) &= \mu\int_Q (T_k(v_j) - \omega^i_{\mu,j})\phi((T_k(u_{n\sigma}) - \omega^i_{\mu,j}))dxdt \\ &\geq \mu\int_Q (T_k(v_j) - T_k(u_{n\sigma}))\phi((T_k(u_{n\sigma}) - \omega^i_{\mu,j}))dxdt.\end{aligned}$$

Since, as $\sigma \to \infty$, the last side goes to

$$\mu\int_Q (T_k(v_j) - T_k(u_n))\phi((T_k(u_n) - \omega^i_{\mu,j}))dxdt,$$

which is of form $\varepsilon(n,j)$, we obtain

$$\limsup_{\sigma\to\infty} I_2(\sigma) \geq \varepsilon(n,j).$$

For what concerns $I_3(\sigma)$, one has by integrating by parts

$$\begin{aligned}I_3(\sigma) = &-\int_Q G_k(u_{n\sigma})\phi'(T_k(u_{n\sigma}) - \omega^i_{\mu,j})(T_k(u_{n\sigma}) - \omega^i_{\mu,j})'dxdt \\ &+[\int_\Omega G_k(u_{n\sigma})(t)\phi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})(t)dx]_0^T.\end{aligned}$$

Since $(T_k(u_{n\sigma}))' = 0$ on $\{|u_{n\sigma}| > k\}$ and

$$[\int_\Omega G_k(u_{n\sigma})(t)\phi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})(t)dx]_0^T \geq -\int_\Omega G_k(u_{n\sigma})(0)\phi(T_k(u_{n\sigma})(0) - T_k(w_i)dx,$$

we have

$$\begin{aligned} I_3(\sigma) &\geq \int_Q G_k(u_{n\sigma})\phi'(T_k(u_{n\sigma}) - \omega^i_{\mu,j})(\omega^i_{\mu,j})'dxdt \\ &\quad - \int_\Omega G_k(u_{n\sigma})(0)\phi(T_k(u_{n\sigma})(0) - T_k(w_i))dx \\ &= \mu\int_Q G_k(u_{n\sigma})\phi'(T_k(u_{n\sigma}) - \omega^i_{\mu,j})(T_k(v_j) - \omega^i_{\mu,j})dxdt \\ &\quad - \int_\Omega G_k(u_{n\sigma})(0)\phi(T_k(u_{n\sigma})(0) - T_k(w_i))dx, \end{aligned}$$

which implies that

$$\begin{aligned} \limsup_{\sigma\to\infty} I_3(\sigma) &\geq \mu\int_Q G_k(u_n)\phi'(T_k(u_n) - \omega^i_{\mu,j})(T_k(v_j) - \omega^i_{\mu,j})dxdt \\ &\quad - \int_\Omega G_k(u_0)\phi(T_k(u_0) - T_k(w_i))dx, \end{aligned}$$

and hence, by letting $n \to \infty$ in the first integral of last side,

$$\begin{aligned} \limsup_{\sigma\to\infty} I_3(\sigma) &\geq \mu\int_Q G_k(u)\phi'(T_k(u) - \omega^i_{\mu,j})(T_k(v_j) - \omega^i_{\mu,j})dxdt \\ &\quad - \int_\Omega G_k(u_0)\phi(T_k(u_0) - T_k(w_i))dx + \varepsilon(n) \\ &\geq \mu\int_Q G_k(u)\phi'(T_k(u) - \omega^i_{\mu,j})(T_k(v_j) - T_k(u))dxdt \\ &\quad - \int_\Omega G_k(u_0)\phi(T_k(u_0) - T_k(w_i))dx + \varepsilon(n), \end{aligned} \tag{23}$$

where we have used the fact that (recall that $|\omega^i_{\mu,j}| \leq k$)

$$\begin{aligned} &\int_Q G_k(u)\phi'(T_k(u) - \omega^i_{\mu,j})(T_k(u) - \omega^i_{\mu,j})dxdt \\ &= \int_{\{u>k\}} (u-k)\phi'(k - \omega^i_{\mu,j})(k - \omega^i_{\mu,j})dxdt \\ &+ \int_{\{u<-k\}} (u+k)\phi'(-k - \omega^i_{\mu,j})(-k - \omega^i_{\mu,j})dxdt \geq 0. \end{aligned}$$

Since the first integral of last side of (23) is of the form $\varepsilon(j)$ while the second one is of the form $\varepsilon(i)$, we deduce that

$$\limsup_{\sigma\to\infty} I_3(\sigma) \geq \varepsilon(n, j, i).$$

Combining the estimates on each I_i, we get

$$\langle u_n', \phi(T_k(u_n) - \omega_{\mu,j}^i)\rangle \geq \varepsilon(n,j,i). \tag{24}$$

For $s > 0$, set $Q^s = \{(x,t) \in Q : |\nabla T_k(u)| \leq s\}$ and $Q_j^s = \{(x,t) \in Q : |\nabla T_k(v_j)| \leq s\}$ and denote by χ^s and χ_j^s the characteristic functions of Q^s and Q_j^s, respectively. On the other hand, the second term of the left-hand side of (21) reads as

$$\begin{aligned}
&\int_Q a(u_n, \nabla u_n)[\nabla T_k(u_n) - \nabla \omega_{\mu,j}^i]\phi'(T_k(u_n) - \omega_{\mu,j}^i)dxdt \\
&= \int_Q [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(v_j)\chi_j^s)] \\
&\quad [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] \times \phi'(T_k(u_n) - \omega_{\mu,j}^i)dxdt \\
&\quad + \int_Q a(T_k(u_n), \nabla T_k(v_j)\chi_j^s)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\phi'(T_k(u_n) - \omega_{\mu,j}^i)dxdt \\
&\quad + \int_Q a(T_k(u_n), \nabla T_k(u_n))\nabla T_k(v_j)\chi_j^s\phi'(T_k(u_n) - \omega_{\mu,j}^i)dxdt \\
&\quad - \int_Q a(u_n, \nabla u_n)\nabla\omega_{\mu,j}^i\phi'(T_k(u_n) - \omega_{\mu,j}^i)dxdt \\
&\quad := J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

We shall go to the limit as n, j, μ and $s \to \infty$ in the last three integrals of the last side.
Starting with J_2, we have by letting $n \to \infty$

$$J_2 = \int_Q a(T_k(u), \nabla T_k(v_j)\chi_j^s)[\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s]\phi'(T_k(u) - \omega_{\mu,j}^i)dxdt + \varepsilon(n),$$

since $a(T_k(u_n), \nabla T_k(v_j)\chi_j^s) \to a(T_k(u), \nabla T_k(v_j)\chi_j^s)$ strongly in $(E_\psi(Q))^N$ by using (8) and Lebesgue theorem while $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_\varphi(Q))^N$ by (18). Letting $j \to \infty$ in the first term of last side of of the above equality, one has, since $a(T_k(u), \nabla T_k(v_j)\chi_j^s) \to a(T_k(u), \nabla T_k(u)\chi^s)$ strongly in $(E_\psi(Q))^N$ by using (8), (20) and Lebesgue theorem while $\nabla T_k(v_j)\chi_j^s \to \nabla T_k(u)\chi^s$ strongly in $(L_\varphi(Q))^N$,

$$J_2 = \int_{Q\setminus Q^s} a(T_k(u), 0)\nabla T_k(u)\phi'(T_k(u) - T_k(u)_\mu - \exp(-\mu t)T_k(w_i))dxdt + \varepsilon(n,j)$$

since $\phi'(T_k(u) - T_k(u)_\mu - \exp(-\mu t)T_k(w_i)) \to 1$ a.e in Q and is uniformly bounded by $\phi'(2k)$ we can let $\mu \to \infty$ in the first term of the last side to get

$$J_2 = \int_{Q\setminus Q^s} a(T_k(u), 0)\nabla T_k(u)dxdt + \varepsilon(n,j,\mu)$$

and thus, by letting $s \to \infty$, we conclude that $J_2 = \varepsilon(n,j,\mu,s)$.

About J_3, we can write

$$J_3 = \int_{\{|u_n|\le k\}} a(u_n,\nabla u_n)\nabla T_k(v_j)\chi_j^s\phi'(T_k(u_n)-\omega^i_{\mu,j})dxdt + \int_{\{|u_n|>k\}} a(T_k(u_n),0)\nabla T_k(v_j)\chi_j^s\phi'(T_k(u_n)-\omega^i_{\mu,j})dxdt,$$

which gives by letting $n\to\infty$, thanks to (18),

$$J_3 = \int_{\{|u|\le k\}} h\nabla T_k(v_j)\chi_j^s\phi'(T_k(u)-\omega^i_{\mu,j})dxdt + \int_{\{|u|>k\}} a(T_k(u),0)\nabla T_k(v_j)\chi_j^s\phi'(T_k(u)-\omega^i_{\mu,j})dxdt + \varepsilon(n),$$

so that, by letting $j\to\infty$ in two first integrals last of the last side and using (20),

$$J_3 = \int_{\{|u|\le k\}} h\nabla T_k(u)\chi^s\phi'(T_k(u)-T_k(u)_\mu-\exp(-\mu t)T_k(w_i)dxdt + \varepsilon(n,j),$$

in which we can let $\mu\to\infty$ to obtain

$$J_3 = \int_Q h\nabla T_k(u)\chi^s dxdt + \varepsilon(n,j,\mu).$$

Consequently, by letting $s\to\infty$,

$$J_3 = \int_Q h\nabla T_k(u)dxdt + \varepsilon(n,j,\mu,s).$$

For what concerns J_4 we have, as above, by letting first n then j and finally μ go to infinity :

$$\begin{aligned} J_4 &= \int_Q h\nabla\omega^i_{\mu,j}\phi'(T_k(u)-\omega^i_{\mu,j})dxdt + \varepsilon(n) \\ &= \int_Q h[\nabla T_k(u)_\mu - \exp(-\mu t)T_k(w_i)] \\ &\quad \phi'(T_k(u)-T_k(u)_\mu-\exp(-\mu t)T_k(w_i))dxdt + \varepsilon(n,j) \\ &= -\int_Q h\nabla T_k(u)dxdt + \varepsilon(n,j,\mu). \end{aligned}$$

We conclude then that

$$\begin{aligned} &\int_Q a(u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\phi'(T_k(u_n)-\omega^i_{\mu,j})dxdt \\ &= \int_Q [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(v_j)\chi_j^s)][\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s] \\ &\qquad \times\phi'(T_k(u_n)-\omega^i_{\mu,j})dxdt + \varepsilon(n,j,\mu,s). \end{aligned} \tag{25}$$

The third term of the left-hand side of(21) can be estimated as

$$\begin{aligned}
&| \int_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\phi(T_k(u_n)-\omega^i_{\mu,j})dxdt \,| \\
&\le b(k)\int_Q (c_2(x,t)+\frac{1}{\alpha}d(x,t))|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt \\
&+\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt. \qquad (26)
\end{aligned}$$

Since $c_2(x,t)$ and $d(x,t)$ belong to $L^1(Q)$ it is easy to see that

$$b(k)\int_Q (c_2(x,t)+\frac{1}{\alpha}d(x,t))|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt=\varepsilon(n,j,\mu).$$

On the other hand, the second term of the right-hand side of (26) reads as

$$\begin{aligned}
&\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt \\
&=\frac{b(k)}{\alpha}\int_Q [a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)] \\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt \\
&+\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt \\
&\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n)\nabla T_k(v_j)\chi^s_j]|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt.
\end{aligned}$$

As above, by letting successively first n, then j,μ and finally s go to infinity, we can easily see that each one of last two integrals of the right-hand side of the last equality is of the form $\varepsilon(n,j,\mu)$ and then

$$\begin{aligned}
&| \int_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\phi(T_k(u_n)-\omega^i_{\mu,j})dxdt \,| \\
&\le \frac{b(k)}{\alpha}\int_Q [a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)] \\
&\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt+\varepsilon(n,j,\mu). \qquad (27)
\end{aligned}$$

Combining (21),(22),(24),(25) and (27), we get

$$\begin{aligned}
&\int_Q [a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)][\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j] \\
&\quad\times[\phi'(T_k(u_n)-\omega^i_{\mu,j})-\frac{b(k)}{\alpha}|\phi(T_k(u_n)-\omega^i_{\mu,j})|]dxdt\le\varepsilon(n,j,\mu,i,s).
\end{aligned}$$

and so,thanks to (19),

$$\begin{aligned}
&\int_Q [a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)] \\
&\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]dxdt\le 2\varepsilon(n,j,\mu,i,s). \qquad (28)
\end{aligned}$$

On the other hand, we have

$$\int_Q [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi^s)][\nabla T_k(u_n) - \nabla T_k(u)\chi^s]dxdt$$
$$-\int_Q [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)][\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s_j]dxdt$$
$$= \int_Q a(T_k(u_n),\nabla T_k(u_n))[\nabla T_k(v_j)\chi^s_j - \nabla T_k(u)\chi^s]dxdt$$
$$-\int_Q a(T_k(u_n),\nabla T_k(u)\chi^s)[\nabla T_k(u_n) - \nabla T_k(u)\chi^s]dxdt$$
$$+\int_Q a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s_j]dxdt$$

and,as it can be easily seen, each integral of the right-hand side is of the form $\varepsilon(n, j, s)$, implying that

$$\int_Q [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi^s)][\nabla T_k(u_n) - \nabla T_k(u)\chi^s]dxdt$$
$$= \int_Q [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)] \times[\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s_j]dxdt + \varepsilon(n, j, s). \quad (29)$$

For $r \leq s$, we have

$$\begin{aligned}
0 &\leq \int_{Q^r} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))] \\
&\quad [\nabla T_k(u_n) - \nabla T_k(u)]dxdt \\
&\leq \int_{Q^s} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))] \\
&\quad [\nabla T_k(u_n) - \nabla T_k(u)]dxdt \\
&= \int_{Q^s} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi^s)] \\
&\quad [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]dxdt \\
&\leq \int_{Q} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi^s)] \\
&\quad [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]dxdt \\
&= \int_{Q} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)] \\
&\quad [\nabla T_k(u_n) - \nabla T_k(u)\chi^s_j]dxdt + \varepsilon(n, j, s) \\
&\leq \varepsilon(n, j, \mu, i, s),
\end{aligned}$$

hence, by passing to the limit sup over n, get

$$\begin{aligned} 0 \leq & \limsup_{n\to\infty} \int_{Q^r} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] \\ & [\nabla T_k(u_n) - \nabla T_k(u)]dxdt \\ \leq & \limsup_{n\to\infty} \varepsilon(n, j, \mu, i, s), \end{aligned}$$

in which we let successively $j \to \infty, \mu \to, i \to \infty$ and $s \to \infty$ to obtain

$$\int_{Q^r} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] \\ [\nabla T_k(u_n) - \nabla T_k(u)]dxdt \to 0 \text{ as} n \to \infty$$

and thus,as in the elliptic case(see [1]), there exists a subsequence also denote by u_n such that

$$\nabla u_n \to \nabla u \text{ a.e. in} Q. \tag{30}$$

We deduce then that,for all $k > 0$
$a(x, t, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup a(x, t, T_k(u), \nabla T_k(u))$
and $a(x, t, u_n, \nabla u_n) \rightharpoonup a(x, t, u, \nabla u)$ weakly in $(L_\psi(Q))^N$ for $\sigma(\Pi L_\psi, \Pi E_\varphi)$
Step 3. Modular convergence of the truncations and equi-integrability of the nonlinearities.
Thanks to (28) and (29), we can write

$$\begin{aligned} & \int_Q a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dxdt \\ & \leq \int_Q a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u) \chi^s dxdt \\ & + \int_Q a(T_k(u_n), \nabla T_k(u) \chi^s)[\nabla T_k(u_n) - \nabla T_k(u) \chi^s] dxdt \\ & + \varepsilon(n, j, \mu, i, s), \end{aligned}$$

and then

$$\begin{aligned} & \limsup_{n\to\infty} \int_Q a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dxdt \\ & \leq \int_Q a(T_k(u), \nabla T_k(u)) \nabla T_k(u) \chi^s dxdt \\ & + \int_Q a(T_k(u_n), \nabla T_k(u) \chi^s)[1 - \chi^s] dxdt \\ & + \lim_{n\to\infty} \varepsilon(n, j, \mu, i, s), \end{aligned}$$

in which we can pass to the limit as $j, \mu, i, s \to \infty$ to obtain

$$\limsup_{n\to\infty} \int_Q a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dxdt \leq \int_Q a(T_k(u), \nabla T_k(u)) \nabla T_k(u) dxdt.$$

On the other hand, Fatou's lemma implies

$$\int_Q a(T_k(u),\nabla T_k(u))\nabla T_k(u)dxdt \le \liminf_{n\to\infty}\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt,$$

and thus, as $n\to\infty$,

$$\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \to \int_Q a(T_k(u),\nabla T_k(u))\nabla T_k(u)dxdt.$$

Since $a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\ge d(x,t)\in L^1(Q)$ we deduce that

$$a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \to a(T_k(u),\nabla T_k(u))\nabla T_k(u)dxdt \text{ in } L^1(Q), \quad (31)$$

as $n\to\infty$; implying by using (10) and Vitali's theorem that

$$\nabla T_k(u_n)) \to \nabla T_k(u) \text{ in } (L_\varphi(Q))^N \text{ for the modular convergence .}$$

We shall now prove that $g_n(x,t,u_n,\nabla u_n)\to g(x,t,u_n,\nabla u_n)$ strongly in $L^1(Q)$ by using Vitli's theorem. Since $g_n(x,u_n,\nabla u_n)\to g(x,u_n,\nabla u_n)$ a.e. in Q,thanks to (17)and (29), it suffices to prove that $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q.
Let $E\subset Q$ be a measurable subset of Q. We have for any $m>0$

$$\begin{aligned}\int_E |g_n(x,t,u_n,\nabla u_n)|dxdt &= \int_{E\cap\{|u_n|\le m\}} |g_n(x,t,u_n,\nabla u_n)|dxdt\\ &\quad+\int_{E\cap\{|u_n|>m\}} |g_n(x,t,u_n,\nabla u_n)|dxdt.\end{aligned}$$

On the one hand

$$\int_{E\cap\{|u_n|>m\}} |g_n(x,t,u_n,\nabla u_n)|dxdt \le \frac{1}{m}\int_Q g_n(x,t,u_n,\nabla u_n)u_n dxdt \le \frac{C}{m},$$

where C is the constant in (17). Therefore, there exists $m=m(\varepsilon)$ large enough such that

$$\int_{E\cap\{|u_n|>m\}} |g_n(x,t,u_n,\nabla u_n)|dxdt \le \frac{\varepsilon}{2}\forall n.$$

On the other hand

$$\begin{aligned}&\int_{E\cap\{|u_n|\le m\}} |g_n(x,t,u_n,\nabla u_n)|dxdt\\ &\le \int_E |g_n(x,t,T_m(u_n),\nabla T_m(u_n))|dxdt\\ &\le b(m)\int_E [c_2(x,t)+\varphi(x,|\nabla T_m(u_n)|)]dxdt\\ &\le b(m)\int_E [c_2(x,t)+\frac{1}{\alpha}d(x,t)]dxdt\\ &+\frac{b(m)}{\alpha}\int_E a(T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt.\end{aligned}$$

By virtue of strong convergence (31) and the fact that $c_2(x,t), d(x,t) \in L^1(Q)$, there exists ν such that

$$|E| < \nu \Rightarrow \int_{E\cap\{|u_n|\leq m\}} |g_n(x,t,u_n,\nabla u_n)|dxdt \leq \frac{\varepsilon}{2}\forall n.$$

Consequently,

$$|E| < \nu \Rightarrow \int_E |g_n(x,t,u_n,\nabla u_n)|dxdt \leq \varepsilon\forall n,$$

which shows that $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q as required.
Step 4. Passage to the limit and regularity of the solution.
Let $v \in W_0^{1,x}L_\varphi(Q)\cap L^\infty(Q)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q) + L^2(Q)$. There exists a prolongation $\bar{v}$ of v such that (see proof of Lemma1)

$$\bar{v} = v \text{ on } Q, \bar{v} \in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R})\cap L^2(\Omega\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R}),$$

and

$$\frac{\partial\bar{v}}{\partial t} = v \in W^{-1,x}L_\psi(\Omega\times\mathbb{R}) + L^2(\Omega\times\mathbb{R}). \tag{32}$$

By Theorem1(see also Remark1), there exists a sequence $(w_j \subset \mathcal{D}(\Omega\times\mathbb{R}))$ such that

$$w_j \to \bar{v} \text{ in } W_0^{1,x}L_\varphi(\Omega\times\mathbb{R})\cap L^2(\Omega\times\mathbb{R}),$$

and

$$\frac{\partial w_j}{\partial t} \to \frac{\partial\bar{v}}{\partial t} \text{ in } W^{-1,x}L_\psi(\Omega\times\mathbb{R}) + L^2(\Omega\times\mathbb{R}), \tag{33}$$

for the modular convergence and $||w_j||_{\infty,\Omega\times\mathbb{R}} \leq (N+2)||\bar{v}||_{\infty,\Omega\times\mathbb{R}}$.
Go back to approximate equations (16) and use $w_j\chi_{(0,\tau)}$, for every $\tau\in[0,T]$(which belongs to $W_0^{1,x}L_\varphi(Q)$) as a test function one has

$$\begin{aligned}\langle\frac{\partial u_n}{\partial t}, w_j\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla w_j dxdt\\ + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)w_j dxdt = \langle f, w_j\rangle_{Q_\tau},\end{aligned}$$

which implies that

$$\begin{aligned}[\int_\Omega u_n(t)w_j(t)dx]_0^\tau - \int_{Q_\tau} u_n\frac{\partial w_j}{\partial t}dxdt + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla w_j dxdt\\ + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)w_j dxdt = \langle f, w_j\rangle_{Q_\tau}.\end{aligned} \tag{34}$$

We shall go to the limit as $n \to \infty$ in all terms of (34). Since for all $j, w_j \chi_{(0,\tau)} \in \mathcal{D}(\bar{Q}_\tau)$ we have

$$-\int_{Q_\tau} u_n \frac{\partial w_j}{\partial t} dxdt \to -\int_{Q_\tau} u \frac{\partial w_j}{\partial t} dxdt,$$

$$\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla w_j dxdt \to \int_{Q_\tau} a(x,t,u,\nabla u)\nabla w_j dxdt$$

and

$$\int_{Q_\tau} g_n(x,t,u_n,\nabla u_n) w_j dxdt \to \int_{Q_\tau} g(x,t,u,\nabla u) w_j dxdt.$$

To go to the limit as $n \to \infty$ in the first term of (34), we will first prove that $u_n \to u$ in $C([0,T], L^2(\Omega))$ (implying, in particular, that $u \in C([0,T], L^2(\Omega))$).
To do that,let now $\omega_{j,\mu}^{i,l} = T_l(v_j)_\mu + \exp(-\mu t) T_l(w_i)$ and
$\omega_\mu^{i,l} = T_l(u)_\mu + \exp(-\mu t) T_l(w_i)$, for every $l > 0$.
On one hand, we have for every $\tau \in (0,T]$

$$\begin{aligned} \langle (\omega_{j,\mu}^{i,l})', u_n - \omega_{j,\mu}^{i,l} \rangle_{Q_\tau} &= \mu \int_{Q_\tau} (T_l(v_j) - \omega_{j,\mu}^{i,l})(u_n - \omega_{j,\mu}^{i,l}) dxdt \\ &\to \mu \int_{Q_\tau} (T_l(v_j) - \omega_{j,\mu}^{i,l})(u - \omega_{j,\mu}^{i,l}) dxdt \\ &\to \mu \int_{Q_\tau} (T_l(u) - \omega_{j,\mu}^{i,l})(u - \omega_{j,\mu}^{i,l}) dxdt \geq 0, \end{aligned} \tag{35}$$

as first $n \to \infty$ and then $j \to \infty$ and where we have used the fact that $\omega_\mu^{i,l} \leq l$ to get the positiveness of last integral.
On the other hand, by using (16)

$$\begin{aligned} \langle u_n', u_n - \omega_{j,\mu}^{i,l} \rangle_{Q_\tau} = \langle f, u_n - \omega_{j,\mu}^{i,l} \rangle_{Q_\tau} &+ \int_{Q_\tau} a(u_n, \nabla u_n)[\nabla \omega_{j,\mu}^{i,l} - \nabla u_n] dxdt \\ &+ \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)(\omega_{j,\mu}^{i,l} - u_n) dxdt, \end{aligned}$$

in which we can use Fatou's lemma and Lebesgue theorem to pass to the limit sup first over n and then over j, μ, l, to get

$$\langle u_n', u_n - \omega_{j,\mu}^{i,l} \rangle_{Q_\tau} \leq \varepsilon(n,j,\mu,l) \text{ not depending on } \tau. \tag{36}$$

Therefore, by writing

$$\begin{aligned} \frac{1}{2} ||u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau)||_{L^2(\Omega)}^2 &= \langle u_n' - (\omega_{j,\mu}^{i,l})', u_n - \omega_{j,\mu}^{i,l} \rangle_{Q_\tau} \\ &+ \frac{1}{2} \int_\Omega (u_0 - T_l(w_i))^2 dxdt \\ &= \langle u_n' - (\omega_{j,\mu}^{i,l})', u_n - \omega_{j,\mu}^{i,l} \rangle_{Q_\tau} \\ &+ \frac{1}{2} ||u_0, u_n - T_l(w_i)||_{L^2(\Omega)}^2, \end{aligned}$$

and using (35) and (37), we deduce that $||u_n(\tau)-\omega^{i,l}_{j,\mu}(\tau)||_{L^2(\Omega)} \leq \varepsilon(n,j,\mu,l,i)$ not depending on $\tau \in (0,T]$. This implies that

$$||u_n(\tau) - u_m(\tau)||_{L^2(\Omega)} \leq \varepsilon(n,m) \text{ not depending on } \tau \in [0,T],$$

and thus,u_n is a Cauchy sequence in $C([0,T], L^2(\Omega))$.
Since the limit of u_n in $L^1(Q)$ is u we deduce that

$$u_n \to u \text{ in } C([0,T], \Omega),$$

therefore, by letting $n \to \infty$ in the first term of (34), we have

$$[\int_\Omega u_n(t)w_j(t)dx]_0^\tau \to [\int_\Omega u(t)w_j(t)dx]_0^\tau.$$

Consequently, by letting $n \to \infty$ in (34), we get

$$[\int_\Omega u(t)w_j(t)dx]_0^\tau - \int_{Q_\tau} u\frac{\partial w_j}{\partial t}dxdt + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla w_j dxdt \\ + \int_{Q_\tau} g(x,t,u,\nabla u)w_j dxdt = \langle f, w_j\rangle_{Q_\tau}. \quad (37)$$

We shall now go to the limit as $j \to \infty$ in all terms of (37). In view of (33) and the fact that w_j are uniformly bounded, there is problem to pass to the limit in last four terms of (37). For what concerns the first one,observe that,as in the proof of Lemma 3.4, we have $w_j \to v$ in $C([0,T], L^2(\Omega))$. Therefore, we can let $j \to \infty$ in all terms of (37) to get

$$[\int_\Omega u(t)v(t)dx]_0^\tau - \langle\frac{\partial v}{\partial t}, u\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla v dxdt \\ + \int_{Q_\tau} g(x,t,u,\nabla u)v dxdt = \langle f, v\rangle_{Q_\tau},$$

which shows that u satisfies all properties of Theorem 5.1.
It only remains to prove the energy equality. For that, we use, for a given $k > 0, T_k(u_n)$ as a test function in (16), to get

$$\langle u_n', T_k(u_n)\rangle_{Q_\tau} = -\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)dxdt \\ - \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_k(u_n)dxdt + \langle f, T_k(u_n)\rangle_{Q_\tau},$$

which gives by setting $S_k(s) = \int_0^s T_k(z)dz$,

$$\int_\Omega S_k(u_n(\tau))dx - \int_\Omega S_k(u_0)dx = -\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)dxdt \\ - \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_k(u_n)dxdt + \langle f, T_k(u_n)\rangle_{Q_\tau}. \quad (38)$$

Recall that $|S_k(u_n(\tau))| \leq k|u_n(\tau)| \to k|u(\tau)|$ in $L^2(\Omega)$ as $n \to \infty$, then, by using Lesbegue theorem and (31), we can pass to the limit as $n \to \infty$ each term of (38) to obtain

$$\int_\Omega S_k(u(\tau))dx - \int_\Omega S_k(u_0)dx = -\int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u)dxdt - \int_{Q_\tau} g(x,t,u,\nabla u)T_k(u)dxdt + \langle f, T_k(u)\rangle_{Q_\tau}. \tag{39}$$

Observe that for every $s \in \mathbb{R}$,

$$|S_k(s)| \leq \frac{s^2}{2} \text{ and } S_k(s) \to \frac{s^2}{2} \text{ as } k \to \infty,$$

so that, by using Lebesgue theorem and the fact that $u(\tau) \in L^2(\Omega)$, we have, as $k \to \infty$

$$\int_\Omega S_k(u(\tau))dx \to \frac{1}{2}\int_\Omega u^2(\tau) \text{ and } \int_\Omega S_k(u_0)dx \to \frac{1}{2}\int_\Omega S_k(u_0)^2 dx.$$

Remark also that

$$|a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)| \leq a(x,t,u,\nabla u)\nabla u \in L^1(Q)$$

and

$$|g(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)| \leq g(x,t,u,\nabla u)\nabla u \in L^1(Q),$$

therefore, it is easy to pass the limit as $k \to \infty$ in (39) to get the energy equality

$$[\frac{1}{2}\int_\Omega u(t)^2 dx]_0^\tau + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla u dxdt + \int_{Q_\tau} g(x,t,u,\nabla u)u dxdt = \langle f, u\rangle_{Q_\tau}.$$

This completes the proof of Theorem 5.1.

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